DIFFERENTIAL EQUATIONS
1st order

SEPARATION OF VARIABLES

Question 1 (**)

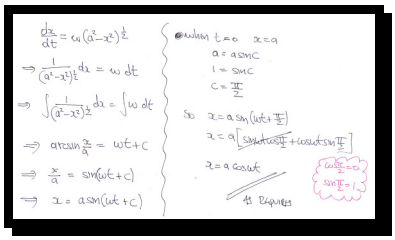
Show that if $y = a$ at $t = 0$, the solution of the differential equation

$$\frac{dy}{dt} = \omega\left(a^2 - y^2\right)^{\frac{1}{2}},$$

where a and ω are positive constants, can be written as

$$y = a\cos\omega t\,.$$

proof



Question 2 **(**+)**

Show that a general solution of the differential equation

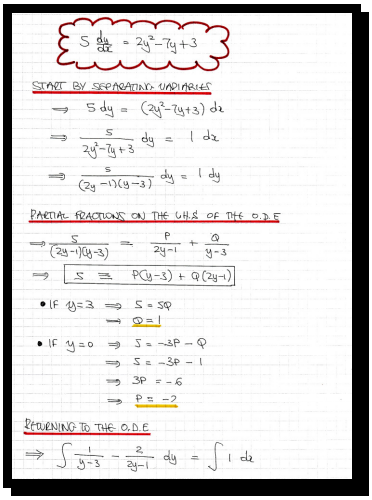
$$5\frac{dy}{dx}=2y^2-7y+3$$

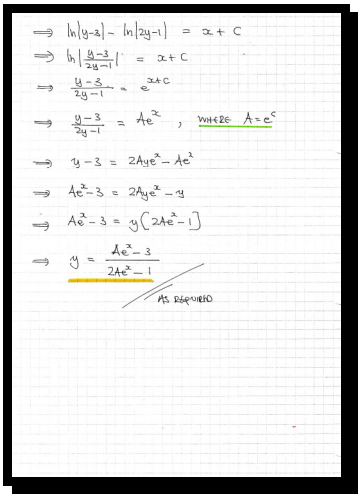
is given by

$$y=\frac{A\mathrm{e}^x-3}{2A\mathrm{e}^x-1},$$

where A is an arbitrary constant.

[ ], [proof]





Question 3 (+)**

Show that a general solution of the differential equation

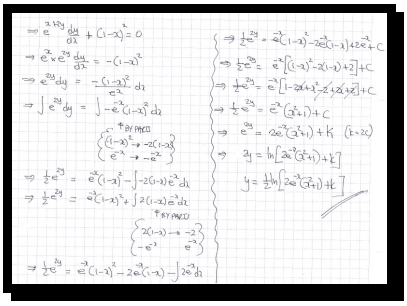
$$\mathrm{e}^{x+2y}\frac{dy}{dx}+(1-x)^2=0$$

is given by

$$y=\frac{1}{2}\ln\left[2\mathrm{e}^{-x}\left(x^2+1\right)+K\right],$$

where K is an arbitrary constant.

proof



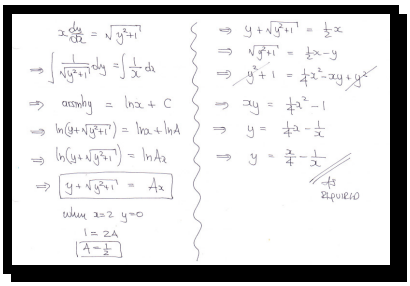
Question 4 **(**+)**

$$x\frac{dy}{dx}=\sqrt{y^2+1}\ ,\ x>0\text{, with }\ y=0\ \text{ at }\ x=2\,.$$

Show that the solution of the above differential equation is

$$y=\frac{x}{4}-\frac{1}{x}\,.$$

proof



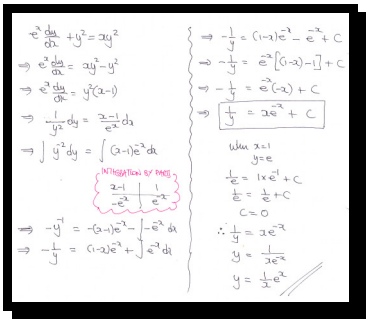
Question 5 **(***)**

$$\mathrm{e}^x\frac{dy}{dx}+y^2=xy^2\,,\ x>0\,,\ y>0$$

Show that the solution of the above differential equation subject to $y=\mathrm{e}$ at $x=1$, is

$$y=\frac{1}{x}\mathrm{e}^x\,.$$

proof



Question 6 (***)

A curve $y=f\left(x\right)$ satisfies the differential equation

$$y=1-\frac{dy}{dx}\frac{x+1}{\left(x-1\right)\left(x+2\right)},\ y>1, x>-1$$

a) Solve the differential equation to show that

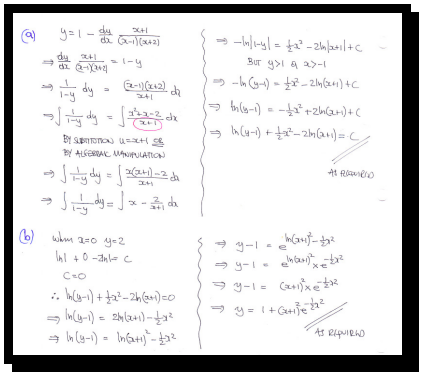
$$\ln\left(y-5\right)+\frac{1}{2}x^2+4x-2\ln\left(x+1\right)=C\,.$$

When $x=0$, $y=2$.

b) Show further that

$$y=1+\left(x+1\right)^2\mathrm{e}^{-\frac{1}{2}x^2}\,.$$

proof



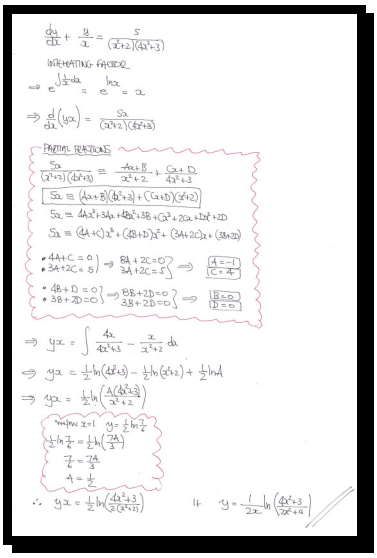
Question 7 (***)

$$\frac{dy}{dx}+\frac{y}{x}=\frac{5}{\left(x^2+2\right)\left(4x^2+3\right)},\ x>0\,.$$

