DIFFERENTIAL EQUATIONS

2nd order or higher

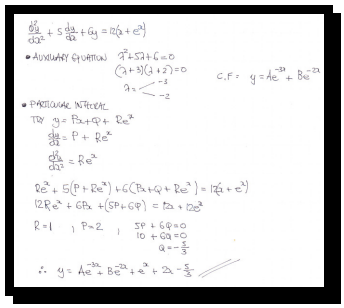
2^{ND} ORDER WITH CONSTANT COEFFICIENTS

Question 1 (**)

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=12\left(x+\mathrm{e}^x\right).$$

$\boxed{}$, $\boxed{y=A\mathrm{e}^{-3x}+B\mathrm{e}^{-2x}+\mathrm{e}^x+2x-\frac{5}{3}}$

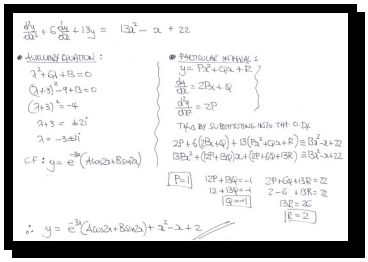


Question 2 (**)

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+13y=13x^2-x+22.$$

$\boxed{}$, $\boxed{y=\mathrm{e}^{-3x}\left(A\cos 2x+B\sin 2x\right)+x^2-x+2}$



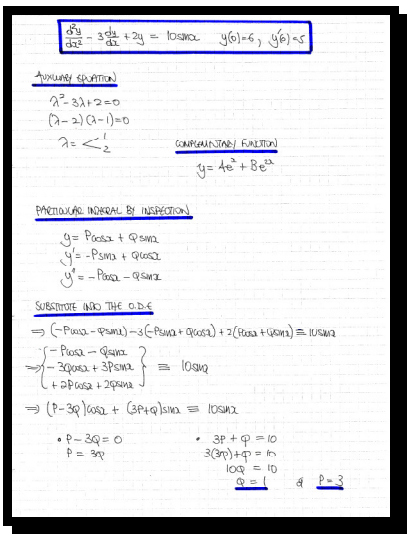
Question 3 (**)

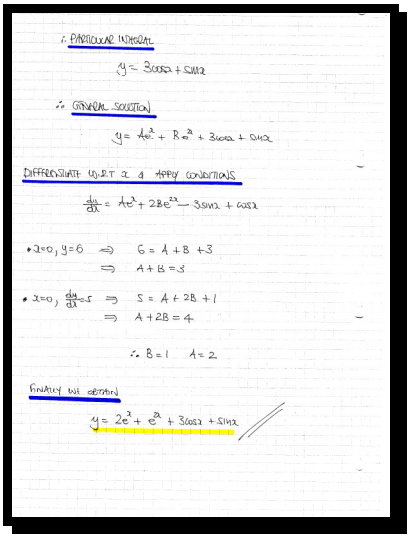
Find a solution of the differential equation

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=10\sin x,$$

subject to the boundary conditions $y=6$ and $\frac{dy}{dx}=5$ at $x=0$.

$\boxed{}$, $\boxed{y=2\mathrm{e}^{x}+\mathrm{e}^{2x}+3\cos x+\sin x}$



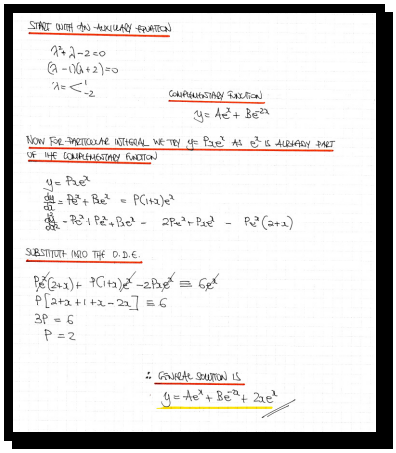


Question 4 (**)

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=6\mathrm{e}^{x}.$$

$\boxed{}$, $\boxed{y=(A+2x)\mathrm{e}^{x}+B\mathrm{e}^{-2x}}$



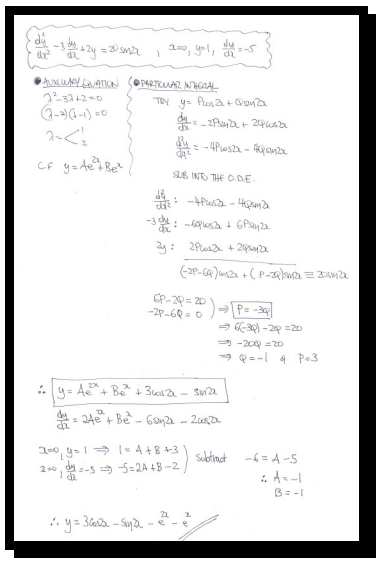
Question 5 (**)

Find a solution of the differential equation

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=20\sin 2x\,,$$

subject to the boundary conditions $y=1$ and $\dfrac{dy}{dx}=-5$ at $x=0$.

$$\boxed{y=3\cos 2x-\sin 2x-\mathrm{e}^{2x}-\mathrm{e}^{x}}$$

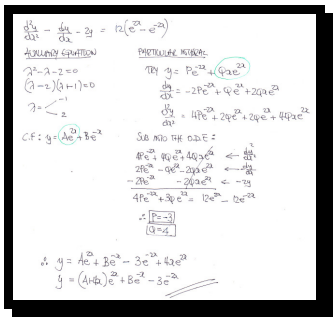


Question 6 (**)

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2}-\frac{dy}{dx}-2y=12\left(\mathrm{e}^{2x}-\mathrm{e}^{-2x}\right).$$

$$\boxed{y=\left(A+4x\right)\mathrm{e}^{2x}+B\mathrm{e}^{-x}-3\mathrm{e}^{-2x}}$$



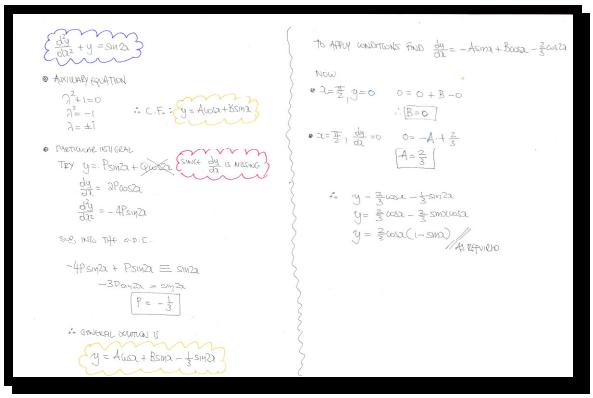
Question 7 (**)

$$\frac{d^2y}{dx^2}+y=\sin 2x, \quad \text{with } y=0 \text{ , } \frac{dy}{dx}=0 \text{ at } x=\frac{\pi}{2}.$$

Show that a solution of the above differential equation is

$$y=\frac{2}{3}\cos x\left(1-\sin x\right).$$

proof



Question 8 **(**+)**

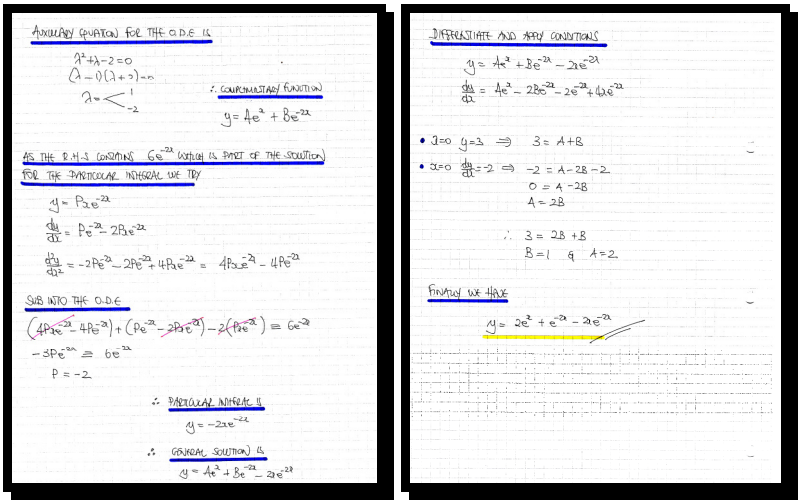
$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=6\mathrm{e}^{-2x},$$

with $y=3$ and $\frac{dy}{dx}=-2$ at $x=0$.

Show that the solution of the above differential equation is

$$y=2\mathrm{e}^{x}+(1-2x)\mathrm{e}^{-2x}.$$

[ ], [proof]

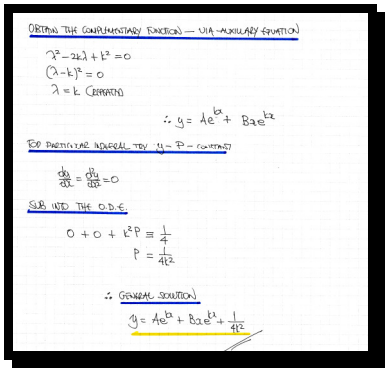


Question 9 (+)**

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2}-2k\frac{dy}{dx}+k^2y=\frac{1}{4},\ k>0.$$

$\boxed{}$, $\boxed{y=A\mathrm{e}^{kx}+Bx\mathrm{e}^{kx}+\frac{1}{4k^2}}$



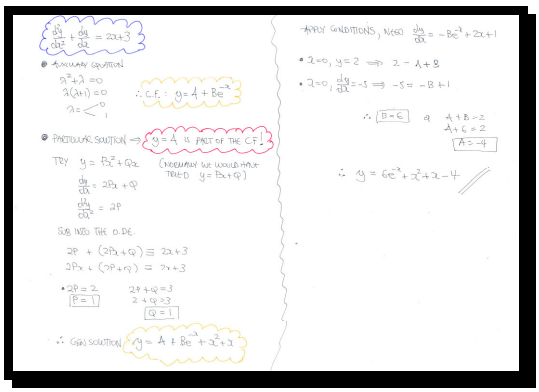
Question 10 (+)**

Find the solution of the differential equation

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}=2x+3,$$

subject to the conditions $y=2$, $\frac{dy}{dx}=-5$ at $x=0$.

$$\boxed{y=x^2+x-4+6\mathrm{e}^{-x}}$$



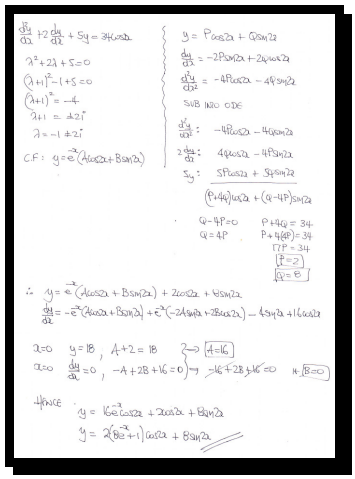
Question 11 (+)**

Find a solution of the differential equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=34\cos 2x,$$

subject to the boundary conditions $y=18$ and $\frac{dy}{dx}=0$ at $x=0$.

$$\boxed{y=2\left(8\mathrm{e}^{-x}+1\right)\cos 2x+8\sin 2x}$$



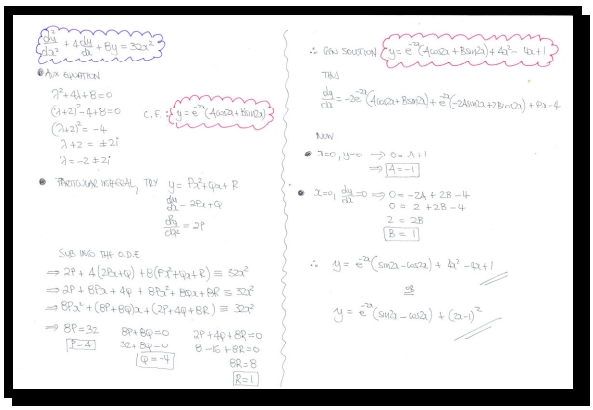
Question 12 (+)**

The curve C has a local minimum at the origin and satisfies the differential equation

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+8y=32x^2 .$$

Find an equation for C.

$$\boxed{y=\mathrm{e}^{x}\left(\sin 2x+\cos 2x\right)+\left(2x-1\right)^2}$$



Question 13 **(**+)**

$$\frac{d^2x}{dt^2}+9x+12\sin 3t=0,\ t\geq 0,$$

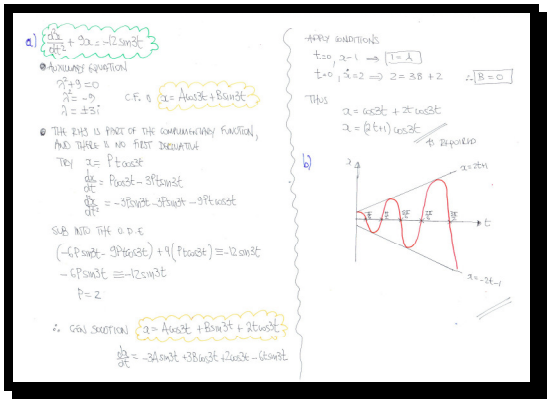
with $x=1$, $\frac{dx}{dt}=2$ at $t=0$.

a) Show that a solution of the differential equation is

$$x=(2t+1)\cos 3t.$$

b) Sketch the graph of x.

proof



Question 14 **(**+)**

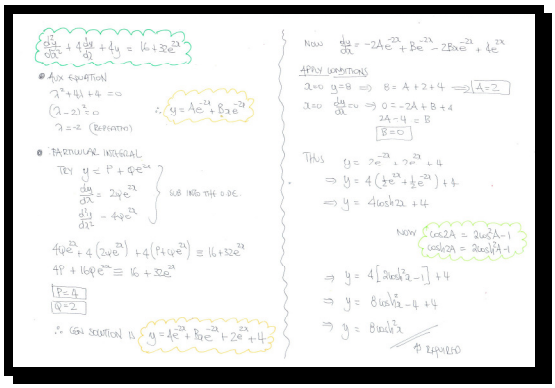
$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=16+32\mathrm{e}^{2x},$$

with $y=8$ and $\frac{dy}{dx}=0$ at $x=0$.

