PARAMETERIZATION

Question 1

A curve C is defined parametrically

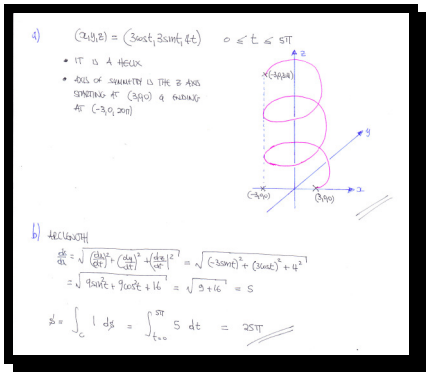
$$(x, y, z) = (3\cos t, 3\sin t, 4t), \quad 0 \le t \le 5\pi .$$

where t is a parameter.

a) Sketch the graph of C.

b) Find the length of C.

25π



Question 2

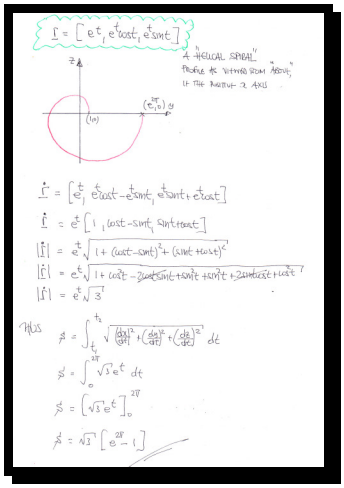
A curve C is defined parametrically

$$(x, y, z) = \left(\mathrm{e}^t, \mathrm{e}^t \cos t, \mathrm{e}^t \sin t\right), \quad 0 \le t \le 2\pi .$$

where t is a parameter.

Describe the graph of C and find its length.

$$\boxed{\text{arclength} = \sqrt{3}\left[\mathrm{e}^{2\pi} - 1\right]}$$



Question 3

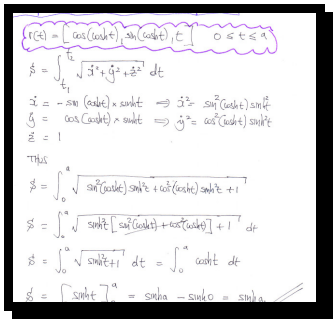
The position vector of a curve C is given by

$$\mathbf{r}(t)=\cos(\cosh t)\,\mathbf{i}+\sin(\cosh t)\,\mathbf{j}+t\,\mathbf{k}\,,$$

where t is a scalar parameter with $0\le t\le a$, $a\in\mathbb{R}$.

Determine the length of C.

$$\boxed{\text{arclength}=\sinh a}$$



Question 4

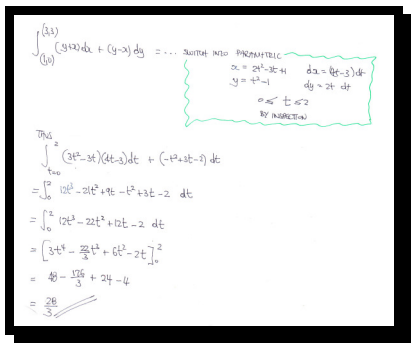
Evaluate the integral

$$\int_{(1,0)}^{(3,3)} \left(y+x\right) dx + \left(y-x\right) dy ,$$

along the curve with parametric equations

$$x = 2t^2 - 3t + 1 \quad \text{and} \quad y = t^2 - 1 .$$

$$\frac{28}{3}$$



Question 5

It is given that

$$\mathbf{F}(x,y,z)\equiv \mathbf{j}\wedge\mathbf{r}\,,$$

where $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

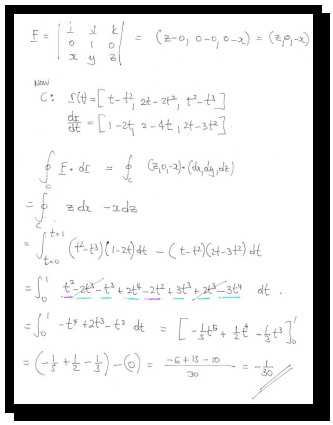
Evaluate the line integral

$$\int_C \mathbf{F}\cdot\mathbf{dr}\,,$$

where C is the closed curve given parametrically by

$$\mathbf{R}(t)=\left(t-t^2\right)\mathbf{i}+\left(2t-2t^2\right)\mathbf{j}+\left(t^2-t^3\right)\mathbf{k}\,,\ 0\le t\le 1\,.$$

$$\boxed{-\frac{1}{30}}$$



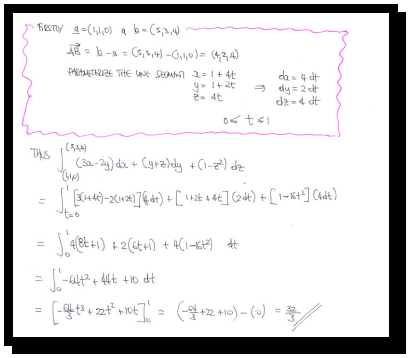
Question 6

Evaluate the integral

$$\int_{(1,1,0)}^{(5,3,4)} (3x-2y)\,dx+(y+z)\,dy+\left(1-z^2\right)dz\,,$$

along the straight line segment joining the points with Cartesian coordinates $(1,1,0)$ and $(5,3,4)$.