Given that $y=\frac{1}{2}\ln\frac{7}{6}$ at $x=1$, show that the solution of the above differential equation can be written as

$$y=\frac{1}{2x}\ln\left(\frac{4x^2+3}{2x^2+4}\right).$$

proof

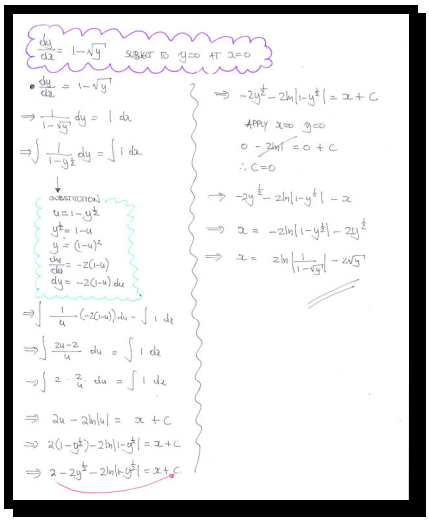


Question 8 **(***)**

$$\frac{dy}{dx}=1-\sqrt{y}\,,\ y\geq 0,\ y\neq 1\,.$$

Find the solution of the above differential equation subject to the condition $y=0$ at $x=0$, giving the answer in the form $x=f\left(y\right)$.

$$\boxed{x=2\ln\left|\frac{1}{1-\sqrt{y}}\right|-2\sqrt{y}}$$



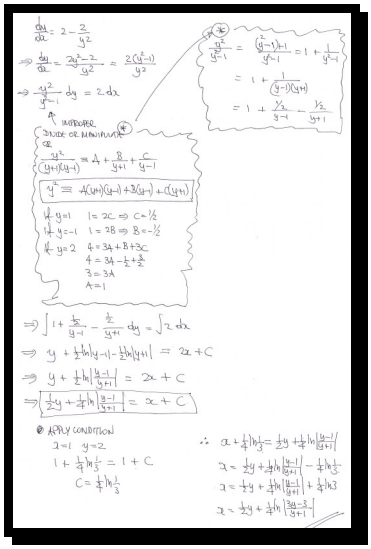
Question 9 (***)

Solve the differential equation

$$\frac{dy}{dx}=2-\frac{2}{y^2},$$

subject to the condition $y=2$ at $x=1$, giving the answer in the form $x=f\left(y\right)$.

$$\boxed{x=\frac{1}{2}y+\frac{1}{4}\ln\left|\frac{3y-3}{y+1}\right|}$$



Question 10 (*+)**

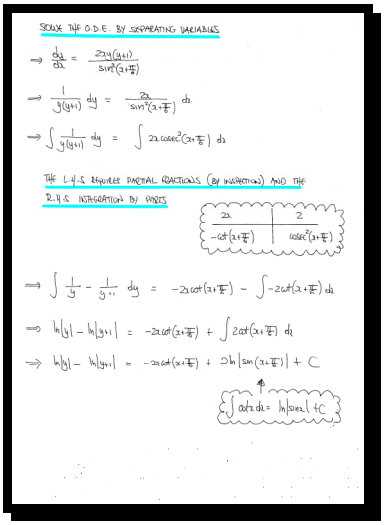
The function $y = f(x)$ satisfies the differential equation

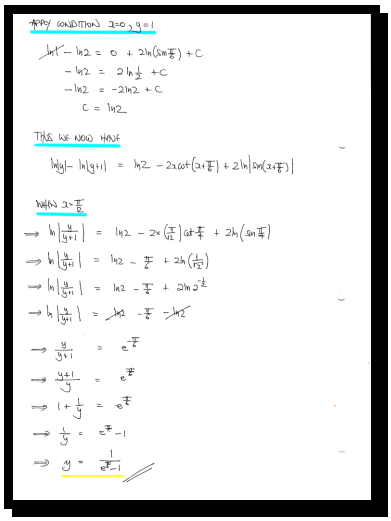
$$\frac{dy}{dx} = \frac{2xy(y+1)}{\sin^2\left(x+\frac{1}{6}\pi\right)},$$

subject to the condition $y = 1$ at $x = 0$.

Find the exact value of y when $x = \frac{\pi}{12}$.

$\square$, $\boxed{y = \dfrac{1}{e^{\frac{1}{6}\pi} - 1}}$





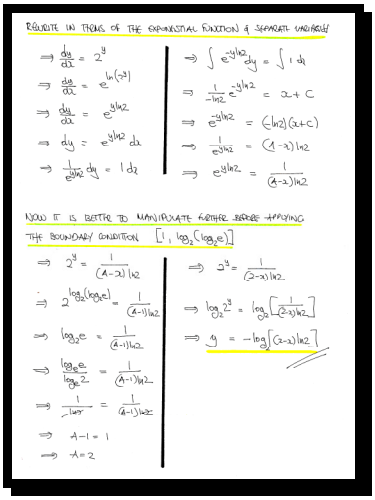
Question 11 (**+)**

A curve passes through the point with coordinates $\left[1\ ,\ \log_2\left(\log_2 \mathrm{e}\right)\right]$ and its gradient function satisfies

$$\frac{dy}{dx}=2^y\ ,\quad x\in\mathbb{R}\ ,\quad x<2\ .$$

Find the equation of the curve in the form $y=f\left(x\right)$

$\square$, $\boxed{y=-\log_2\left[\left(2-x\right)\ln 2\right]}$



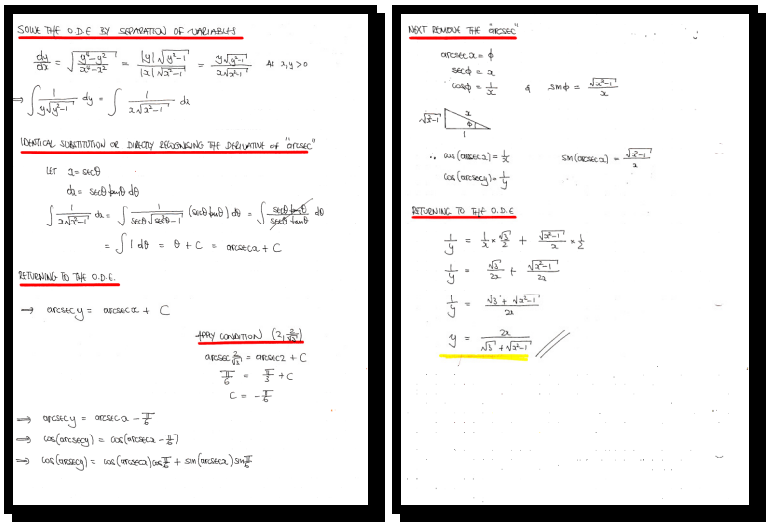
Question 12 (****)

$$\frac{dy}{dx}=\sqrt{\frac{y^4-y^2}{x^4-x^2}}\,,\ x>0\,,\ y>0\,.$$

Find the solution of the above differential equation subject to the boundary condition $y=\dfrac{2}{\sqrt{3}}$ at $x=2$.

Give the answer in the form $y=\dfrac{2x}{f(x)}$, where $f(x)$ is a function to be found.