Show that the solution of the above differential equation is

$$y=8\cosh^2 x.$$

proof



Question 15 (+)**

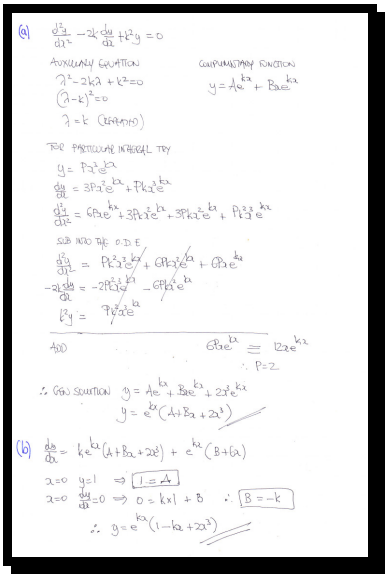
$$\frac{d^2y}{dx^2}-2k\frac{dy}{dx}+k^2y=12x\mathrm{e}^{kx},\ k>0$$

a) Find a general solution of the differential equation given that $y=Px^3\mathrm{e}^{kx}$, where P is a constant, is part of the solution.

b) Given further that $y=1$, $\frac{dy}{dx}=0$ at $x=0$ show that

$$y=\mathrm{e}^{kx}\left(2x^3-kx+1\right).$$

$$\boxed{y=\mathrm{e}^{kx}\left(2x^3+Ax+B\right)}$$



Question 16 **(**+)**

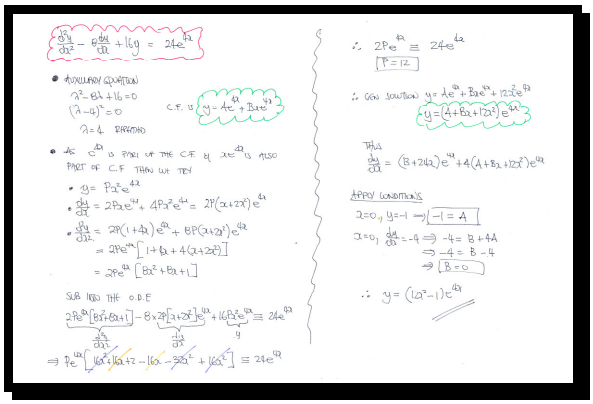
Show that the solution of the differential equation

$$\frac{d^2y}{dx^2}-8\frac{dy}{dx}+16y=24\mathrm{e}^{4x},$$

subject to the boundary conditions $y=-1$, $\frac{dy}{dx}=-4$ at $x=0$, can be written as

$$y=\left(12x^2-1\right)\mathrm{e}^{4x}.$$

proof



Question 17 (***)

$$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=4\mathrm{e}^{3x}.$$

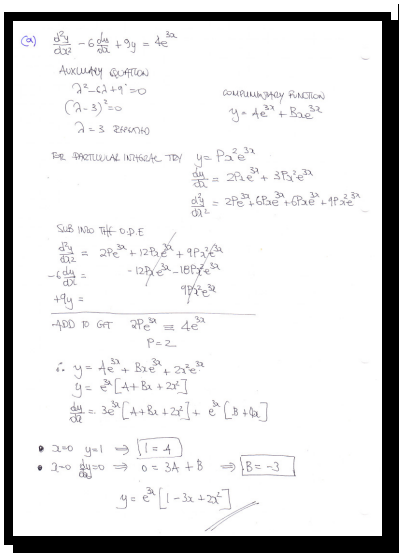
a) Find a solution of the differential equation given that $y=1$, $\dfrac{dy}{dx}=0$ at $x=0$.

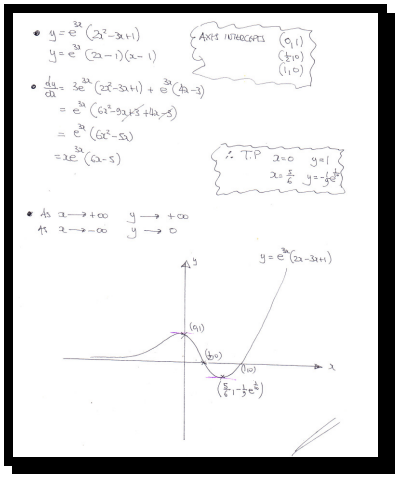
b) Sketch the graph of y.

The sketch must include …

- the coordinates of any points where the graph meets the coordinate axes.
- the coordinates of any stationary points of the curve.
- clear indications of how the graph looks for large positive or negative values of x.

$$\boxed{y=\mathrm{e}^{3x}\left(2x^2-3x+1\right)}$$





Question 18 (*+)**

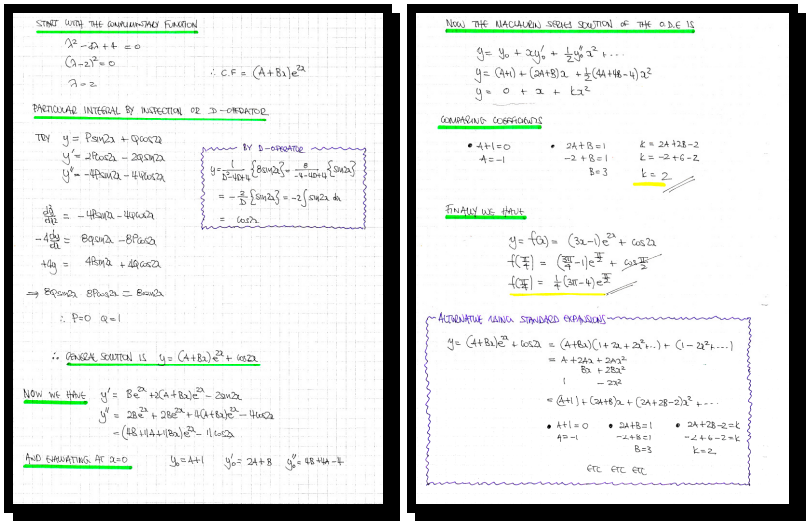
The curve with equation $y = f(x)$ is the solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 8\sin 2x .$$

The first two non zero terms in Maclaurin series expansion of $f(x)$ are $x + kx^2$, where k is a constant.

Determine in any order the value of k and the exact value of $f\left(\frac{1}{4}\pi\right)$.

$\boxed{}$, $\boxed{k = 2}$, $\boxed{f\left(\frac{1}{4}\pi\right) = \frac{1}{2}(3\pi - 4)\mathrm{e}^{\frac{1}{2}\pi}}$



Question 19 (*+)**

The function $y = f(x)$ satisfies the following differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 2e^{-x}(\sin 2x - 2\cos 2x),$$

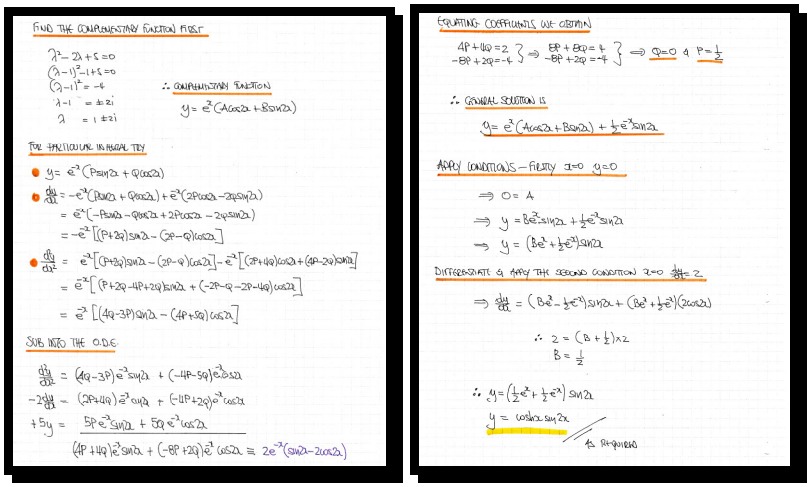
subject to the boundary conditions $y = 0$, $\frac{dy}{dx} = 2$ at $x = 0$.

Solve the differential equation to show that

$$y = \cosh x \sinh 2x .$$

No credit will be given for verification methods.

$\boxed{}$, $\boxed{\text{proof}}$



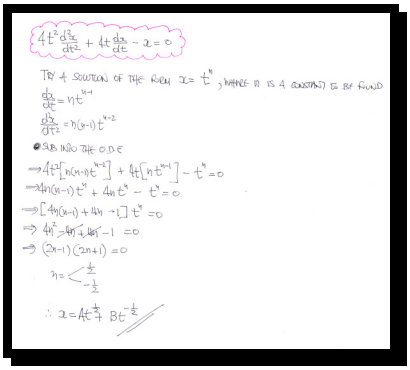
2^{ND} ORDER or HIGHER CAUCHY EULER TYPE

Question 1 (**)

Find the general solution of the following differential equation.

$$4t^2\frac{d^2x}{dt^2}+4t\frac{dx}{dt}-x=0\,.$$

$$\boxed{x=At^{\frac{1}{2}}+Bt^{-\frac{1}{2}}}$$

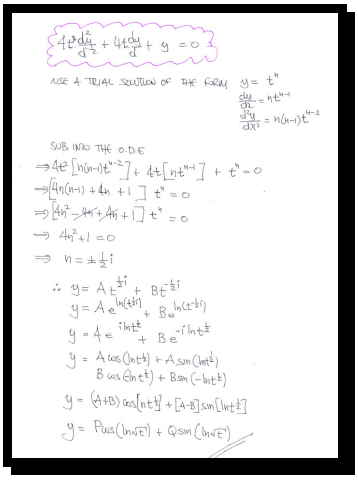


Question 2 (+)**

Find the general solution of the following differential equation.

$$4t^2\frac{d^2y}{dt^2}+4t\frac{dy}{dt}+y=0\,.$$

$$\boxed{y=P\cos\left[\ln\sqrt{t}\right]+P\sin\left[\ln\sqrt{t}\right]}$$



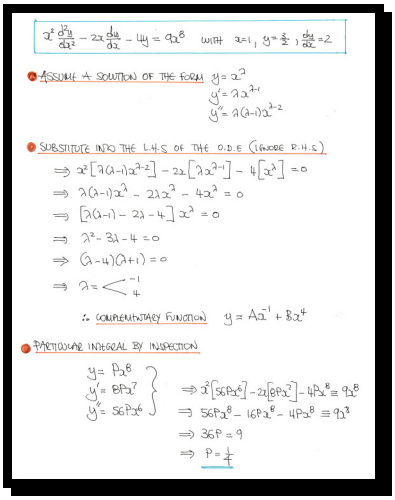
Question 3 (***)

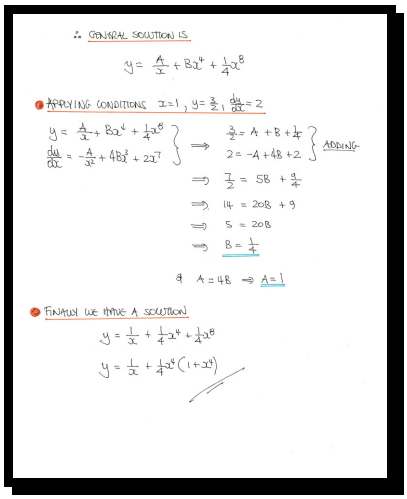
$$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=9x^8.$$

Determine the solution of the above differential equation subject to the boundary conditions

$$y=\frac{3}{2},\ \frac{dy}{dx}=2 \text{ at } x=1.$$

$\square$, $\boxed{y=\frac{1}{4}x^4\left(x^4+1\right)+\frac{1}{x}}$



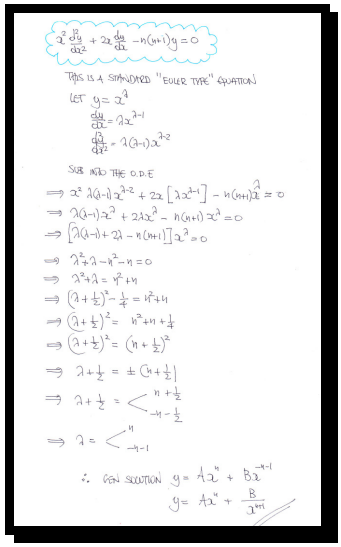


Question 4 (***)

Find the general solution of the following differential equation

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-n(n+1)\,y=0\,.$$

$$\boxed{y=Ax^n+\frac{B}{x^{n+1}}}$$



Question 5 (*+)**

Given that if $x = \mathrm{e}^t$ and $y = f(x)$, show clearly that …

a) … $x\dfrac{dy}{dx} = \dfrac{dy}{dt}$.

b) … $x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}$.