$$\frac{32}{3}$$



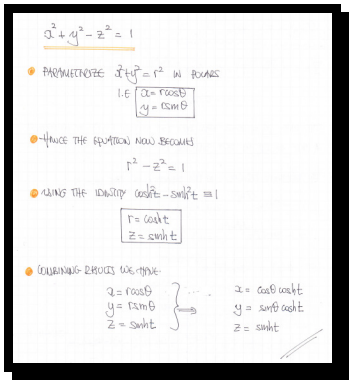
Question 7

A surface S has equation

$$x^2 + y^2 - z^2 = 1 .$$

Find a suitable parameterization of S.

$$\boxed{x = \cos\theta\cosh t, \quad y = \sin\theta\cosh t, \quad z = \sinh t}$$



Question 8

The position vector of a curve C is given by

$$\mathbf{r}(t)=\left(\frac{2}{1+t^2}-1\right)\mathbf{i}+\left(\frac{2t}{1+t^2}\right)\mathbf{j},$$

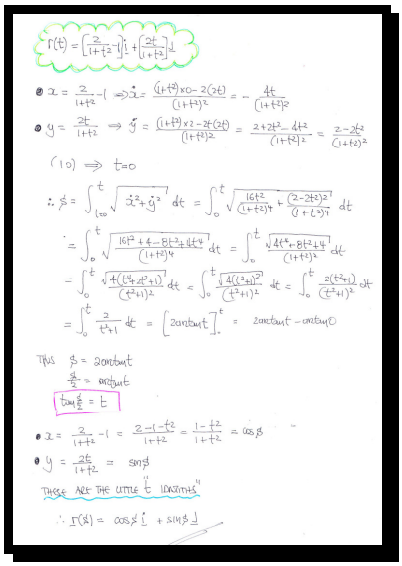
where t is a scalar parameter with $t\in\mathbb{R}$.

Find an expression for the position vector of C, giving the answer in the form

$$\mathbf{r}(s)=f(s)\mathbf{i}+g(s)\mathbf{j},$$

where s is the arc length of a general point on C, measured from the point $(1,0)$.

$$\boxed{\mathbf{r}(s)=(\cos s)\mathbf{i}+(\sin s)\mathbf{j}}$$



Question 9

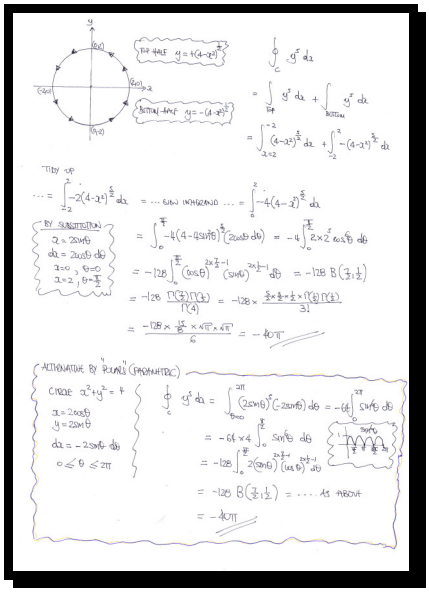
Evaluate the line integral

$$\oint_C y^5 \, dx,$$

where C is a circle of radius 2, centre at the origin O, traced anticlockwise.

You may not use Green's theorem in this question.