$\square$, $\boxed{f(x)=\sqrt{3}+\sqrt{x^2-1}}$



1^{ST} ORDER BY STANDARD INTEGRATING FACTORS

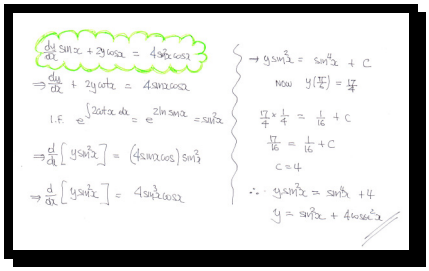
Question 1 (**)

Solve the differential equation

$$\frac{dy}{dx}\sin x+2y\cos x=4\sin^2 x\cos x\,,\qquad y\left(\frac{1}{6}\pi\right)=\frac{17}{4}\,.$$

Give the answer in the form $y=f\left(x\right)$.

$$\boxed{y=\sin^2 x+4\operatorname{cosec}^2 x}$$

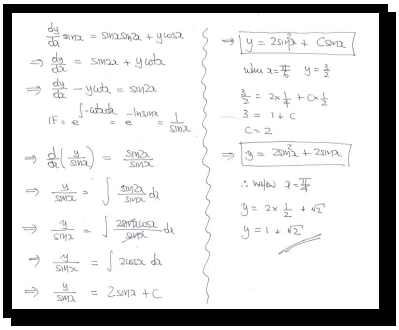


Question 2 (**)

$$\frac{dy}{dx}\sin x=\sin x\sin 2x+y\cos x\,.$$

Given that $y=\frac{3}{2}$ at $x=\frac{\pi}{6}$, find the exact value of y at $x=\frac{\pi}{4}$.

$$\boxed{1+\sqrt{2}}$$



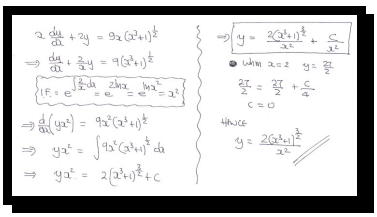
Question 3 (**)

$$x\frac{dy}{dx}+2y=9x\left(x^3+1\right)^{\frac{1}{2}} \text{ , with } y=\frac{27}{2} \text{ at } x=2 .$$

Show that the solution of the above differential equation is

$$y=\frac{2}{x^2}\left(x^3+1\right)^{\frac{3}{2}} .$$

proof



Question 4 (**)

20 grams of salt are dissolved into a beaker containing 1 litre of a certain chemical.

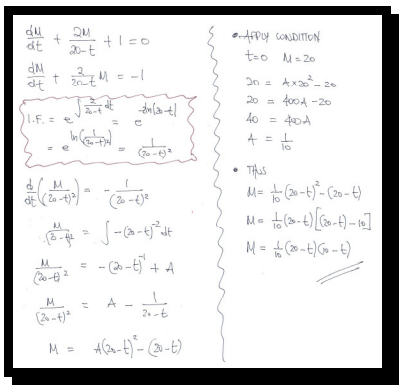
The mass of salt, M grams, which remains undissolved t seconds later, is modelled by the differential equation

$$\frac{dM}{dt}+\frac{2M}{20-t}+1=0\,,\ t\geq 0\,.$$

Show clearly that

$$M=\frac{1}{10}(10-t)(20-t)\,.$$

proof

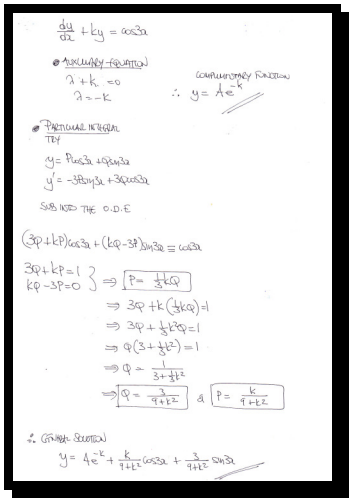


Question 5 (+)**

$$\frac{dy}{dx}+ky=\cos 3x\,,\ k \text{ is a non zero constant.}$$

By finding a complimentary function and a particular integral, or otherwise, find the general of the above differential equation.

$$\boxed{y=A\mathrm{e}^{-x}+\frac{k}{9+k^2}\cos 3x+\frac{3}{9+k^2}\sin 3x}$$



Question 6 (+)**

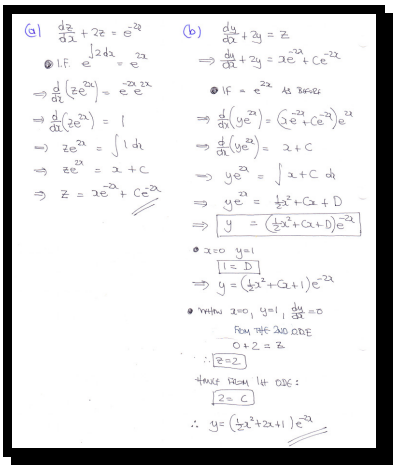
Given that $z=f(x)$ and $y=g(x)$ satisfy the following differential equations

$$\frac{dz}{dx}+2z=\mathrm{e}^{-2x} \quad \text{and} \quad \frac{dy}{dx}+2y=z\,,$$

a) Find z in the form $z=f(x)$

b) Express y in the form $y=g(x)$, given further that at $x=0$, $y=1$, $\dfrac{dy}{dx}=0$

$$\boxed{z=(x+C)\mathrm{e}^{-2x}}\,, \quad \boxed{y=\left(\frac{1}{2}x^2+2x+1\right)\mathrm{e}^{-2x}}$$



Question 7 (***)

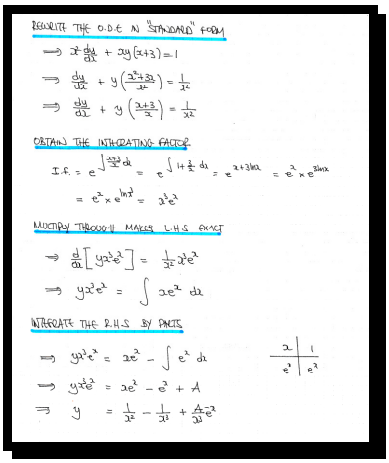
A curve C, with equation $y=f(x)$, passes through the points with coordinates $(1,1)$ and $(2,k)$, where k is a constant.

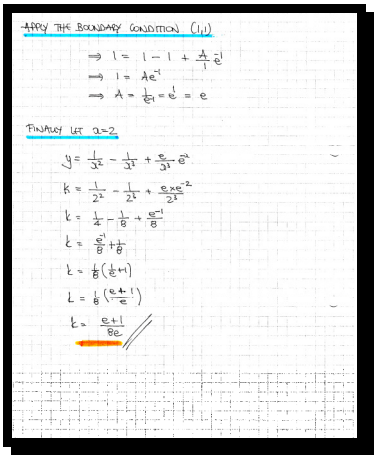
Given further that the equation of C satisfies the differential equation

$$x^2\frac{dy}{dx}+xy(x+3)=1,$$

determine the exact value of k.

$$\boxed{k=\frac{e+1}{8e}}$$





Question 8 **(***)**

A curve C, with equation $y=f(x)$, meets the y axis the point $(0,1)$.