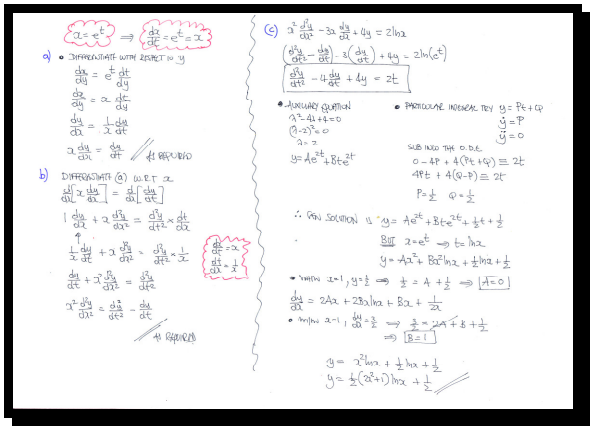
The following differential equation is to be solved

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2\ln x$$

subject to the boundary conditions $y = \dfrac{1}{2}, \dfrac{dy}{dx} = \dfrac{3}{2}$ at $x = 1$.

c) Use the substitution $x = \mathrm{e}^t$ to solve the above differential equation.

$$\boxed{y = \frac{1}{2} + \frac{1}{2}\left(2x^2 + 1\right)\ln x}$$

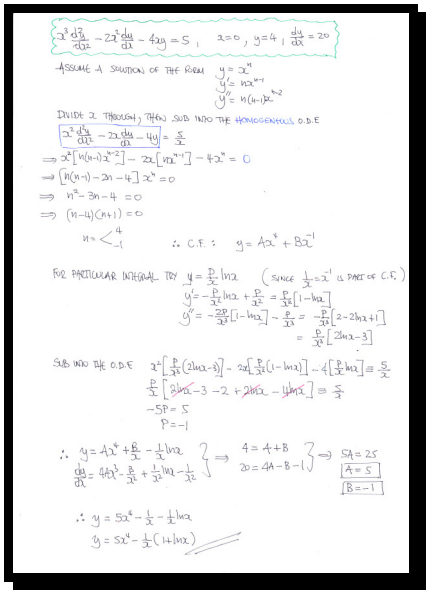


Question 6 (*+)**

$$x^3\frac{d^2y}{dx^2}-2x^2\frac{dy}{dx}-4xy=5\,.$$

Find the solution of the above differential equation subject to the boundary conditions $y=4$, $\dfrac{dy}{dx}=20$ at $x=0$.

$$\boxed{y=5x^4-\frac{1}{x}(1+\ln x)}$$

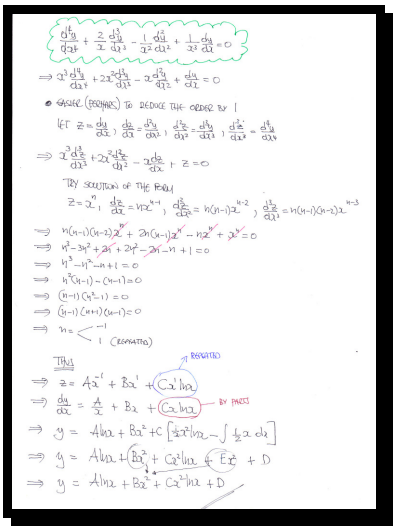


Question 7 (*+)**

Find the general solution of the following differential equation.

$$\frac{d^4y}{dx^4}+\frac{2}{x}\frac{d^3y}{dx^3}-\frac{1}{x^2}\frac{d^2y}{dx^2}+\frac{1}{x^3}\frac{dy}{dx}=0,\ x>0.$$

$\boxed{}$, $\boxed{y=A\ln x+Bx^2+Cx^2\ln x+D}$



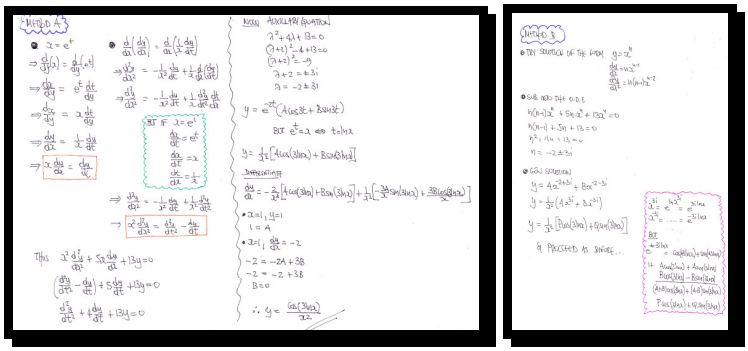
Question 8 (*+)**

The curve with equation $y = f(x)$ satisfies

$$x^2 \frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 13y = 0, \ x > 0.$$

By using the substitution $x = \mathrm{e}^t$, or otherwise, determine an equation for $y = f(x)$, given further that $y = 1$ and $\frac{dy}{dx} = -2$ at $x = 1$.

$$\boxed{y = \frac{\cos(3\ln x)}{x^2}}$$

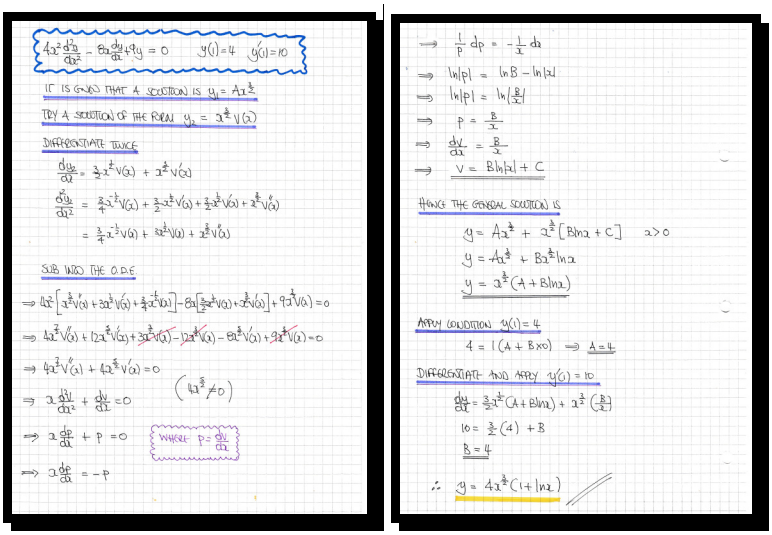


Question 9 (*+)**

$$x^2\frac{d^2y}{dx^2}-8x\frac{dy}{dx}+9y=0,\ x>0.$$

Use the fact that $y=Ax^{\frac{3}{2}}$ satisfies the above differential equation, to find the full solution subject to $y=4$ and $\frac{dy}{dx}=10$ at $x=1$.

$\boxed{}$, $\boxed{y=4x^{\frac{3}{2}}\left(1+\ln x\right)}$

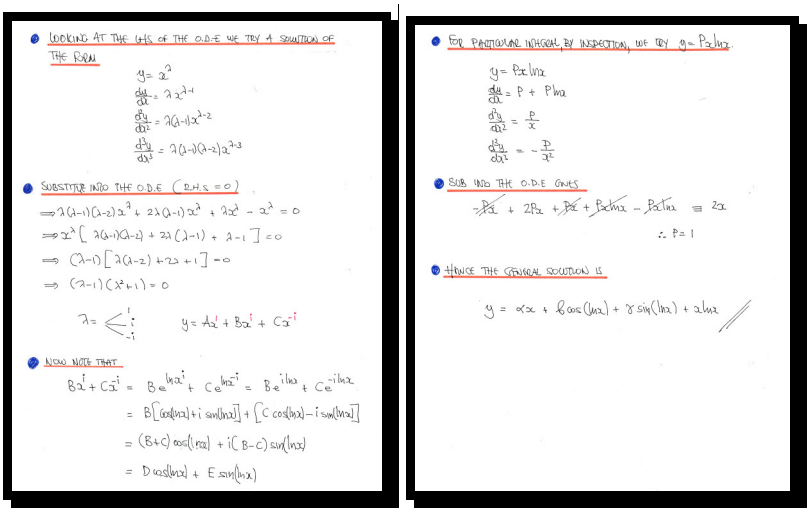


Question 10 (****)

Find the general solution of the following differential equation.

$$x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=2x\,,\quad x>0\,.$$

$\boxed{}$, $\boxed{y=Ax+B\cos\left(\ln x\right)+C\sin\left(\ln x\right)+x\ln x}$



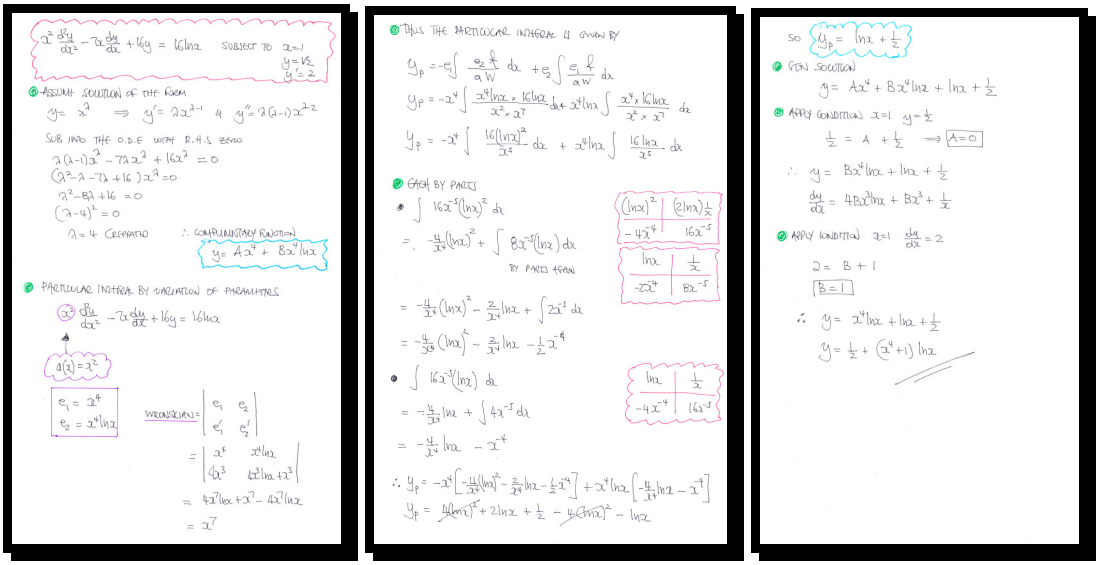
Question 11 (****)

Use variation of parameters to determine the specific solution of the following differential equation

$$x^2\frac{d^2y}{dx^2}-7x\frac{dy}{dx}+16y=16\ln x\,,$$

given further that $y=\frac{1}{2}$, $\frac{dy}{dx}=2$ at $x=1$.

$$\boxed{y=\frac{1}{2}+\left(1+x^4\right)\ln x}$$



2^{ND} ORDER ODEs WITH MISSING INDEPENDENT VARIABLE

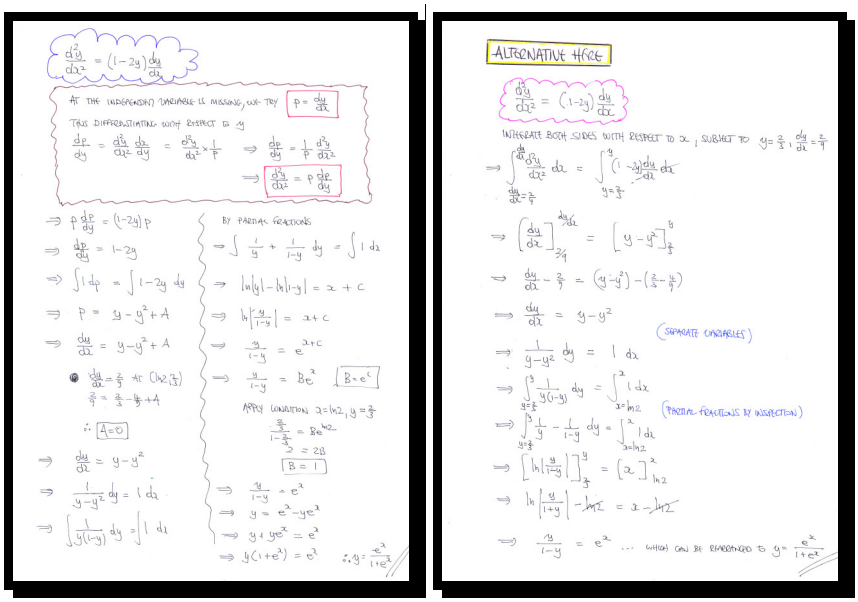
Question 1 **(****+)**

The curve C, has gradient $\frac{2}{9}$ at the point with coordinates $\left(\ln 2, \frac{2}{3}\right)$, and satisfies the differential relationship

$$\frac{d^2y}{dx^2} = (1-2y)\frac{dy}{dx}, \quad y < \frac{1}{2}.$$

Find an equation for C, giving the answer in the form $y = f(x)$.

$$\boxed{y = \frac{e^x}{1+e^x} = \frac{1}{e^x+e^{-x}} = \frac{1}{2}\operatorname{sech} x}$$

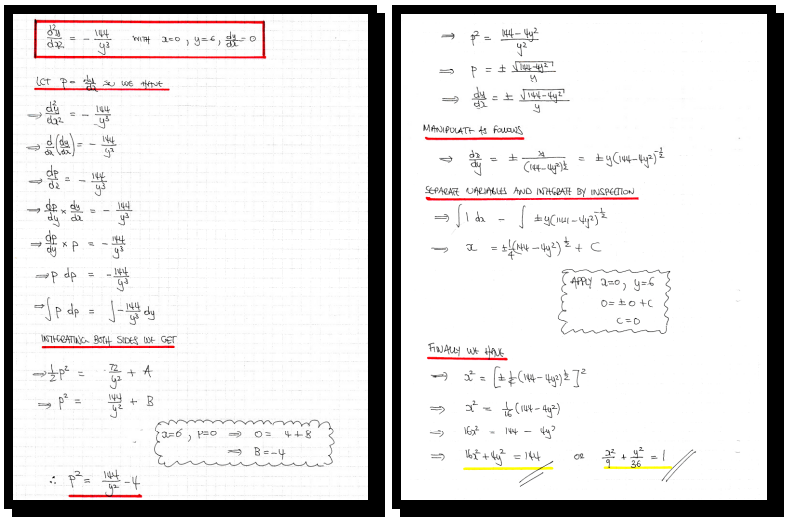


Question 2 (****)

Use appropriate techniques to solve the following differential equation.

$$\frac{d^2y}{dx^2}=-\frac{144}{y^3}, \qquad y(0)=6, \qquad \left.\frac{dy}{dx}\right|_{x=0}=0.$$