$\boxed{-40\pi}$



Question 10

A curve C is defined by $\mathbf{r}=\mathbf{r}(t)$, $0\le t\le 2\pi$ as

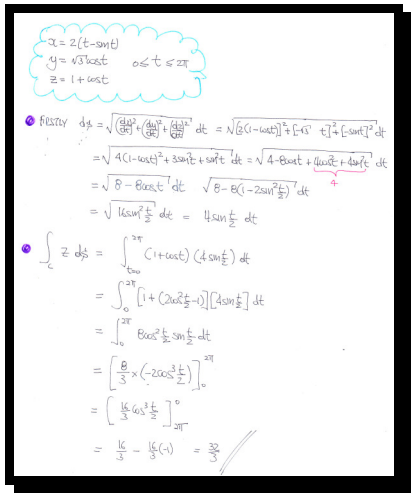
$$\mathbf{r}(t)=(x,y,z)=\left[2(t-\sin t)\,,\,\sqrt{3}\cos t\,,\,1+\cos t\right].$$

Evaluate the integral

$$\int_C z\,ds\,,$$

where s is the arclength along C.

$$\frac{32}{3}$$



Question 11

$$V(x,y,z)=60xyz^2 .$$

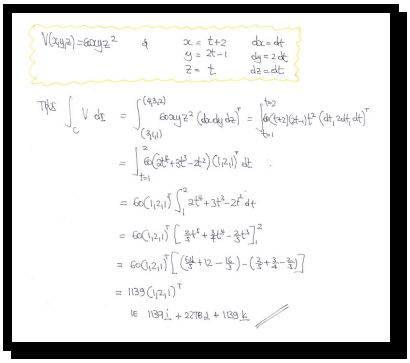
Evaluate the following integral along C, from $(3,1,1)$ to $(4,3,2)$,

$$\int_C V\ \mathbf{dr}, \qquad \mathbf{dr}=(dx,dy,dz)^{\mathrm{T}},$$

where C is the curve with parametric equations

$$x=t+2, \quad y=2t-1, \quad z=t .$$

$$\boxed{1139\mathbf{i}+2278\mathbf{j}+1139\mathbf{k}}$$



Question 12

$$\varphi(x,y,z) \equiv 3x+2y+z\,.$$

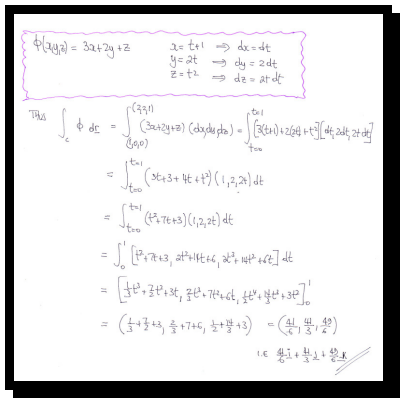
Evaluate the following integral along C, from $(1,0,0)$ to $(2,2,1)$,

$$\int_C \varphi\,\mathbf{dr}\,, \qquad \mathbf{dr}=(dx,dy,dz)^{\mathrm{T}}\,,$$

where C is the curve with parametric equations

$$x=t+1\,, \quad y=2t\,, \quad z=t^2\,.$$

$$\boxed{\frac{41}{6}\mathbf{i}+\frac{41}{3}\mathbf{j}+\frac{49}{6}\mathbf{k}}$$



Question 13

$$F(x,y,z)=xyz\,.$$

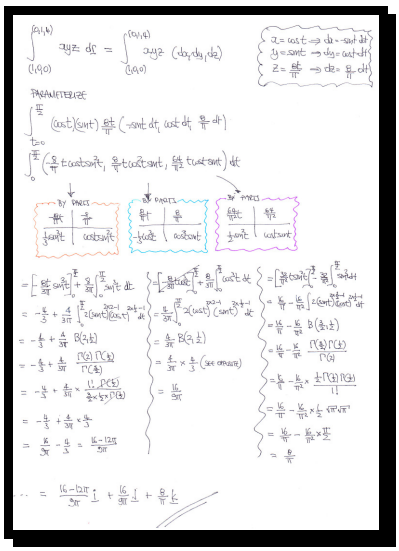
Evaluate the following integral along C, from $(1,0,0)$ to $(0,1,4)$,

$$\int_C F\ \mathbf{dr}\,,\qquad \mathbf{dr}=(dx,dy,dz)^{\mathrm{T}}\,,$$

where C is the curve with parametric equations

$$x=\cos t\,,\qquad y=\sin t\,,\qquad z=\frac{8t}{\pi}\,.$$

$$\boxed{\frac{16-12\pi}{9\pi}\mathbf{i}+\frac{16}{9\pi}\mathbf{j}+\frac{8}{\pi}\mathbf{k}}$$



Question 14

A curve C has equation

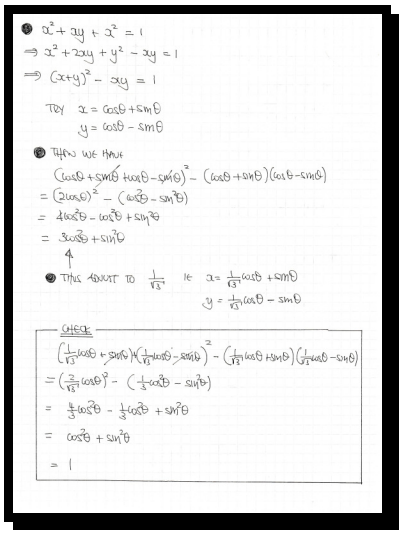
$$x^2 + xy + y^2 = 1. \quad , \quad 0 \le x \le 3.$$