It is further given that the equation of C satisfies the differential equation

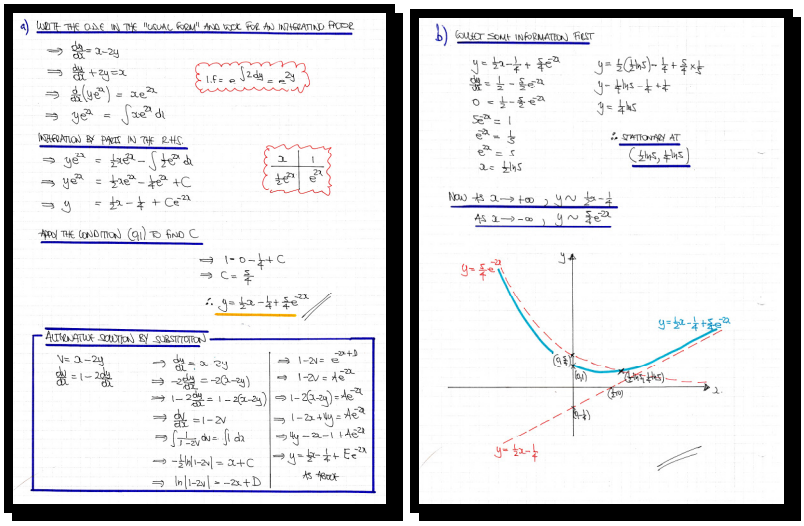
$$\frac{dy}{dx}=x-2y\,.$$

a) Determine an equation of C.

b) Sketch the graph of C.

The graph must include in exact simplified form the coordinates of the stationary point of the curve and the equation of its asymptote.

$\boxed{}$, $\boxed{y=\frac{1}{2}x-\frac{1}{4}+\frac{5}{4}\mathrm{e}^{-2x}}$



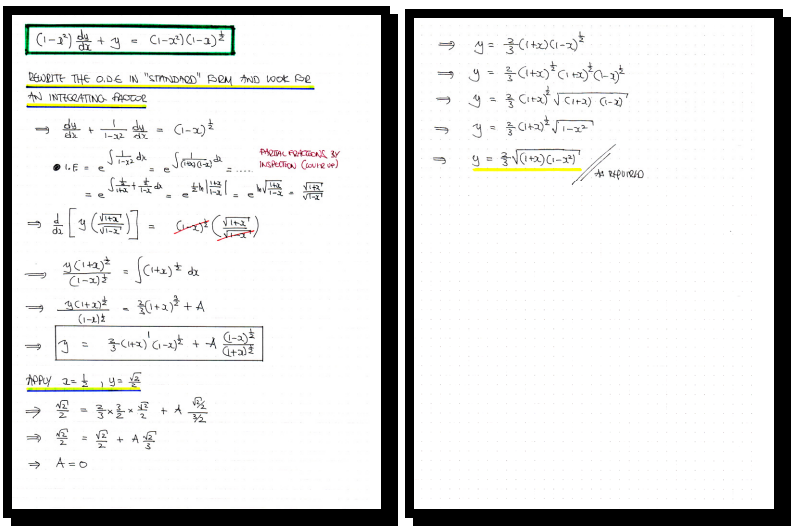
Question 9 **(***)**

$$\left(1-x^2\right)\frac{dy}{dx}+y=\left(1-x^2\right)(1-x)^{\frac{1}{2}}\ ,\ -1<x<1\,.$$

Given that $y=\dfrac{\sqrt{2}}{2}$ at $x=\dfrac{1}{2}$, show that the solution of the above differential equation can be written as

$$y=\frac{2}{3}\sqrt{\left(1-x^2\right)(1+x)}\,.$$

, proof



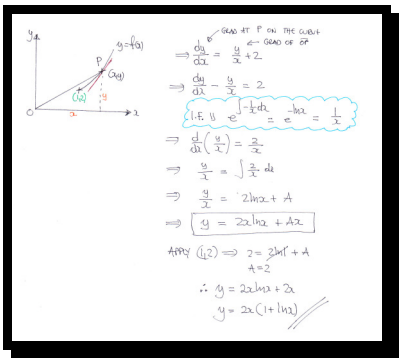
Question 10 (***)

The general point P lies on the curve with equation $y = f(x)$.

The gradient of the curve at P is 2 more than the gradient of the straight line segment OP.

Given further that the curve passes through $Q(1,2)$, express y in terms of x.

$$\boxed{y = 2x(1+\ln x)}$$



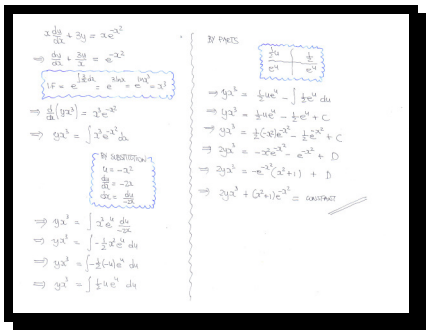
Question 11 (***)

$$x\frac{dy}{dx}+3y=x\mathrm{e}^{-x^2}\ ,\ x>0\,.$$

Show clearly that the general solution of the above differential equation can be written in the form

$$2yx^3+\left(x^2+1\right)\mathrm{e}^{-x^2}=\text{constant}\,.$$

proof



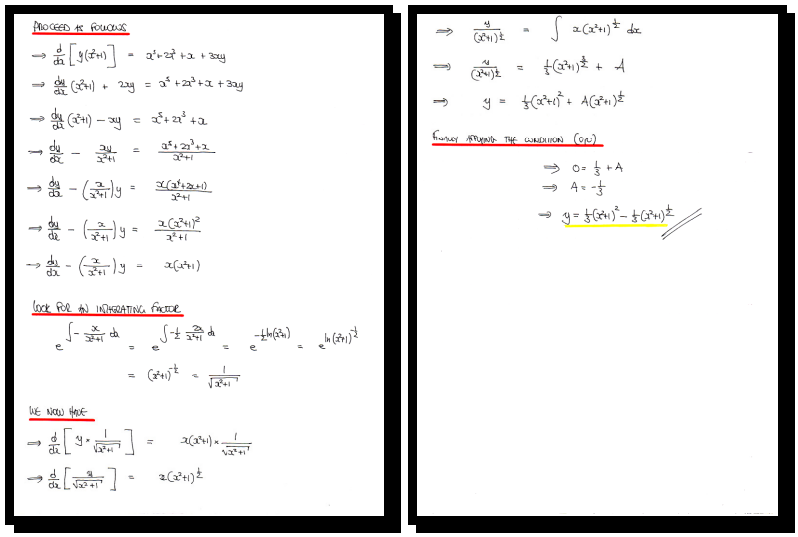
Question 11 (*+)**

The curve with equation $y=f\left(x\right)$ passes through the origin, and satisfies the relationship

$$\frac{d}{dx}\left[y\left(x^2+1\right)\right]=x^5+2x^3+x+3xy\,.$$

Determine a simplified expression for the equation of the curve.