$\square$, $\boxed{\dfrac{x^2}{9}+\dfrac{y^2}{36}=1}$



Question 3 **(****+)**

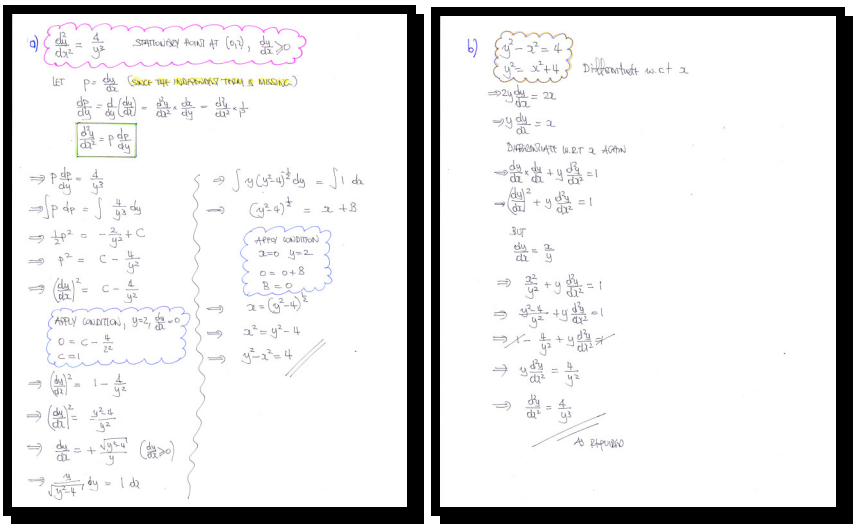
The curve C, has a stationary point at $(0,2)$ and satisfies the differential relationship

$$\frac{d^2y}{dx^2}=\frac{4}{y^3},\ y\neq 0.$$

a) Given further that $\frac{dy}{dx}\geq 0$ along C, determine a simplified expression for the Cartesian equation of C.

b) Verify by differentiation the answer to part **(a)**.

$$\boxed{y^2-x^2=4}$$



Question 4 **(****+)**

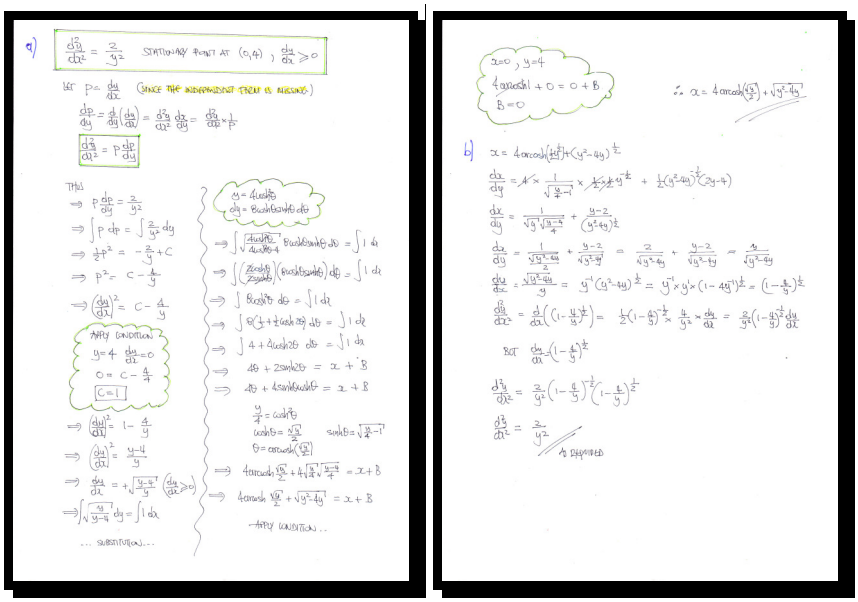
The curve C, has a stationary point at $(0,4)$ and satisfies the differential equation

$$\frac{d^2y}{dx^2}=\frac{2}{y^2},\ y\neq 0.$$

a) Given further that $\frac{dy}{dx}\geq 0$ along C, determine a simplified expression for the Cartesian equation of C, giving the answer in the form $x=f(y)$.

b) Verify by differentiation the answer to part **(a)**.

$$\boxed{x=4\operatorname{arcosh}\left(\frac{1}{2}\sqrt{y}\right)+\sqrt{y^2-4y}}$$



Question 5 **(****+)**

The curve C with Cartesian equation $f(x,y)=0$, satisfies the differential equation

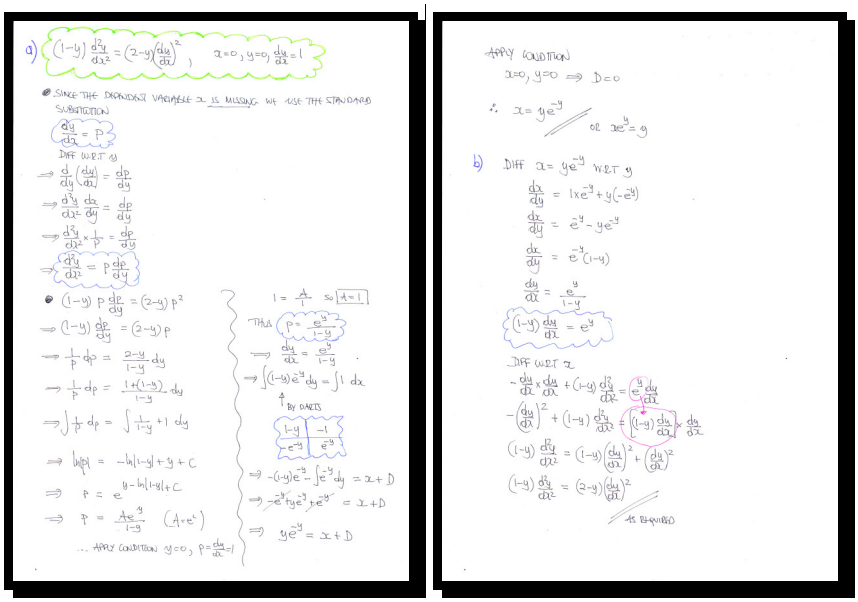
$$(1-y)\,y''=(2-y)(y')^2.$$

It is further given that $y(0)=0$ and $y'(0)=1$

a) Determine a simplified expression for the Cartesian equation of C.

b) Verify by differentiation the answer to part **(a)**.

$$\boxed{x=ye^{-y}}$$



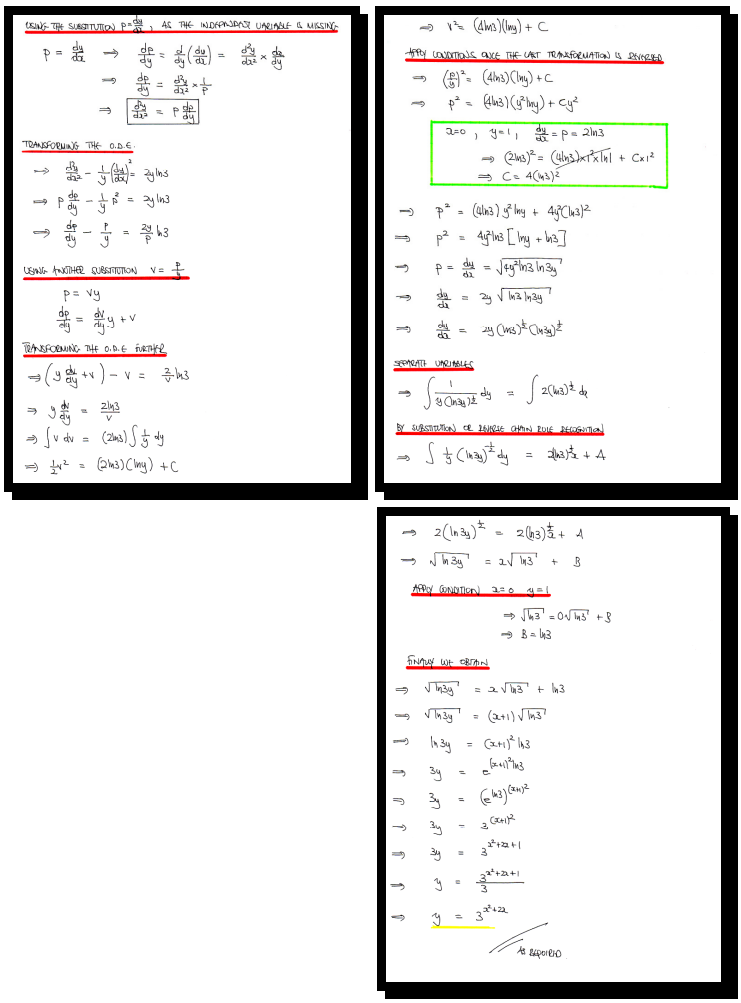
Question 6 (*****)

The function with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 = 2y\ln 3, \quad y(0) = 1, \quad \frac{dy}{dx}(0) = 2\ln 3 .$$

Solve the above differential equation to show that $y = 3^{x^2+2x}$.

$\boxed{}$, $\boxed{\text{proof}}$



Question 7 (*****)

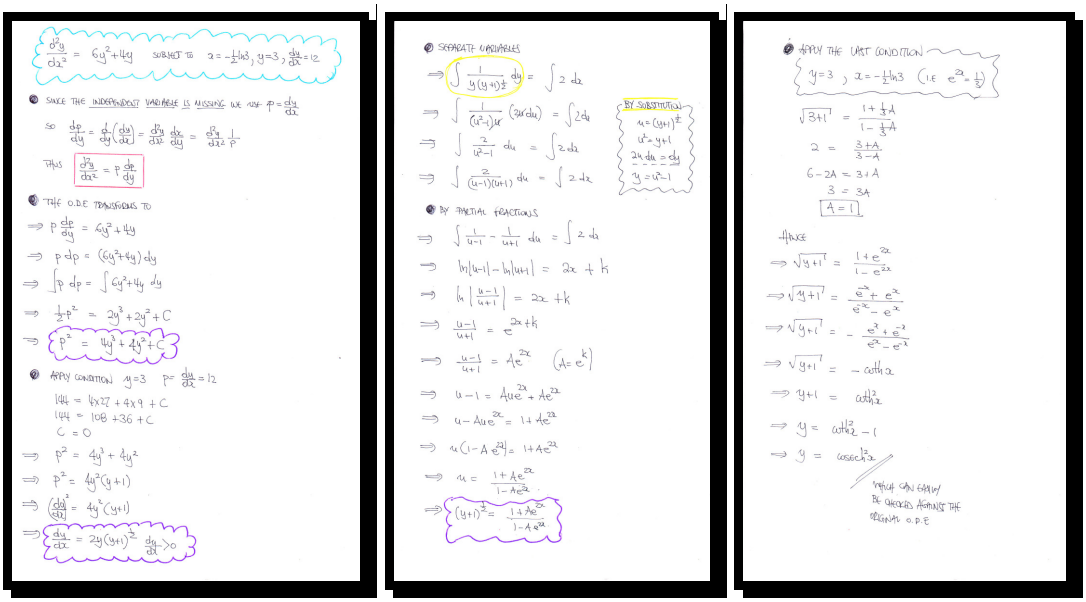
The curve with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} = 6y^2 + 4y, \quad \frac{dy}{dx} \geq 0.$$

If $y = 3$, $\frac{dy}{dx} = 12$ at $x = -\frac{1}{2}\ln 3$, solve the differential equation to show that

$$y = \operatorname{cosech}^2 x.$$

proof



Question 8 **(*****)**

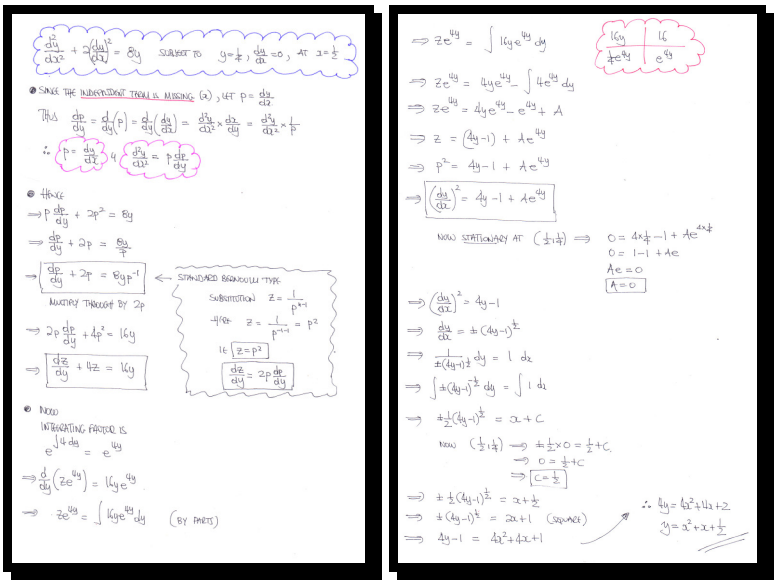
The curve with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 8y\,.$$

Given further that the curve has a stationary point at $\left(\frac{1}{2}, \frac{1}{4}\right)$, solve the differential equation to show that

$$y = x^2 + x + \frac{1}{2}\,.$$

proof



Question 9 **(*****)**

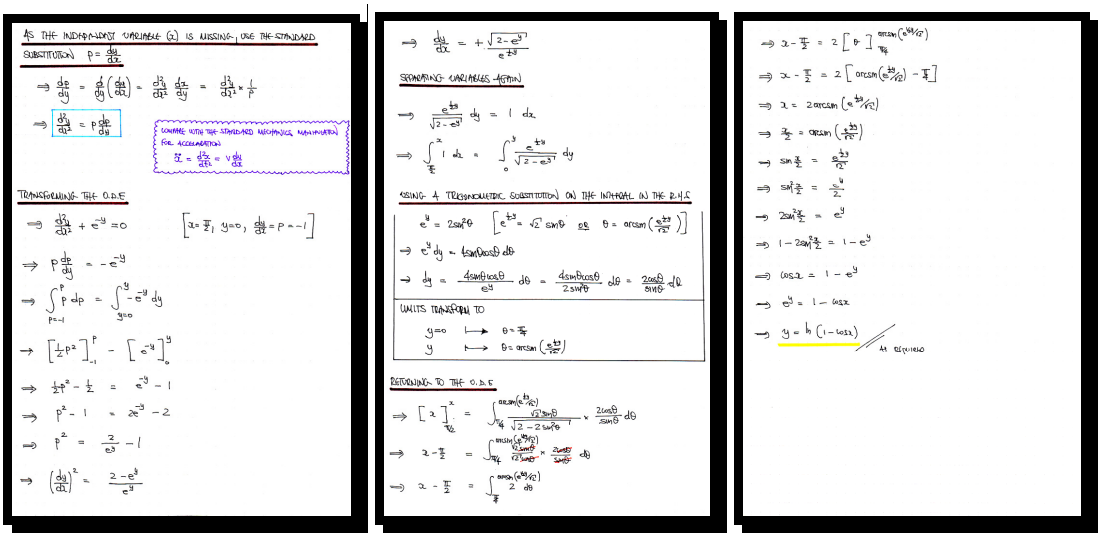
The curve with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + \mathrm{e}^{-y} = 0, \quad \frac{dy}{dx} \geq 0.$$