Find a suitable parameterization of C in the form

$$x = A\cos\theta + B\sin\theta \quad \text{and} \quad y = A\cos\theta - B\sin\theta,$$

where A and B are suitable constants.

$$\boxed{x = \frac{1}{\sqrt{3}}\cos\theta + \sin\theta}, \quad \boxed{x = \frac{1}{\sqrt{3}}\cos\theta - \sin\theta}$$



Question 15

A surface S is given parametrically by

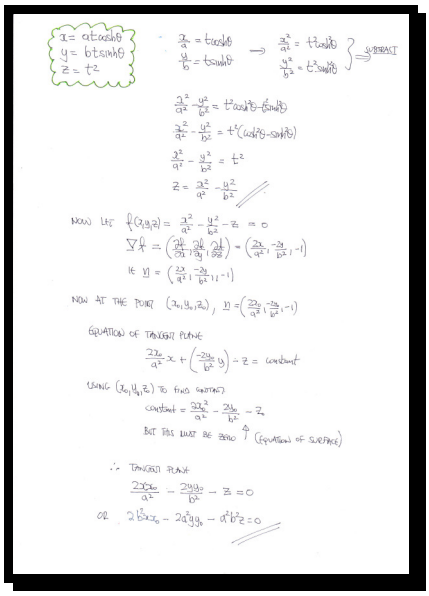
$$x = at\cosh\theta\,,\ x = bt\sinh\theta\,,\ z = t^2\,,$$

where t and θ are real parameters, and a and b are non zero constants .

a) Find a Cartesian equation for S .

b) Determine an equation of the tangent plane on S at the point with Cartesian coordinates (x_0, y_0, z_0).

$$\boxed{z = \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,,\quad \boxed{2b^2 x\,x_0 + 2a^2 y\,y_0 - a^2b^2 z = 0}$$



Question 16

In standard notation used for tori, r is the radius of the tube and R is the distance of the centre of the tube from the centre of the torus.

The surface of a torus has parametric equations

$$x(\theta,\varphi)=(R+r\cos\theta)\cos\varphi,\qquad y(\theta,\varphi)=(R+r\cos\theta)\sin\varphi,\qquad z(\theta,\varphi)=r\sin\theta,$$

where $0\le\theta\le 2\pi$ and $0\le\varphi\le 2\pi$.

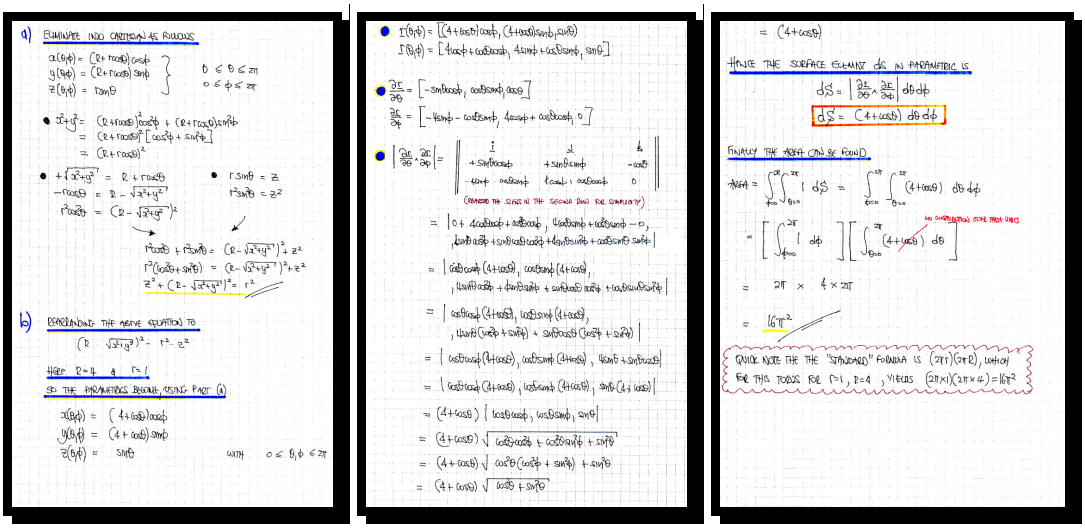
a) Find a general Cartesian equation for the surface of a torus.