$\square$, $y=\frac{1}{3}\left(x^2+1\right)^2-\frac{1}{3}\left(x^2+1\right)^{\frac{1}{2}}$



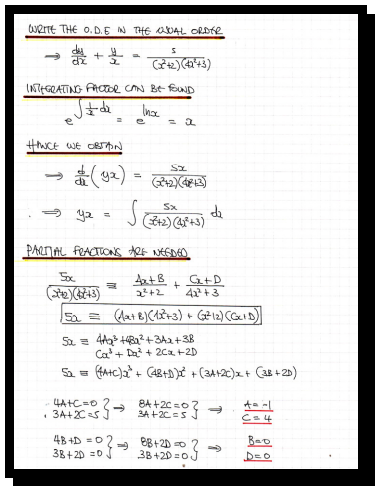
Question 12 (*+)**

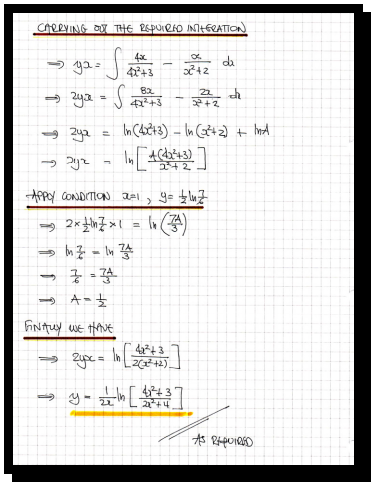
$$\frac{dy}{dx}+\frac{y}{x}=\frac{5}{\left(x^2+2\right)\left(4x^2+3\right)},\ x>0\,.$$

Given that $y=\frac{1}{2}\ln\frac{7}{6}$ at $x=1$, show that the solution of the above differential equation can be written as

$$y=\frac{1}{2x}\ln\left(\frac{4x^2+3}{2x^2+4}\right).$$

$\boxed{}$, $\boxed{\text{proof}}$



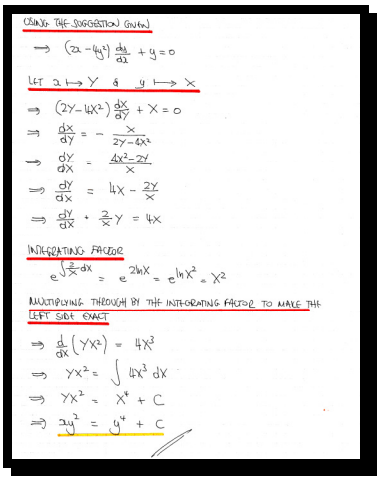


Question 13 **(***+)**

$$\left(2x-4y^2\right)\frac{dy}{dx}+y=0\,.$$

By reversing the role of x and y in the above differential equation, or otherwise, find its general solution.

$\boxed{}$, $\boxed{xy^2=y^4+C}$



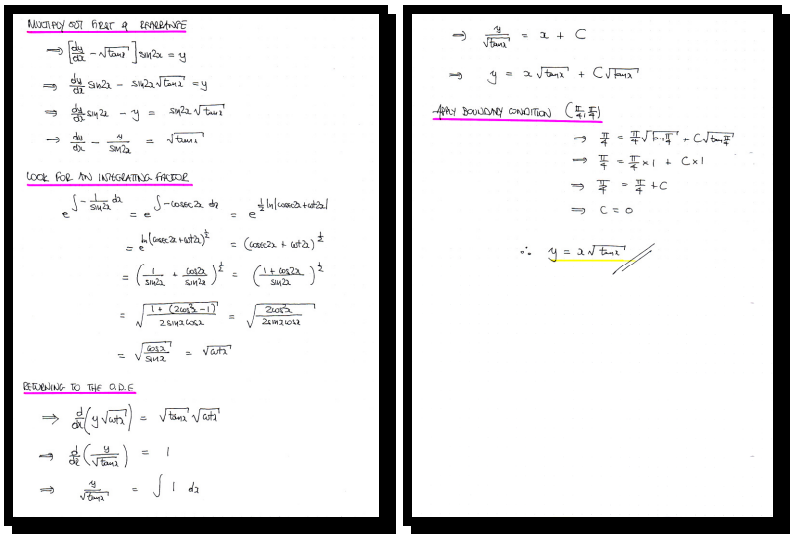
Question 14 (****)

It is given that a curve with equation $y = f(x)$ passes through the point $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ and satisfies the differential equation

$$\left(\frac{dy}{dx} - \sqrt{\tan x}\right)\sin 2x = y .$$

Find an equation for the curve in the form $y = f(x)$.

$\square$, $\boxed{y = x\sqrt{\tan x}}$

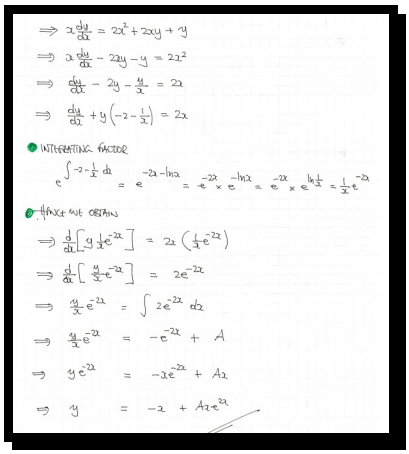


Question 15 (****)

Find a simplified general solution for the following differential equation.

$$x\frac{dy}{dx}=2x^2+2xy+y\,.$$

$$\boxed{y=Ax\mathrm{e}^{-x}-x}$$



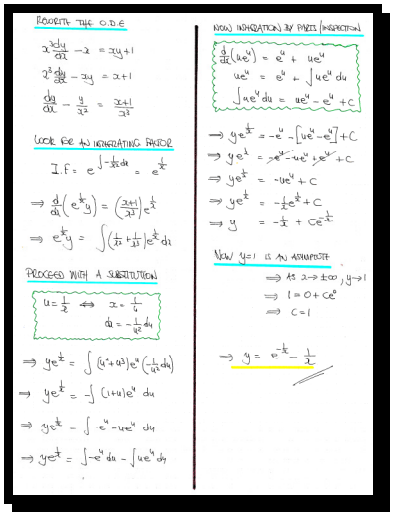
Question 16 (****)

The curve with equation $y = f(x)$ has the line $y = 1$ as an asymptote and satisfies the differential equation

$$x^3 \frac{dy}{dx} - x = xy + 1, \quad x \neq 0.$$

Solve the above differential equation, giving the solution in the form $y = f(x)$.