If $y = 0$, $\dfrac{dy}{dx} = -1$ at $x = \dfrac{1}{2}\pi$, solve the differential equation to show that

$$y = \ln(1 - \cos x).$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 10 **(*****)**

The curve C, has gradient 1 at the origin and satisfies the differential relationship

$$\frac{d^2y}{dx^2}\sqrt{1-2y}=\frac{dy}{dx}(3y-2), \quad y<\frac{1}{2}.$$

Find an equation for C, giving the answer in the form $y=f(x)$.

$$\boxed{y=\frac{\sin x}{1+\sin x}=(\sec x-\tan x)\tan x}$$



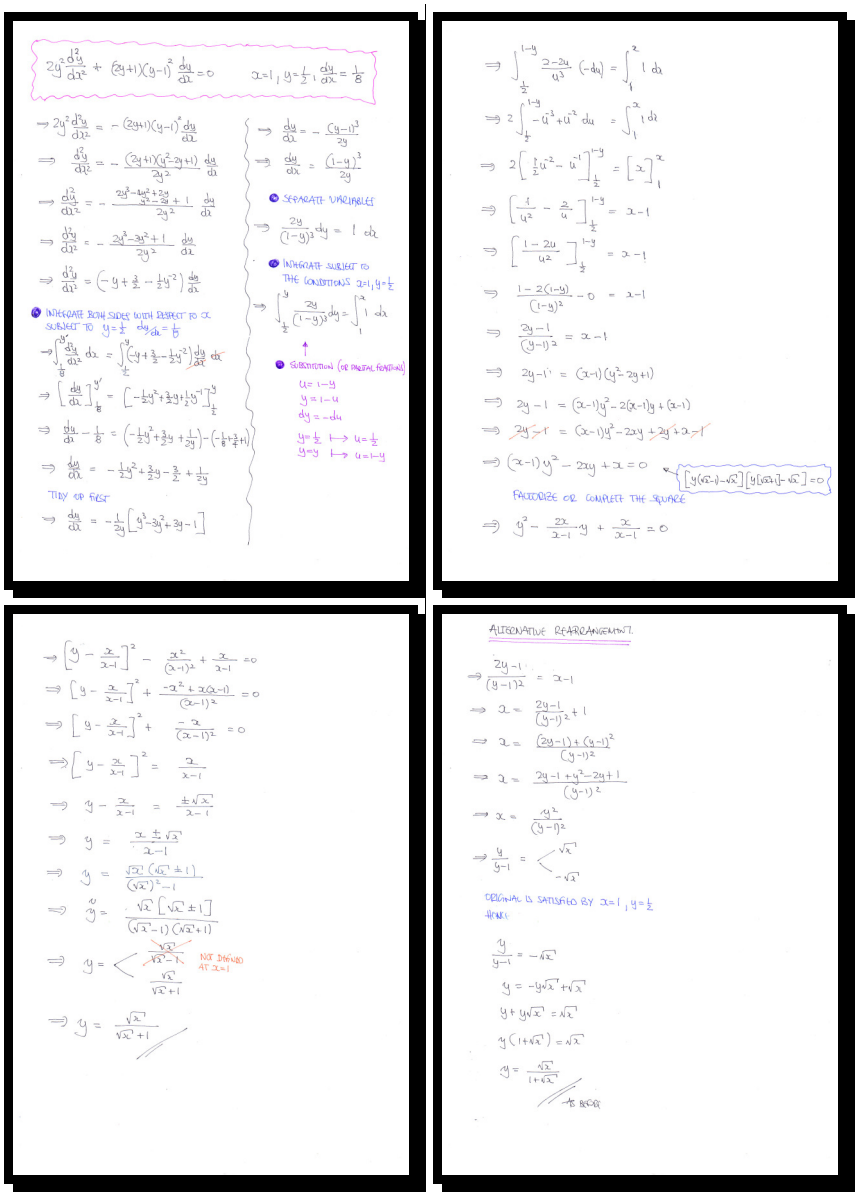
Question 11 (*****)

The curve C, has gradient $\frac{1}{8}$ at the point with coordinates $\left(1,\frac{1}{2}\right)$ and further satisfies the differential relationship

$$2y^2\frac{d^2y}{dx^2}+(2y+1)(y-1)^2\frac{dy}{dx}=0, \qquad y\neq 0.$$

Find an equation for C, giving the answer in the form $y=f(x)$.

$$\boxed{y=\frac{\sqrt{x}}{1+\sqrt{x}}}$$



2^{ND} ORDER BY SUBSTITUTIONS

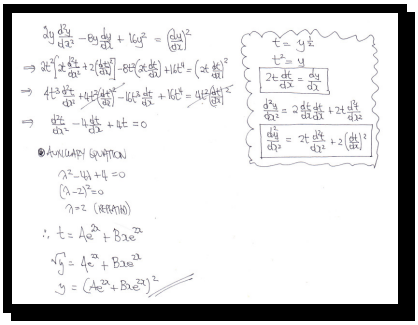
Question 1 **(***)**

$$2y\frac{d^2y}{dx^2}-8y\frac{dy}{dx}+16y^2=\left(\frac{dy}{dx}\right)^2,\ y\neq 0,$$

Find the general solution of the above differential equation by using the transformation equation $t=\sqrt{y}$.

Give the answer in the form $y=f(x)$.

$$\boxed{y=\left(A\mathrm{e}^{2x}+Bx\mathrm{e}^{2x}\right)^2}$$



Question 2 (***)

The differential equation

$$x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=3x\,,\ x\neq 0\,,$$

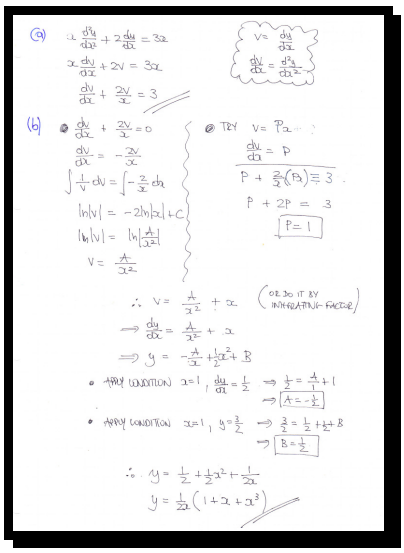
is to be solved subject to the boundary conditions $y=\frac{3}{2}$, $\frac{dy}{dx}=\frac{1}{2}$ at $x=1$.

a) Show that the substitution $v=\frac{dy}{dx}$, transforms the above differential equation into

$$\frac{dv}{dx}+\frac{2v}{x}=3\,.$$

b) Hence find the solution of the original differential equation, giving the answer in the form $y=f(x)$.

$$\boxed{y=\frac{1}{2}\left(x^2+\frac{1}{x}+1\right)}$$



Question 3 **(***)**

The curve C has equation $y=f(x)$ and satisfies the differential equation

$$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-2y\left(2x^2-1\right)=3x^3\,\mathrm{e}^x,\ x\neq 0$$

is to be solved subject to the boundary conditions $y=\frac{3}{2}$, $\frac{dy}{dx}=\frac{1}{2}$ at $x=1$.

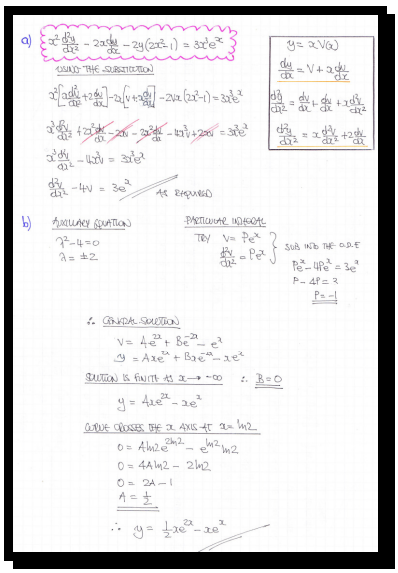
a) Show that the substitution $y=xv$, where v is a function of x transforms the above differential equation into

$$\frac{d^2v}{dx^2}-4v=3\mathrm{e}^x.$$

It is further given that C meets the x axis at $x=\ln 2$ and has a finite value for y as x gets infinitely negatively large.

b) Express the equation of C in the form $y=f(x)$.

$$\boxed{y=\frac{1}{2}x\mathrm{e}^{2x}-x\mathrm{e}^x}$$



Question 4 (*+)**

The differential equation

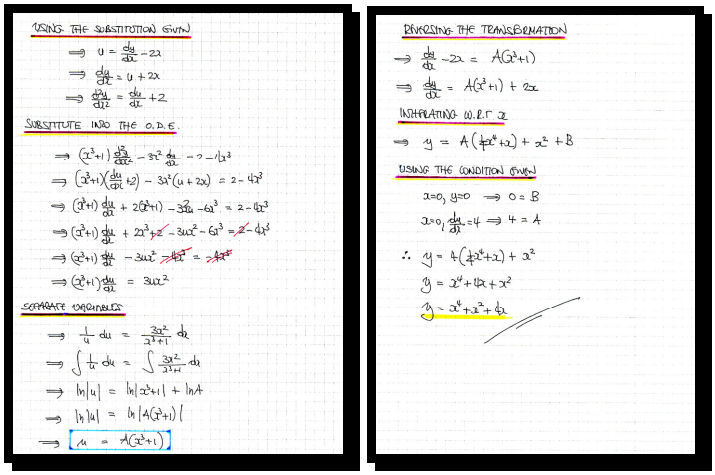
$$\left(x^3+1\right)\frac{d^2y}{dx^2}-3x^2\frac{dy}{dx}=2-4x^3,$$

is to be solved subject to the boundary conditions $y=0$, $\frac{dy}{dx}=4$ at $x=0$.

Use the substitution $u=\frac{dy}{dx}-2x$, where u is a function of x, to show that the solution of the above differential equation is

$$y=x^4+x^2+4x.$$

, proof



Question 5 (****)

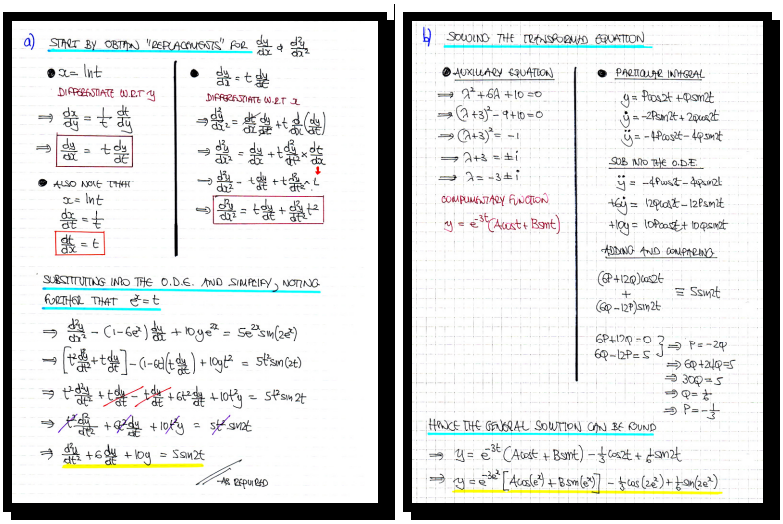
$$\frac{d^2y}{dx^2}-\left(1-6\mathrm{e}^{x}\right)\frac{dy}{dx}+10y\mathrm{e}^{2x}=5\mathrm{e}^{2x}\sin\left(2\mathrm{e}^{x}\right).$$

a) By using the substitution $x=\ln t$ or otherwise, show that the above differential equation can be transformed to

$$\frac{d^2y}{dt^2}+6\frac{dy}{dt}+10y=5\sin 2t\,.$$

b) Hence find a general solution for the original differential equation.

$\square$, $\boxed{y=\mathrm{e}^{-3\mathrm{e}^{x}}\left[A\cos\left(\mathrm{e}^{x}\right)+B\sin\left(\mathrm{e}^{x}\right)\right]+\frac{1}{6}\sin\left(2\mathrm{e}^{x}\right)-\frac{1}{3}\cos\left(2\mathrm{e}^{x}\right)}$



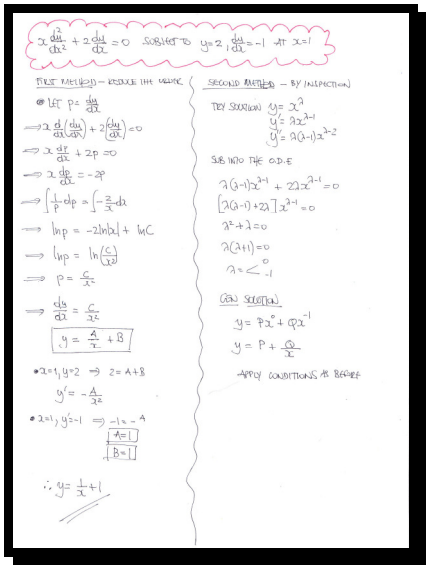
Question 6 (*+)**

Solve the differential equation

$$x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0,$$

subject to the boundary conditions $y=2$, $\frac{dy}{dx}=-1$ at $x=1$.

$$\boxed{y=\frac{2\mathrm{e}^{2x}}{\mathrm{e}^{2x}+x}}$$

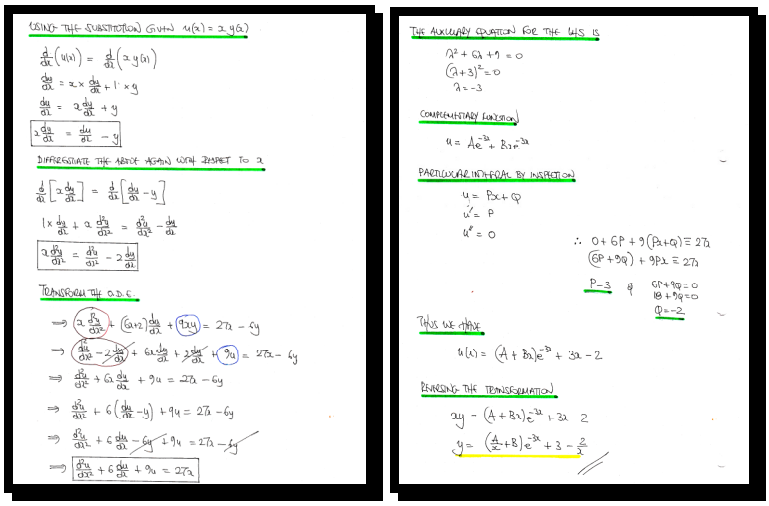


Question 7 (*+)**

$$x\frac{d^2y}{dx^2}+(6x+2)\frac{dy}{dx}+9xy=27x-6y\,.$$

Use the substitution $u=xy$, where u is a function of x, to find a general solution of the above differential equation.