A torus T has Cartesian equation

$$\left(4-\sqrt{x^2+y^2}\right)^2=1-z^2.$$

b) Use a suitable parameterization of T to find its surface area.

$$\boxed{},\qquad \boxed{z^2+\left(R-\sqrt{x^2+y^2}\right)^2=r^2},\qquad \boxed{\text{area}=(2\pi r)(2\pi R)=16\pi^2}$$



Question 17

A spiral ramp is modelled by the surface S defined by the vector function

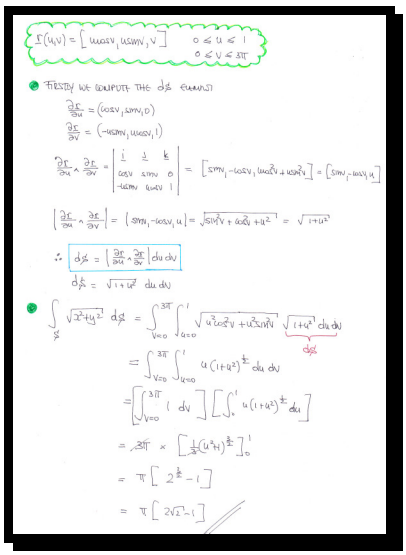
$$\mathbf{r}(u,v)=(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=(u\cos v)\mathbf{i}+(u\sin v)\mathbf{j}+v\mathbf{k}\,,$$

where $0\le u\le 1$, $0\le v\le 3\pi$.

Determine the value of

$$\int_S \sqrt{x^2+y^2}\;dS$$

$$\boxed{\pi\left[\sqrt{8}-1\right]}$$



Question 18

The surface S is defined by the vector equation

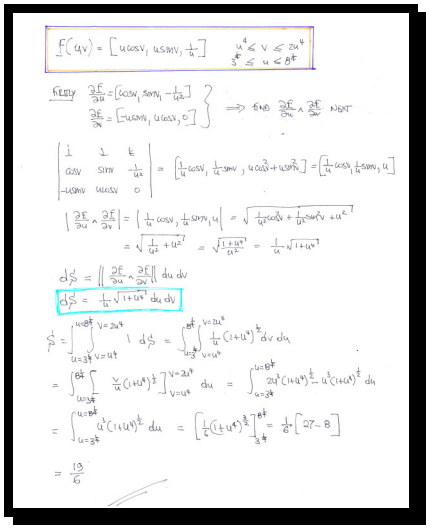
$$\mathbf{F}(u,v)=\left[u\cos v, u\sin v, \frac{1}{u}\right]^T,\ u\neq 0.$$

Find the area of S lying above the region in the uv plane bounded by the curves

$$v=u^4,\ v=2u^4,$$

and the straight lines with equations $u=3^{\frac{1}{4}}$ and $u=8^{\frac{1}{4}}$.

$$\frac{19}{6}$$



Question 19

The surface S is defined by the parametric equations

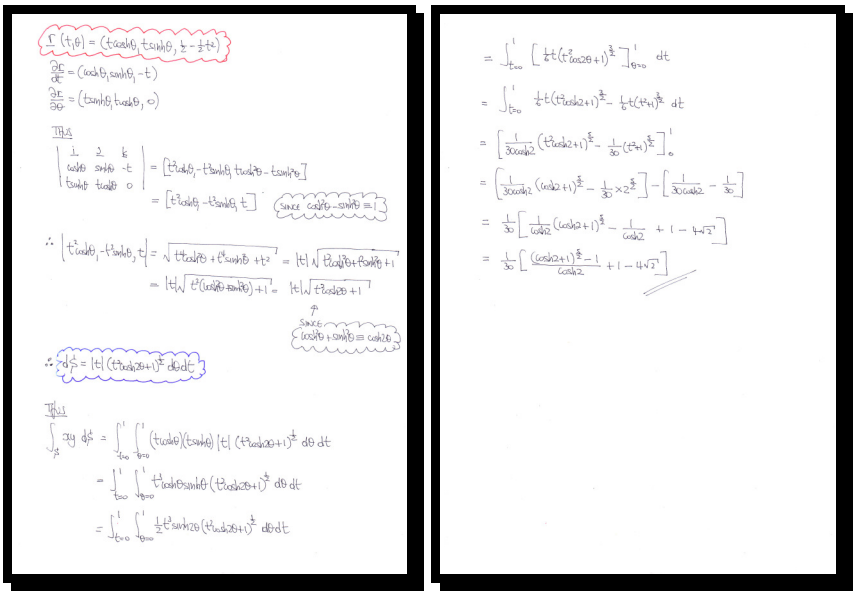
$$x = t\cosh\theta,\ y = t\sinh\theta,\ z = \frac{1}{2}\left(1-t^2\right),$$

where t and θ are real parameters such that $0 \le t \le 1$ and $0 \le \theta \le 1$.