$\square$, $\boxed{y = e^{-\frac{1}{x}} - \frac{1}{x}}$



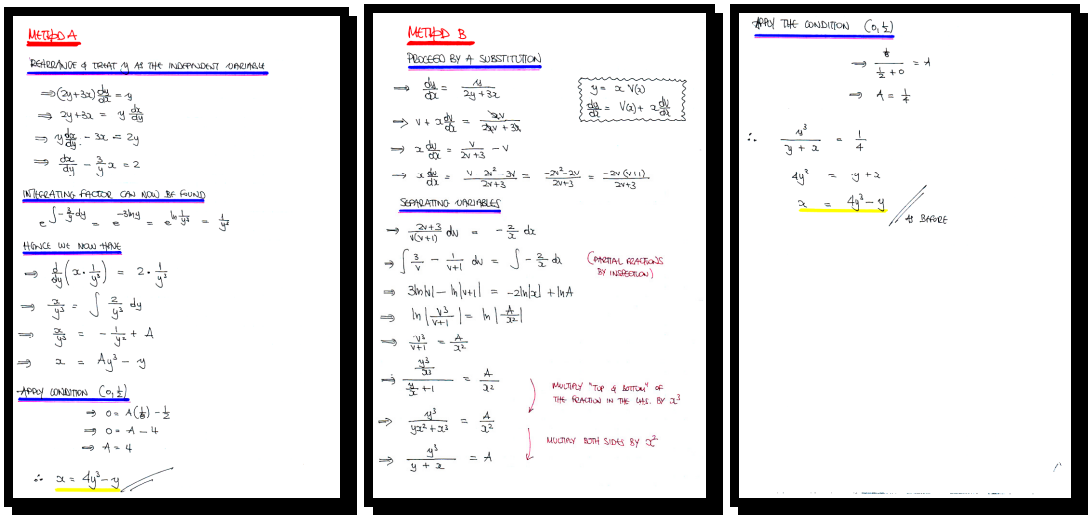
Question 17 (****)

It is given that a curve with equation $x=f(y)$ passes through the point $\left(0,\frac{1}{2}\right)$ and satisfies the differential equation

$$(2y+3x)\frac{dy}{dx}=y\,.$$

Find an equation for the curve in the form $x=f(y)$.

$\boxed{}$, $\boxed{x=4y^3-y}$



Question 18 (****)

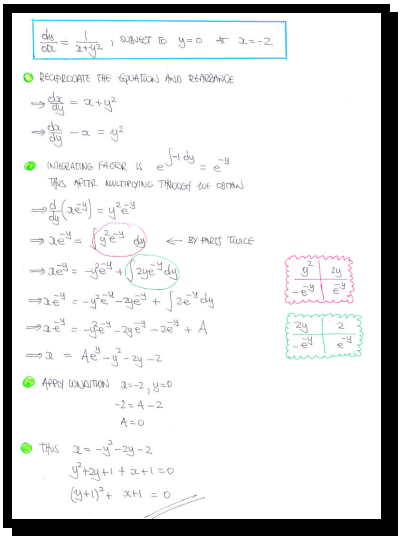
It is given that a curve passes through the point $(-2,0)$ and satisfies the ordinary differential equation

$$\frac{dy}{dx}=\frac{1}{x+y^2}.$$

Show that an equation of C is

$$(y+1)^2+x+1=0.$$

proof



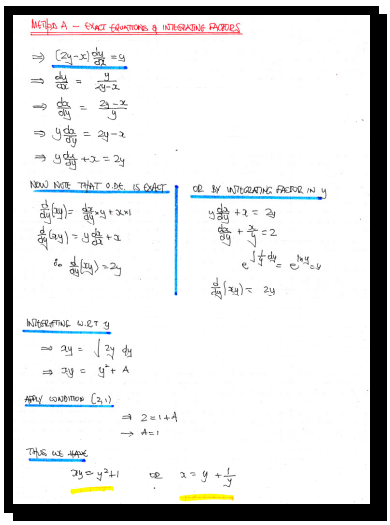
Question 19 (****)

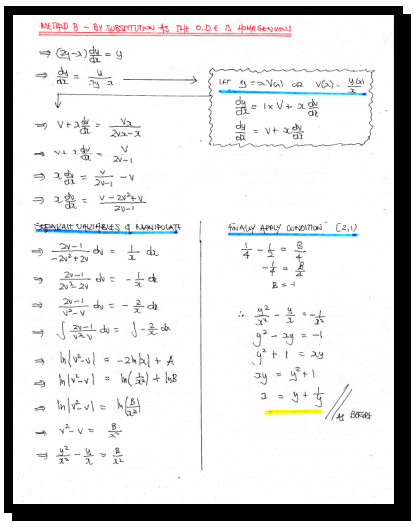
The variables x and y satisfy

$$(2y-x)\frac{dy}{dx}=y\,,\quad y>0\,,\ x>0\,.$$

If $y=1$ at $x=2$, show that $x=y+\dfrac{1}{y}$.

V, [ ], proof





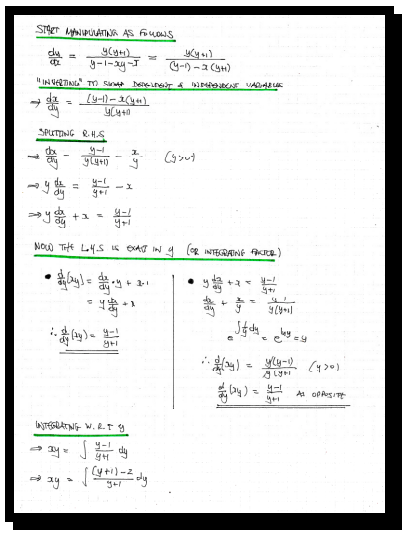
Question 20

The variables x and y satisfy

$$\frac{dy}{dx}=\frac{y(y+1)}{y-x-xy-1},\quad y>0.$$

If $y=1$ at $x=1-\ln 4$, show that $y+\ln(y+1)=0$ at $x=3$.