$\square$, $\boxed{y=\frac{A}{x}\mathrm{e}^{-3x}+B\mathrm{e}^{-3x}+3-\frac{2}{x}}$



Question 8 **(***+)**

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}\tan x-y\sec^4 x=0.$$

The above differential equation is to be solved by a substitution.

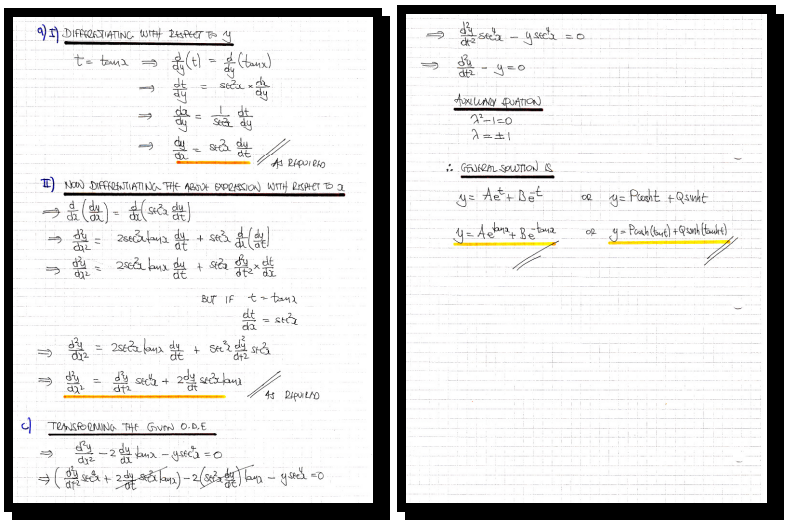
a) If $t=\tan x$ show that …

i. … $\dfrac{dy}{dx}=\dfrac{dy}{dt}\sec^2 x$

ii. … $\dfrac{d^2y}{dx^2}=\dfrac{d^2y}{dt^2}\sec^4 x+2\dfrac{dy}{dt}\sec^2 x\tan x$

b) Use the results obtained in part **(a)** to find a general solution of the differential equation in the form $y=f(x)$.

$\boxed{}$, $\boxed{y=A\mathrm{e}^{\tan x}+B\mathrm{e}^{-\tan x}}$



Question 9 **(***+)**

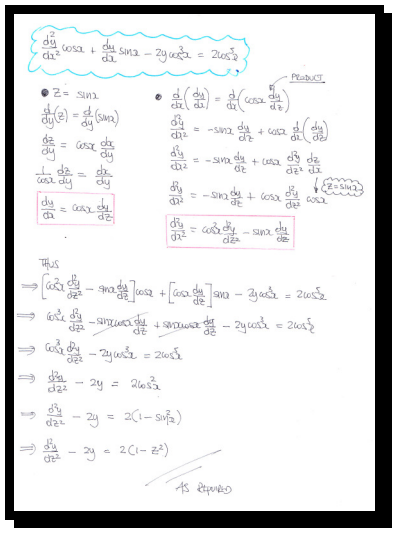
Show clearly that the substitution $z=\sin x$, transforms the differential equation

$$\frac{d^2y}{dx^2}\cos x+\frac{dy}{dx}\sin x-2y\cos^3 x=2\cos^5 x,$$

into the differential equation

$$\frac{d^2y}{dz^2}-2y=2\left(1-z^2\right)$$

proof



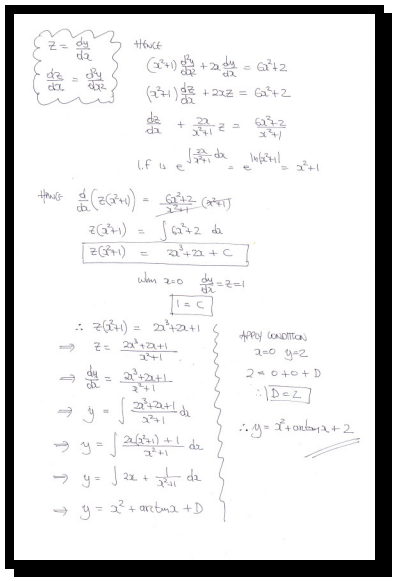
Question 10 (*+)**

By using the substitution $z=\dfrac{dy}{dx}$, or otherwise, solve the differential equation

$$\left(x^2+1\right)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=6x^2+2\,,$$

subject to the conditions $x=0$, $y=2$, $\dfrac{dy}{dx}=1$

$$\boxed{y=x^2+2+\arctan x}$$



Question 11 (****)

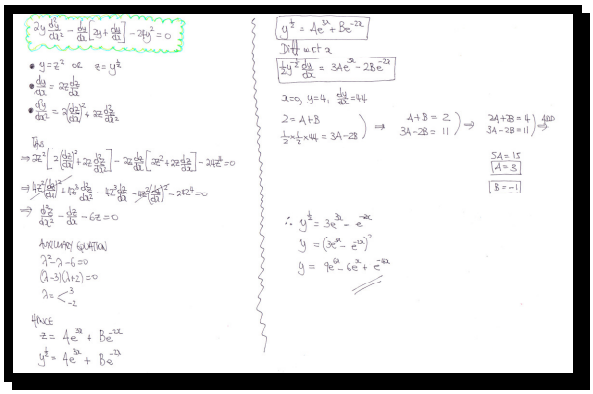
Use the substitution $z=\sqrt{y}$, where $y=f(x)$, to solve the differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{y}\left(\frac{dy}{dx}\right)^2-5\frac{dy}{dx}+2y=0,$$

subject to the boundary conditions $y=4, \frac{dy}{dx}=44$ at $x=0$.

Give the answer in the form $y=f(x)$.

$$\boxed{y=9\mathrm{e}^{6x}-6\mathrm{e}^{x}+\mathrm{e}^{-4x}}$$



Question 12 (****)

$$2x\frac{d^2y}{dx^2}+\left(1-3x^{\frac{1}{2}}\right)\frac{dy}{dx}+y=0.$$

The above differential equation is to be solved by a substitution.

a) Given that $y=f\left(x\right)$ and $t=x^{\frac{1}{2}}$, show clearly that …

i. … $\dfrac{dy}{dx}=\dfrac{1}{2t}\dfrac{dy}{dt}$.

ii. … $\dfrac{d^2y}{dx^2}=\dfrac{1}{4t^2}\dfrac{d^2y}{dt^2}-\dfrac{1}{4t^3}\dfrac{dy}{dt}$.

b) Hence show further that the differential equation

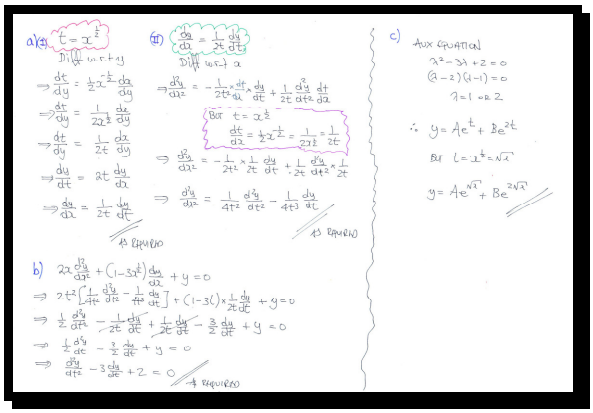
$$2x\frac{d^2y}{dx^2}+\left(1-3x^{\frac{1}{2}}\right)\frac{dy}{dx}+y=0,$$

can be transformed to the differential equation

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=0.$$

c) Find a general solution of the **original** differential equation, giving the answer in the form $y=f\left(x\right)$.

$$\boxed{y=A\mathrm{e}^{\sqrt{x}}+B\mathrm{e}^{2\sqrt{x}}}$$



Question 13 (****)

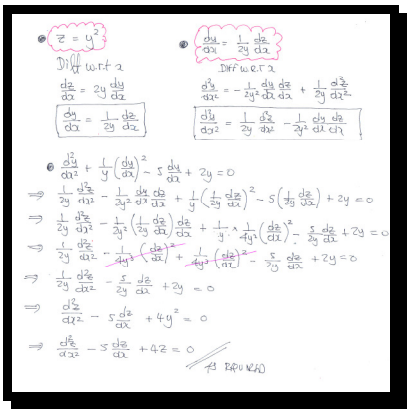
Show clearly that the substitution $z = y^2$, where $y = f(x)$, transforms the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 2y = 0,$$

into the differential equation

$$\frac{d^2z}{dx^2} - 5\frac{dz}{dx} + 4z = 0$$

proof



Question 14 (****)

Given that if $x = t^{\frac{1}{2}}$, where $y = f(x)$, show clearly that

a) $\dfrac{dy}{dx} = 2t^{\frac{1}{2}} \dfrac{dy}{dt}$.

b) $\dfrac{d^2y}{dx^2} = 4t\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt}$.

The following differential equation is to be solved

$$x\frac{d^2y}{dx^2} - \left(8x^2+1\right)\frac{dy}{dx} + 12x^3y = 12x^5,$$

subject to the boundary conditions $y = \dfrac{10}{3}$, $\dfrac{d^2y}{dx^2} = 10$ at $x = 0$.

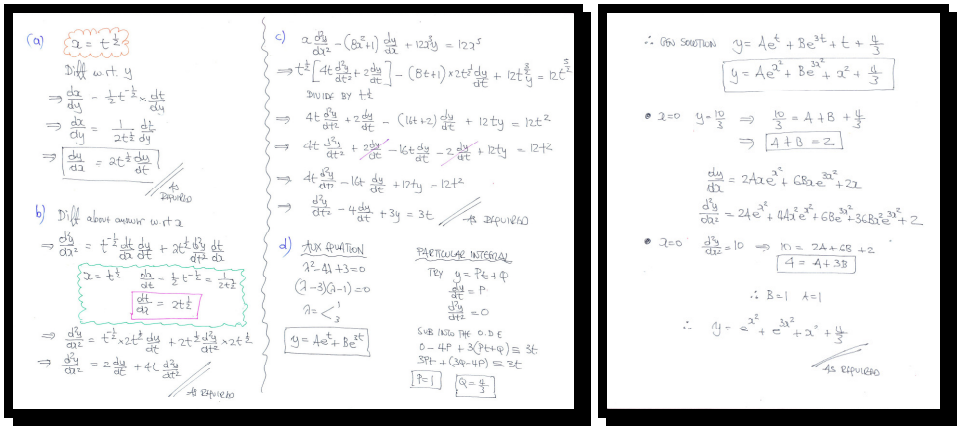
c) Show further that the substitution $x = t^{\frac{1}{2}}$, where $y = f(x)$, transforms the above differential equation into the differential equation

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 3t.$$

d) Show that a solution of the **original** differential equation is

$$y = \mathrm{e}^{3x^2} + \mathrm{e}^{x^2} + x^2 + \frac{4}{3}.$$

proof



Question 15 (****)

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}\cot x+2y\operatorname{cosec}^2 x=2\cos x-2\cos^3 x\,.$$

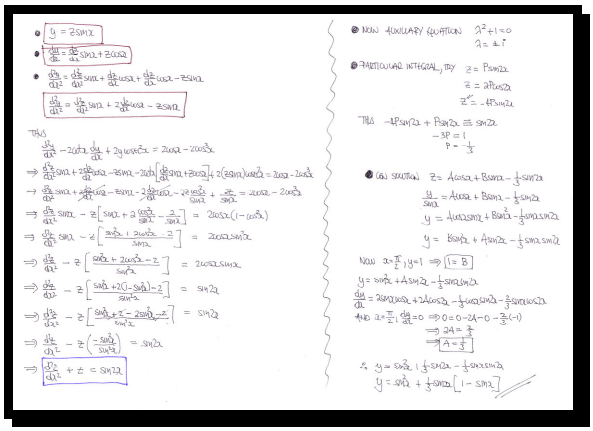
Use the substitution $y=z\sin x$, where z is a function of x, to solve the above differential equation subject to the boundary conditions $y=1$, $\dfrac{dy}{dx}=0$ at $x=\dfrac{\pi}{2}$.

Give the answer in the form

$$y=a\sin^2 x+b(1-\sin x)\sin 2x\,,$$

where a and b are constants to be found.

$$\boxed{a=1}\,,\quad \boxed{b=\frac{1}{3}}$$

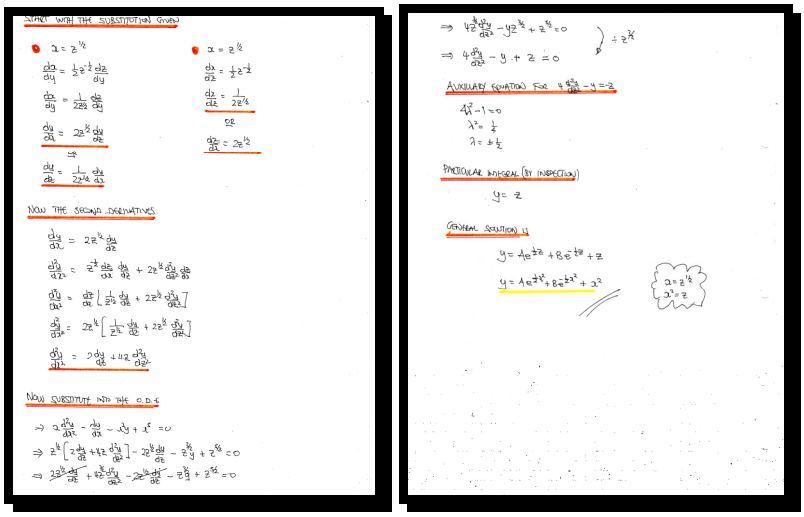


Question 16 (****)

$$x\frac{d^2y}{dx^2}-\frac{dy}{dx}-x^3y+x^5=0\,.$$

Use the substitution $x=z^{\frac{1}{2}}$, where $y=f\left(x\right)$, to find a general solution of the above differential equation.