Find, in exact form, the value of

$$\int_S xy\ dS .$$

$$\boxed{\frac{1}{30}\left[\frac{(\cosh 2+1)^{\frac{5}{2}}-1}{\cosh 2}+1-4\sqrt{2}\right] \approx 0.274397...}$$



Question 20

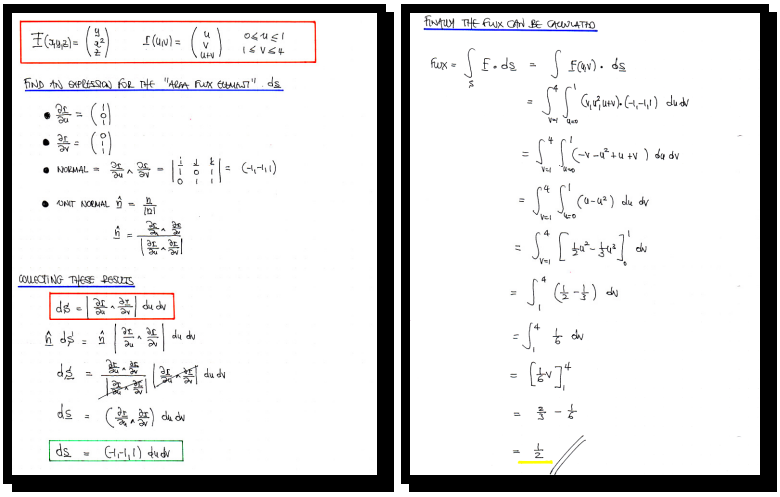
$$\mathbf{F}(x,y,z) \equiv y\,\mathbf{i} + x^2\,\mathbf{j} + z\,\mathbf{k}\,.$$

Find the magnitude of the flux through the surface with parametric equations

$$\mathbf{r}(u,v) = u\,\mathbf{i} + v\,\mathbf{j} + (u+v)\,\mathbf{k}\,, \quad 0 \le u \le 1,\ 1 \le v \le 4\ .$$

All integrations must be carried out in parametric.

$\square$, $\boxed{\frac{1}{2}}$



Question 21

Evaluate the surface integral

$$\int_S \mathbf{F}\cdot\mathbf{dS},$$

where S is the surface represented parametrically by

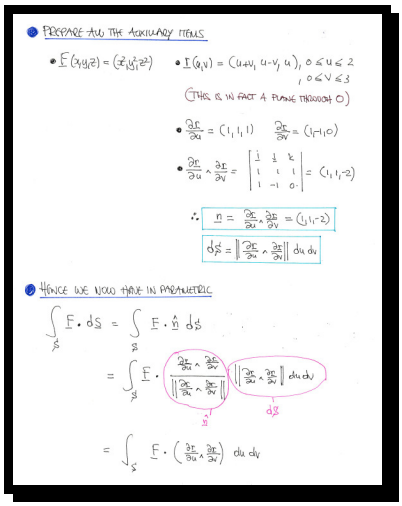
$$\mathbf{r}(u,v)=\begin{bmatrix} u+v \\ u-v \\ u \end{bmatrix}, \quad 0\le u\le 2, \quad 0\le v\le 3,$$

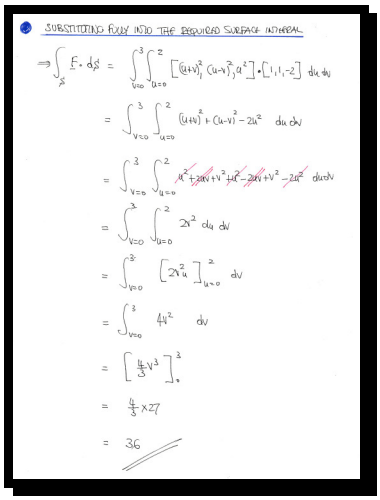
and $\mathbf{F}$ is the vector field

$$x^2\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}.$$

All integrations must be carried out in parametric.