V, [ ], proof


START MANIPULATING AS FRACTIONS

$\frac{dy}{dx}=\frac{y(y+1)}{y-1-xy-x}=\frac{y(y+1)}{(y-1)-x(y+1)}$

"INVERTING" TO CREATE DERIVATIVE OF INDEPENDENT VARIABLE

$\Rightarrow \frac{dx}{dy}=\frac{(y-1)-x(y+1)}{y(y+1)}$

SPLITTING R.H.S

$\Rightarrow \frac{dx}{dy}=\frac{y-1}{y(y+1)}-\frac{x}{y}$ $(y>0)$

$\Rightarrow y\frac{dx}{dy}=\frac{y-1}{y+1}-x$

$\Rightarrow y\frac{dx}{dy}+x=\frac{y-1}{y+1}$

NOW THE L.H.S IS EXACT IN y (OR INTEGRATING FACTOR)

$\frac{d}{dy}(xy)=\frac{dx}{dy}\cdot y+x\cdot 1$

$=y\frac{dx}{dy}+x$

$\therefore \frac{d}{dy}(xy)=\frac{y-1}{y+1}$

$y\frac{dx}{dy}+x=\frac{y-1}{y+1}$

$\frac{dx}{dy}+\frac{x}{y}=\frac{y-1}{y(y+1)}$

$\int \frac{1}{y}dy=e^{\ln y}=y$

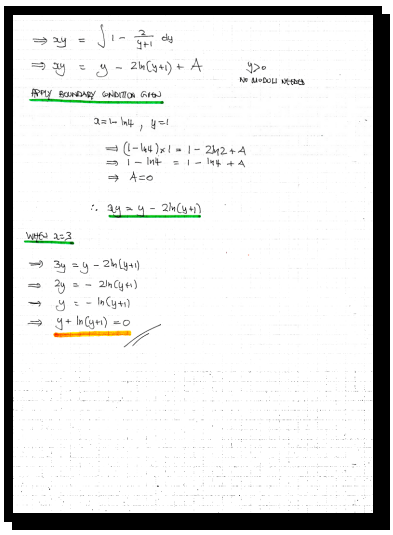
$\therefore \frac{d}{dy}(xy)=\frac{y(y-1)}{y(y+1)}$ $(y>0)$

$\frac{d}{dy}(xy)=\frac{y-1}{y+1}$ AS REQUIRED

INTEGRATING W.R.T y

$\Rightarrow xy=\int \frac{y-1}{y+1}\,dy$

$\Rightarrow xy=\int \frac{(y+1)-2}{y+1}\,dy$


$\Rightarrow xy=\int 1-\frac{2}{y+1}\,dy$

$\Rightarrow xy=y-2\ln(y+1)+A$ $y>0$ NO MODULI NEEDED

APPLY BOUNDARY CONDITION GIVEN

$x=1-\ln 4,\ y=1$

$\Rightarrow (1-\ln 4)\times 1=1-2\ln 2+A$

$\Rightarrow 1-\ln 4=1-\ln 4+A$

$\Rightarrow A=0$

$\therefore xy=y-2\ln(y+1)$

WHEN $x=3$

$\Rightarrow 3y=y-2\ln(y+1)$

$\Rightarrow 2y=-2\ln(y+1)$

$\Rightarrow y=-\ln(y+1)$

$\Rightarrow y+\ln(y+1)=0$

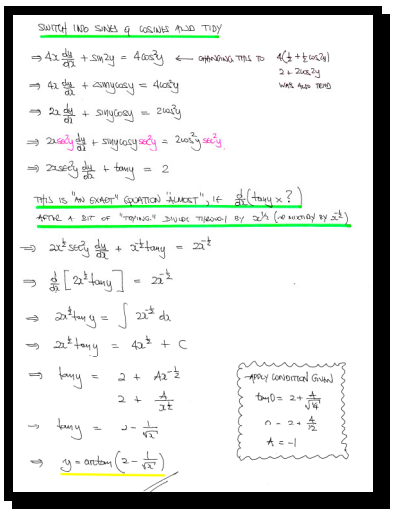
Question 21 **(*****)**

Use suitable manipulations to solve this **exact** differential equation.

$$4x\frac{dy}{dx}+\sin 2y=4\cos^2 y\,,\quad y\left(\frac{1}{4}\right)=0\,.$$

Given the answer in the form $y=f\left(x\right)$.

V, ☐, $\boxed{y=\arctan\left[2-\frac{1}{\sqrt{x}}\right]}$



1^{ST} ORDER HOMOGENEOUS

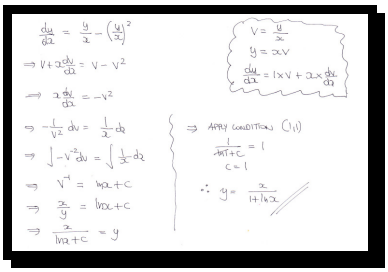
Question 1 (+)**

Solve the differential equation

$$\frac{dy}{dx}=\frac{y}{x}-\left(\frac{y}{x}\right)^2,\ x>0,$$

subject to the condition $y=1$ at $x=1$.

$$\boxed{y=\frac{x}{1+\ln x}}$$



Question 2 **(**+)**

$$\frac{dy}{dx}=\frac{(4x+y)(x+y)}{x^2},\ x>0.$$

a) Use a suitable substitution to show that the above differential equation can be transformed to

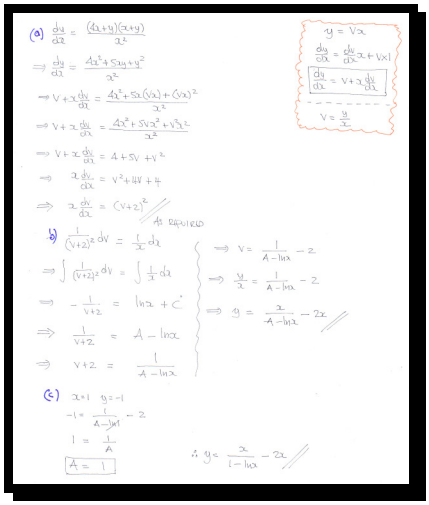
$$x\frac{dv}{dx}=(v+2)^2.$$

b) Hence find the general solution of the original differential equation, giving the answer in the form $y=f(x)$.

c) Use the boundary condition $y=-1$ at $x=1$, to show that a specific solution of the original differential equation is

$$y=\frac{x}{1-\ln x}-2x.$$

$$\boxed{y=\frac{x}{A-\ln x}-2x}$$



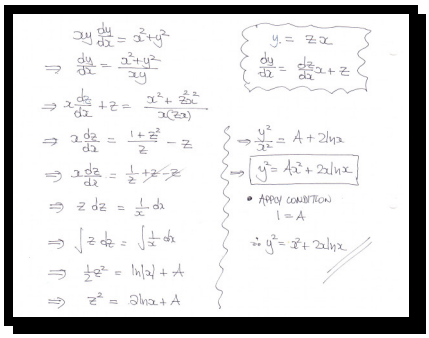
Question 3 (+)**

By using a suitable substitution, solve the differential equation

$$xy\frac{dy}{dx}=x^2+y^2,\ x>0,$$

subject to the boundary condition $y=1$ at $x=1$.