V, [ ], $\boxed{y=A\mathrm{e}^{\frac{1}{2}x^2}+B\mathrm{e}^{-\frac{1}{2}x^2}+x^2}$



Question 17 **(****)**

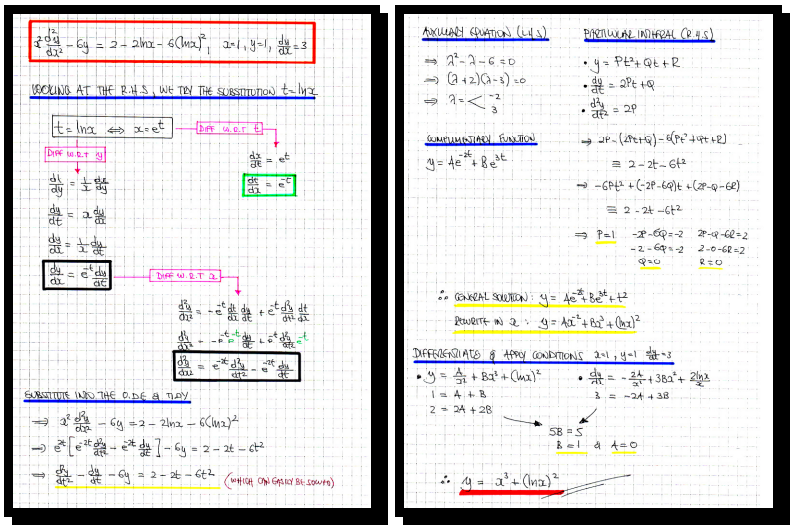
Use a suitable substitution to solve the differential equation

$$x^2\frac{d^2y}{dx^2}-6y=2-2\ln x-6(\ln x)^2,$$

subject to the boundary conditions $y(1)=1$, $\frac{dy}{dx}(1)=3$

Give a simplified answer in the form $y=f(x)$.

$\square$, $\boxed{y=x^3+(\ln x)^2}$

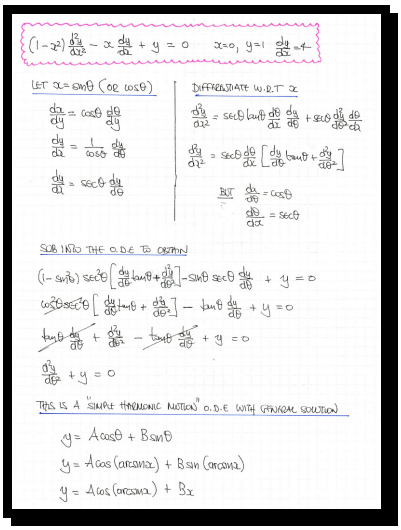


Question 18 (****)

Use a suitable trigonometric substitution to solve the following differential equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0, \quad y(0)=1 \quad \frac{dy}{dx}(0)=4.$$

$$\boxed{y=3x-\cos\left(\arcsin x\right)}$$



Question 19 **(****)**

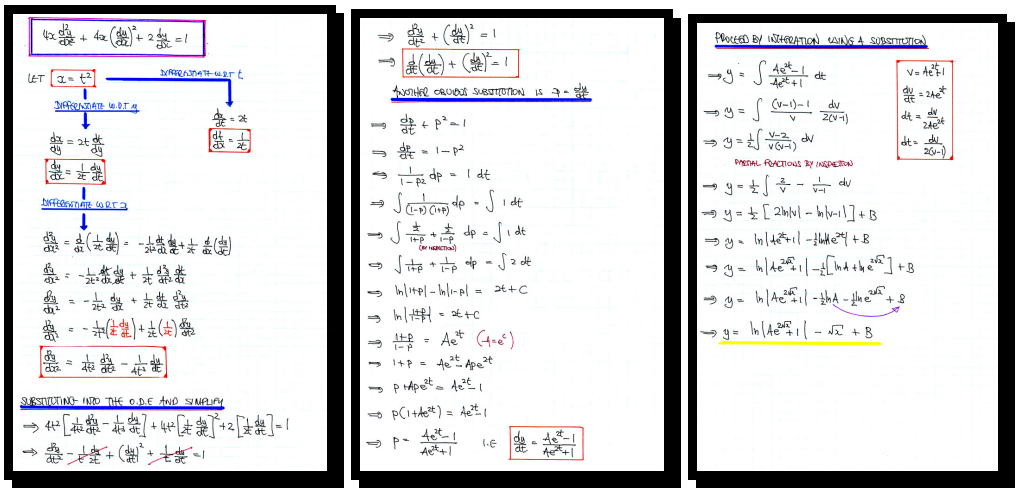
$$4x\frac{d^2y}{dx^2}+4x\left(\frac{dy}{dx}\right)^2+2\frac{dy}{dx}=1\,.$$

By using the substitution $t=\sqrt{x}$, or otherwise, show that the general solution of the above differential equation is

$$y=A-\sqrt{x}+\ln\left[1+B\,\mathrm{e}^{2\sqrt{x}}\right],$$

where A and B are arbitrary constants.

, proof



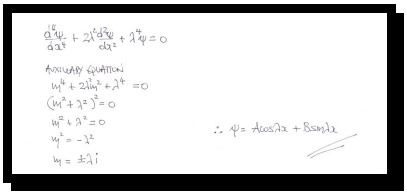
VARIOUS TYPES

Question 1 **(**+)**

Find the general solution of the following differential equation.

$$\frac{d^4\psi}{dx^4}+2\lambda\frac{d^2\psi}{dx^2}+\lambda^4\psi=0.$$

$$\boxed{\psi=A\cos\lambda x+B\sin\lambda x}$$



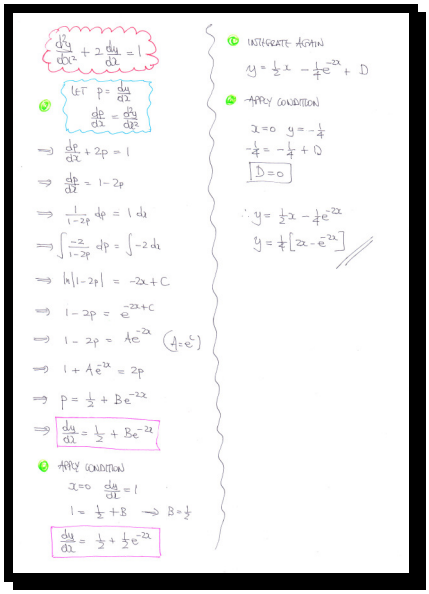
Question 2 **(***)**

Solve the differential equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}=1,$$

given that $y=-\frac{1}{4}$ and $\frac{dy}{dx}=1$ at $x=0$, giving the answer in the form $y=f(x)$.

$$\boxed{y=\frac{1}{2}\left[2x-\mathrm{e}^{-2x}\right]}$$



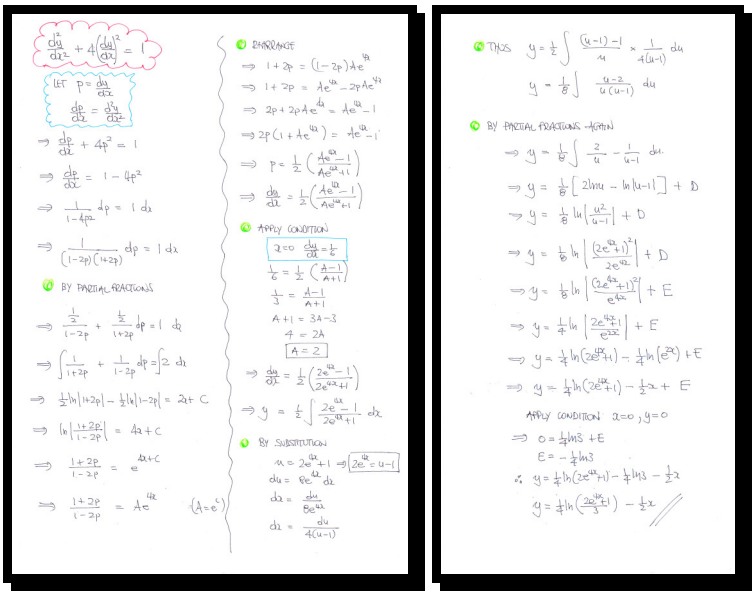
Question 3 (*+)**

Solve the differential equation

$$\frac{d^2y}{dx^2}+4\left(\frac{dy}{dx}\right)^2=1,$$

given that $y=0$ and $\frac{dy}{dx}=\frac{1}{6}$ at $x=0$, giving the answer in the form $y=f(x)$.

$$\boxed{y=\frac{1}{4}\ln\left[\frac{1+2\mathrm{e}^{4x}}{3}\right]-\frac{1}{2}x}$$



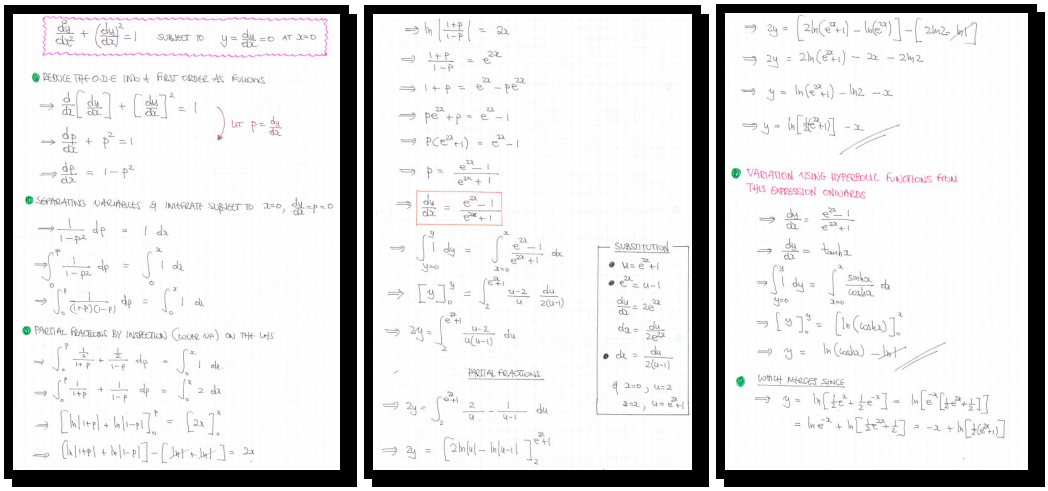
Question 4 **(***+)**

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=1\,.$$

Given that $y=\dfrac{dy}{dx}=0$ at $x=0$, show that

$$y=-x+\ln\left[\frac{1}{2}\left(1+\mathrm{e}^{2x}\right)\right].$$

proof



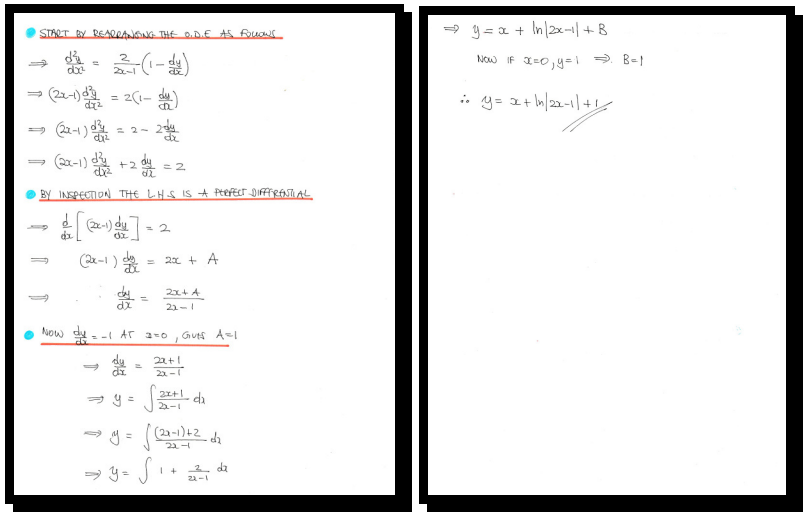
Question 5 **(***+)**

The function with equation $y=f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2}=\frac{2}{2x-1}\left(1-\frac{dy}{dx}\right), \quad y(0)=1, \quad \frac{dy}{dx}(0)=-1.$$

Solve the above differential equation giving the answer in the form $y=f(x)$.

$$\boxed{y=x+1+\ln|2x-1|}$$



Question 6 (****)

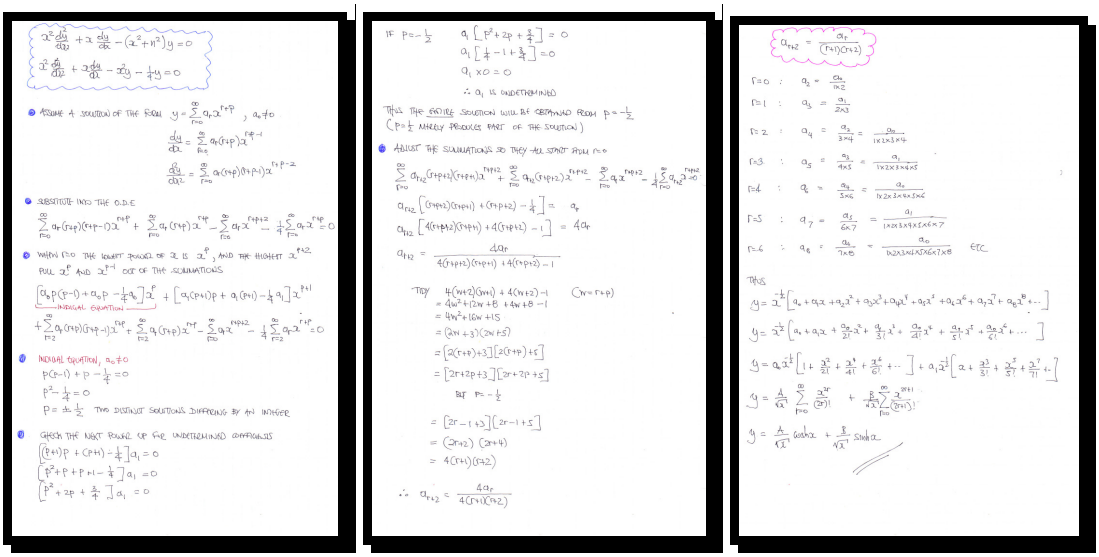
$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-\left(x^2+n^2\right)y=0.$$

The above differential equation is known as modified Bessel's Equation.