, 36





Question 22

Evaluate the surface integral

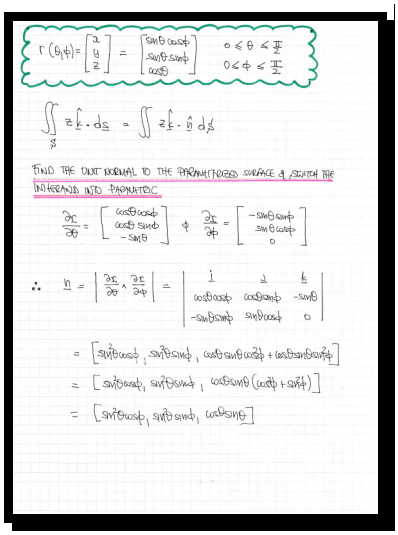
$$\int_S z\,\mathbf{k}\cdot\mathbf{dS}\,,$$

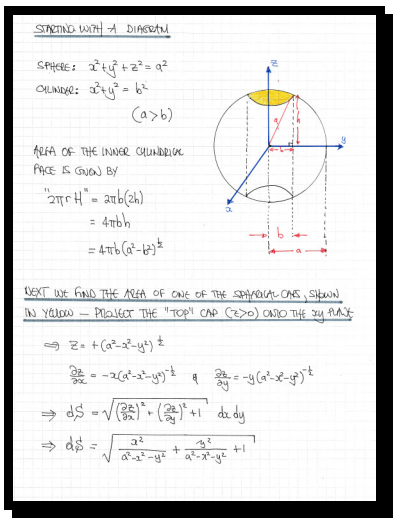
where S is the surface represented parametrically by

$$\mathbf{r}(\theta,\varphi)=\begin{bmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{bmatrix},\ 0\le\theta\le\tfrac{1}{2}\pi\,,\ 0\le\varphi\le\tfrac{1}{2}\pi\,.$$

All integrations must be carried out in parametric.

$\square$, $\boxed{\frac{1}{6}\pi}$





Question 23

$$\mathbf{F}(x,y,z)\equiv x\mathbf{i}+y\mathbf{j}+2z\mathbf{k}\,.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F}\cdot\mathbf{dS}\,,$$

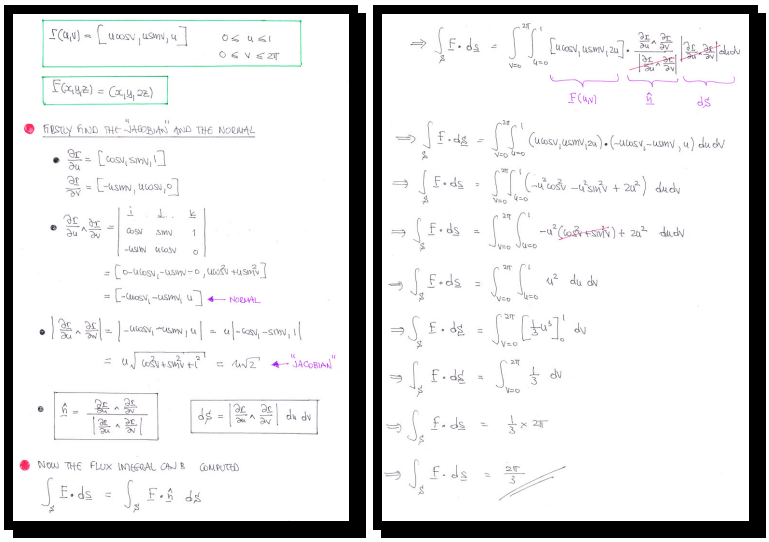
where S is the surface with parametric equations

$$\mathbf{r}(u,v)=(u\cos v)\mathbf{i}+(u\sin v)\mathbf{j}+u\mathbf{k}\,,$$

such that $0\le u\le 1$, $0\le v\le 2\pi$.

All integrations must be carried out in parametric.

$$\boxed{\frac{2}{3}\pi}$$



Question 24

$$\mathbf{F}(x,y,z)\equiv x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\,.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F}\cdot\mathbf{dS}\,,$$

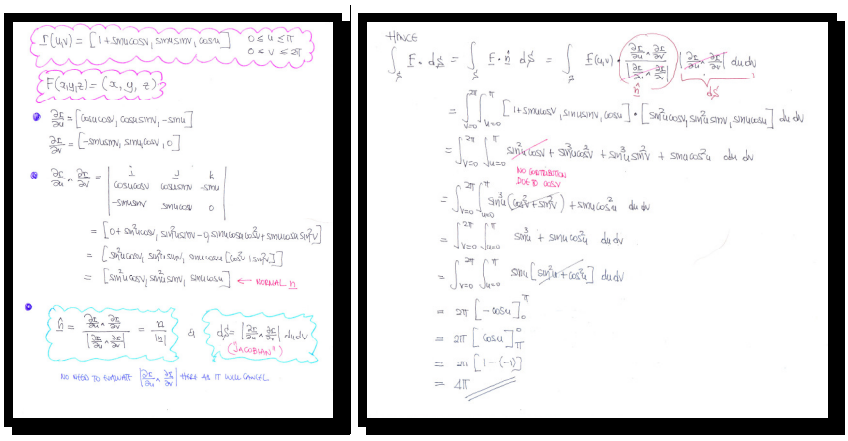
where S is the surface with parametric equations

$$\mathbf{r}(u,v)=(1+\sin u\cos v)\mathbf{i}+(\sin u\sin v)\mathbf{j}+(\cos u)\mathbf{k}\,,$$

such that $0\le u\le\pi$, $0\le v\le 2\pi$.