Use the Frobenius method to show that the general solution of this differential equation, for $n=\frac{1}{2}$, is

$$y=x^{-\frac{1}{2}}\left[A\cosh x+B\sinh x\right].$$

proof



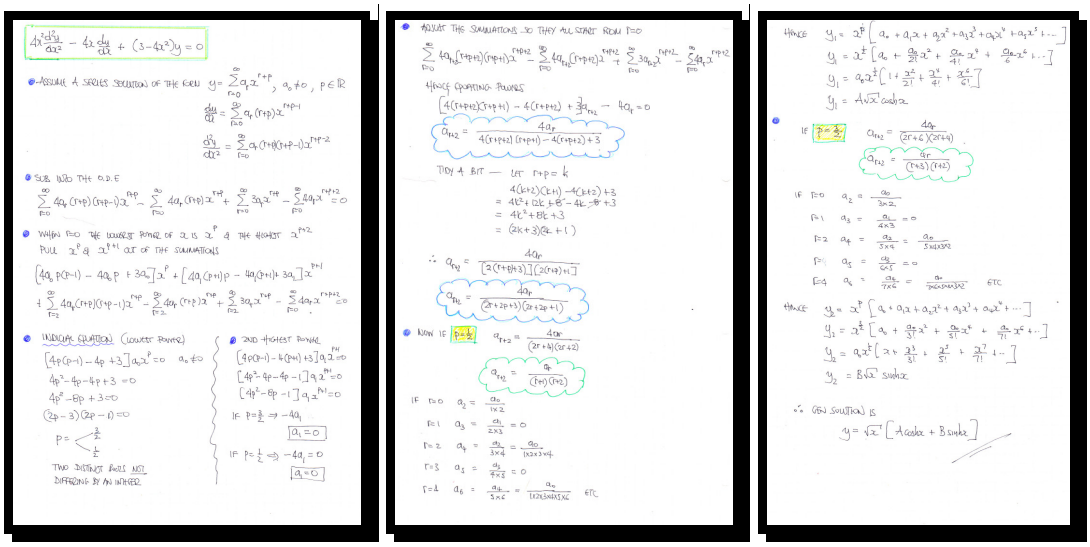
Question 7 (****)

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$4x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+\left(3-4x^2\right)y=0\,.$$

Give the final answer in terms of elementary functions.

$$\boxed{y=\sqrt{x}\left(A\cosh x+B\sinh x\right)}$$



Question 8 (****)

Find the solution of following differential equation

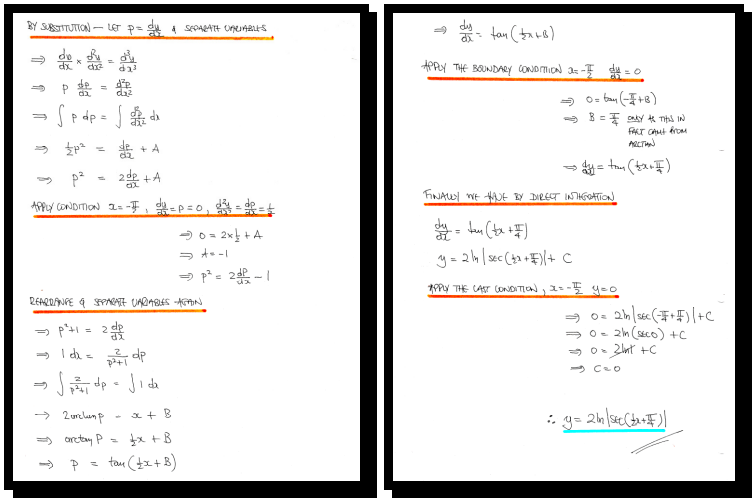
$$\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dx^3},$$

subject to the boundary conditions.

$$y\left(-\tfrac{1}{2}\pi\right)=y'\left(-\tfrac{1}{2}\pi\right)=0, \qquad y''\left(-\tfrac{1}{2}\pi\right)=\frac{1}{2}.$$

Given the answer in the form $y=f(x)$.

$\square$, $\boxed{y=2\ln\left|\sec\left(\frac{1}{2}x+\frac{1}{4}\pi\right)\right|}$



Question 9 **(****+)**

A curve has a stationary point at $\left(-\frac{1}{2},-\frac{1}{2}\right)$.

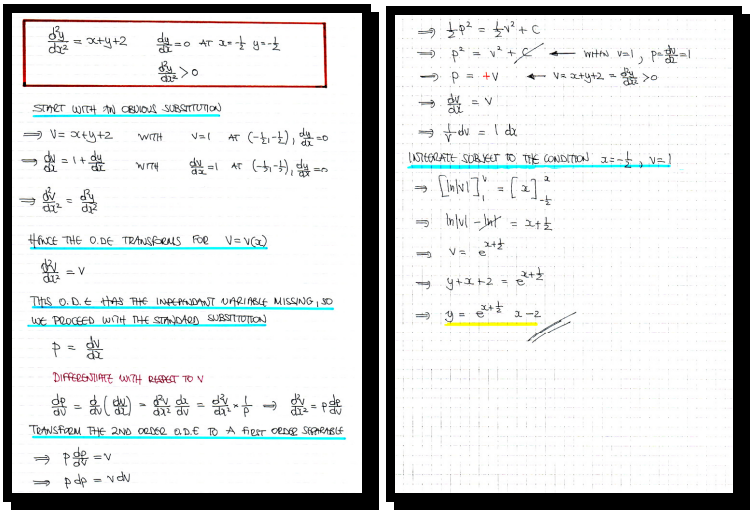
The rate of change of the gradient function of the curve is given by

$$x+y+2,$$

where $x+y+2>0$.

Determine the equation of the curve, giving the answer in the form $y=f(x)$.

$\square$, $\boxed{y=e^{x+\frac{1}{2}}-x-2}$



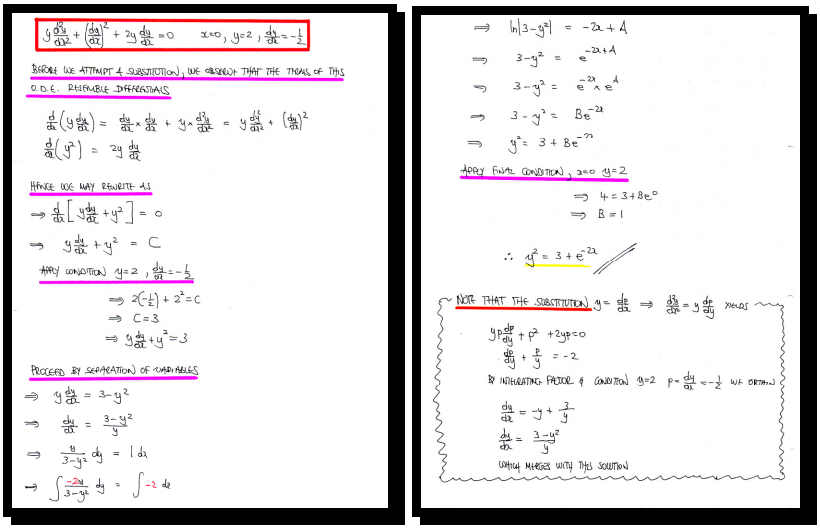
Question 10 (**+)**

Solve the following differential equation

$$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+2y\frac{dy}{dx}=0,\quad y(0)=2,\quad \frac{dy}{dx}(0)=-\frac{1}{2}.$$

Give the answer in the form $y^2=f(x)$.

$\square$, $\boxed{y^2=3+\mathrm{e}^{-2x}}$



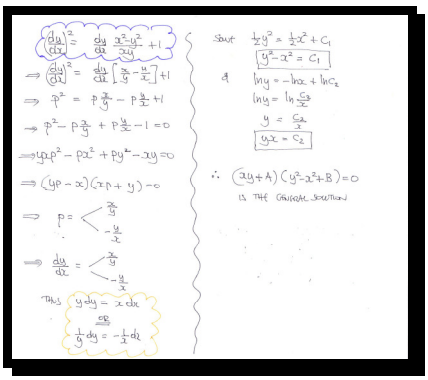
Question 11 (**+)**

By writing $\frac{dy}{dx}=p$ and seeking a suitable factorization find a general solution for the non linear differential equation

$$\left(\frac{dy}{dx}\right)^2=\frac{dy}{dx}\left(\frac{x^2-y^2}{xy}\right)+1.$$

Give the solution in the form $F(x,y)G(x,y)=0$.

$$\boxed{(xy+A)\left(x^2-y^2+B\right)=0}$$



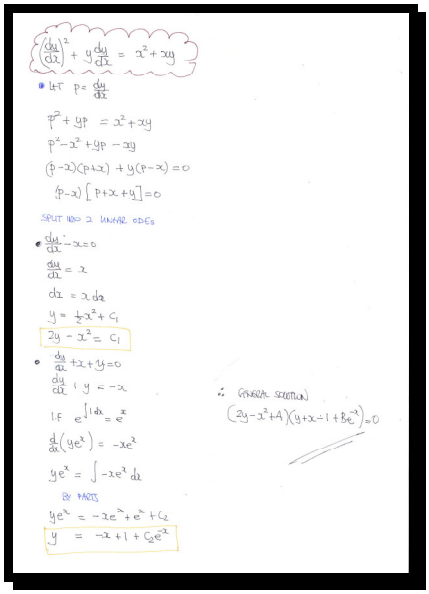
Question 12 (**+)**

By writing $\dfrac{dy}{dx}=p$ and seeking a suitable factorization find a general solution for the non linear differential equation

$$\left(\frac{dy}{dx}\right)^2+y\frac{dy}{dx}=x^2+xy\,.$$

Give the solution in the form $F\left(x,y\right)G\left(x,y\right)=0$.

$$\boxed{\left(2y-x^2+A\right)\left(x+y-1+B\,\mathrm{e}^{-x}\right)=0}$$



Question 13 (*****)

A curve C is described implicitly by the equation

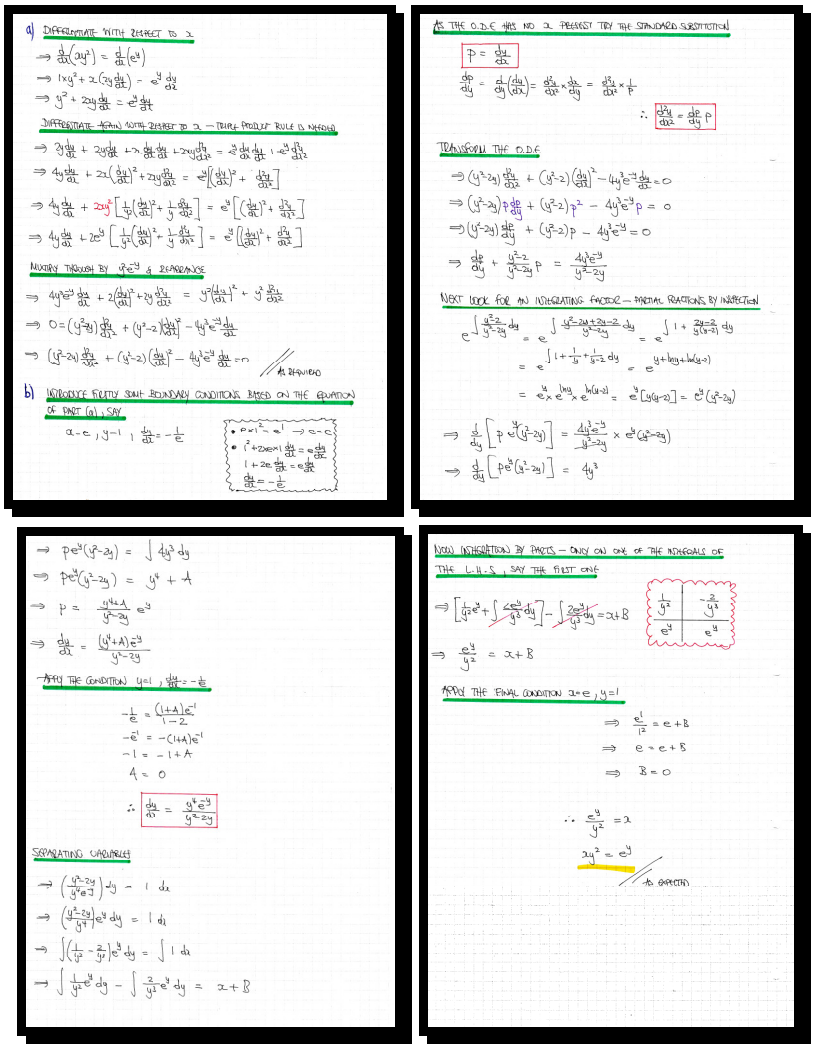
$$xy^2 = \mathrm{e}^y .$$

a) Show, by a detailed method, that

$$\left(y^2-2y\right)\frac{d^2y}{dx^2}+\left(y^2-2\right)\left(\frac{dy}{dx}\right)^2-4y^3\frac{dy}{dx}\mathrm{e}^{-y}=0 .$$

b) Use an analytical method, with suitable boundary conditions, to obtain the equation of C by solving the above differential equation.

V , [ ] , proof



Question 14 (*****)

Find a general solution of the following differential equation.

$$y = x\frac{dy}{dx} + \mathrm{e}^{\frac{dy}{dx}}.$$

$\boxed{}$, $\boxed{(y + Ax + B)(y - x\ln x + Cx) = 0}$