All integrations must be carried out in parametric.

$\boxed{4\pi}$



Question 25

$$\mathbf{F}(x,y,z)\equiv x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\,.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F}\cdot\mathbf{dS}\,,$$

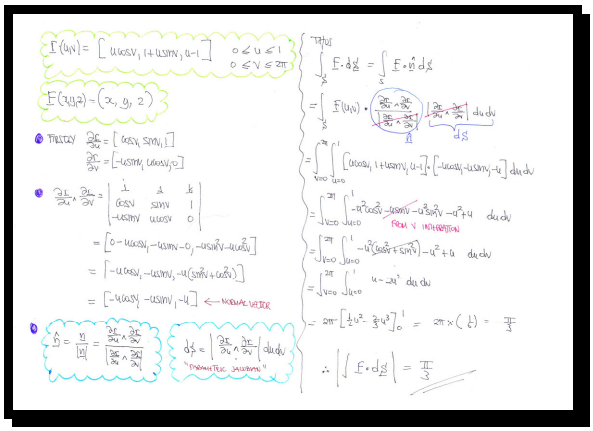
where S is the surface with parametric equations

$$\mathbf{r}(u,v)=(u\cos v)\mathbf{i}+(1+u\sin v)\mathbf{j}+(u-1)\mathbf{k}\,,$$

such that $0\le u\le 1$, $0\le v\le 2\pi$.

All integrations must be carried out in parametric.

$$\boxed{\frac{1}{3}\pi}$$



Question 26

$$\mathbf{F}(x,y,z)\equiv x\mathbf{i}+y\mathbf{j}+2z\mathbf{k}\,.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F}\cdot\mathbf{dS}\,,$$

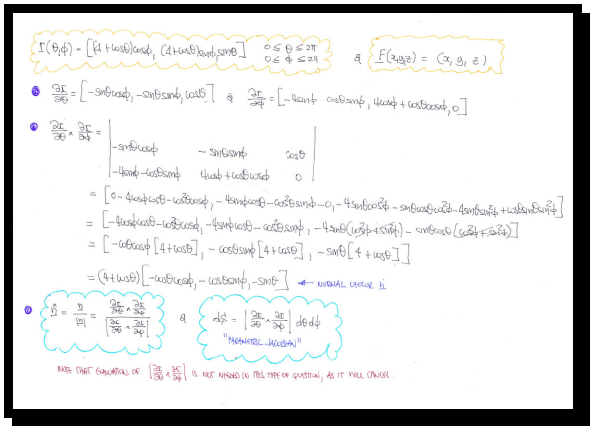
where S is the surface with parametric equations

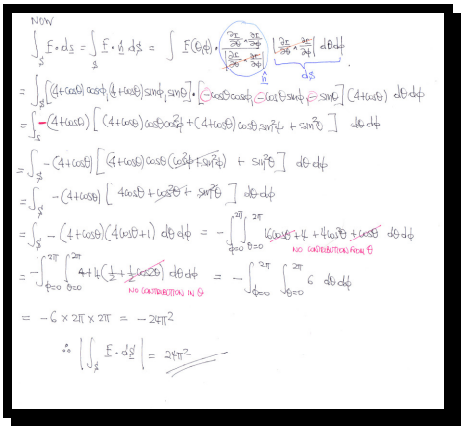
$$\mathbf{r}(\theta,\varphi)=\big[(4+\cos\theta)\cos\varphi\big]\mathbf{i}+\big[(4+\cos\theta)\sin\varphi\big]\mathbf{j}+(\sin\theta)\mathbf{k}\,,$$

such that $0\le\theta\le 2\pi$, $0\le\varphi\le 2\pi$.

All integrations must be carried out in parametric.

$24\pi^2$





Question 27

It is given that the vector field $\mathbf{F}$ satisfies

$$\mathbf{F} = 2y\mathbf{i} - 2x\mathbf{j} + \mathbf{k}\,.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F}\cdot\mathbf{dS}\,,$$

where S is the surface with Cartesian equation

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0\ ,$$

cut off by the cylinder with cartesian equation

$$x^2 + y^2 = x\,.$$

You **must** find a suitable parameterization for S, and carry out the **integration in parametric**, without using any integral theorems.

$$\frac{\pi}{4}$$