$$\boxed{y=x^2\left(1+2\ln x\right)}$$



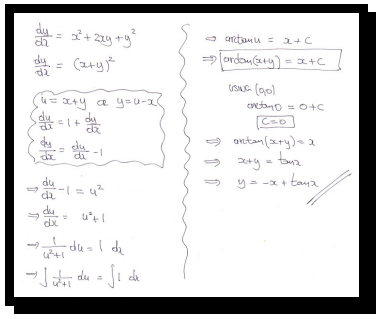
Question 4 (+)**

By using a suitable substitution, or otherwise, solve the differential equation

$$\frac{dy}{dx}=x^2+2xy+y^2,$$

subject to the condition $y(0)=0$.

$$\boxed{y=-x+\tan x}$$



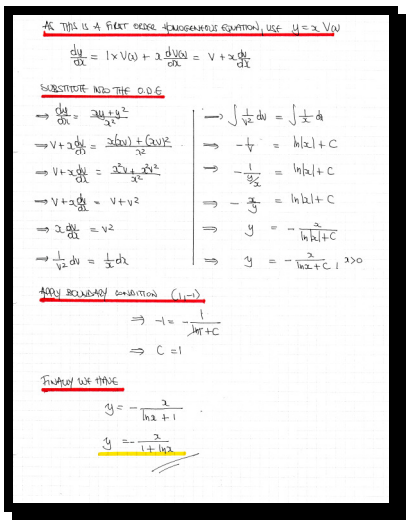
Question 5 (+)**

By using a suitable substitution, solve the differential equation

$$\frac{dy}{dx}=\frac{xy+y^2}{x^2},\ x>0,$$

subject to the condition $y=-1$ at $x=1$.

$$\boxed{y=-\frac{x}{1+\ln x}}$$



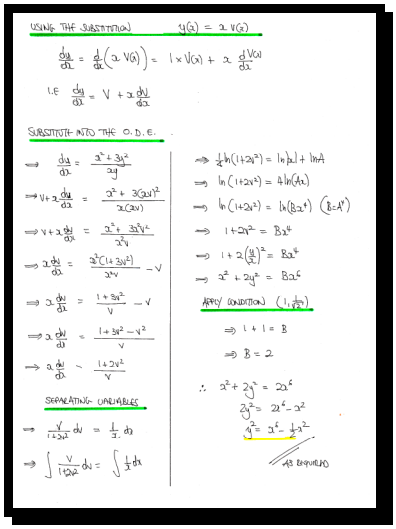
Question 6 (***)

$$\frac{dy}{dx}=\frac{x^2+3y^2}{xy},\ x>0,\ y>0.$$

Given the boundary condition $y(1)=\frac{1}{\sqrt{2}}$, show that

$$y^2=x^6-\frac{1}{2}x^2.$$

, proof



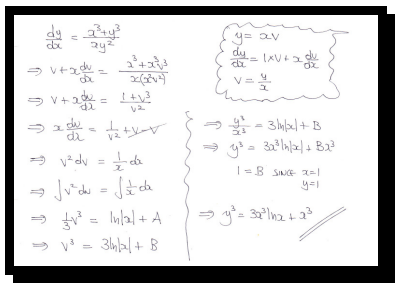
Question 7 (***)

By using a suitable substitution, solve the differential equation

$$\frac{dy}{dx}=\frac{x^3+y^3}{xy^2},$$

subject to the condition $y=1$ at $x=1$.

$$\boxed{y^3=x^3\left(3\ln x+1\right)}$$



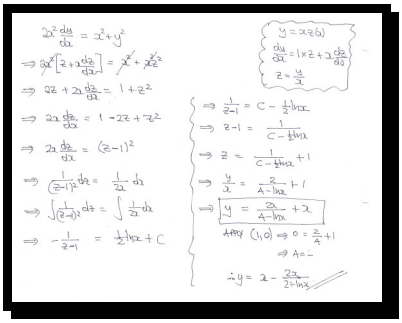
Question 8 (***)

By using a suitable substitution, solve the differential equation

$$2x^2\frac{dy}{dx}=x^2+y^2,\ x>0,$$

subject to the condition $y(1)=0$.

$$\boxed{y=x-\frac{2x}{2+\ln x}}$$



Question 9 (***)

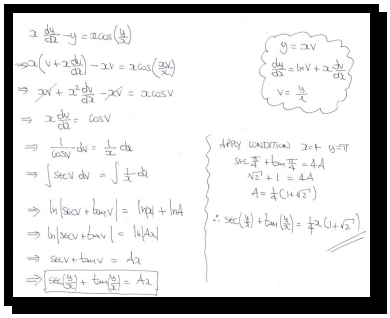
By using a suitable substitution, solve the differential equation

$$x\frac{dy}{dx}-y=x\cos\left(\frac{y}{x}\right),\ x\neq 0,$$

subject to the condition $y(4)=\pi$.

The final answer may not involve natural logarithms.

$$\boxed{\sec\left(\frac{y}{x}\right)+\tan\left(\frac{y}{x}\right)=\frac{1}{4}x\left(1+\sqrt{2}\right)}$$



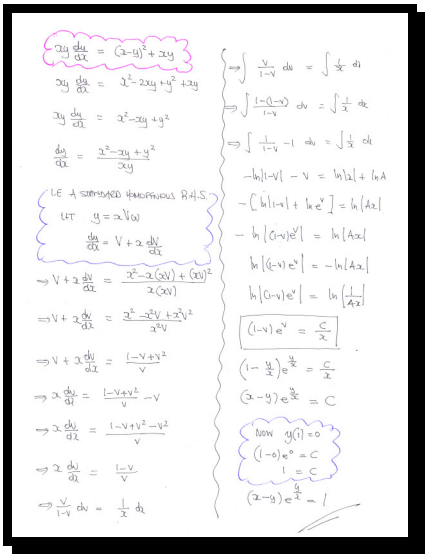
Question 10 (***)

$$xy\frac{dy}{dx}=(x-y)^2+xy\,,\quad y(1)=0\,.$$

Show that the solution of the above differential equation is

$$(x-y)\mathrm{e}^{\frac{y}{x}}=1\,.$$

proof

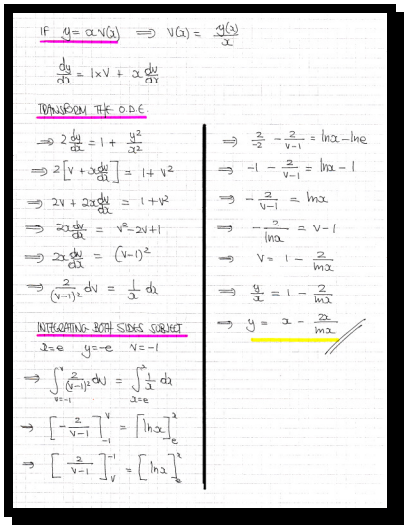


Question 11 (***)

Use the substitution $y=xv$, where $v=v(x)$, to solve the following differential equation

$$2\frac{dy}{dx}=1+\frac{y^2}{x^2}, \quad y(\mathrm{e})=-\mathrm{e}.$$

$\square$, $\boxed{y=x-\dfrac{2x}{\ln x}}$



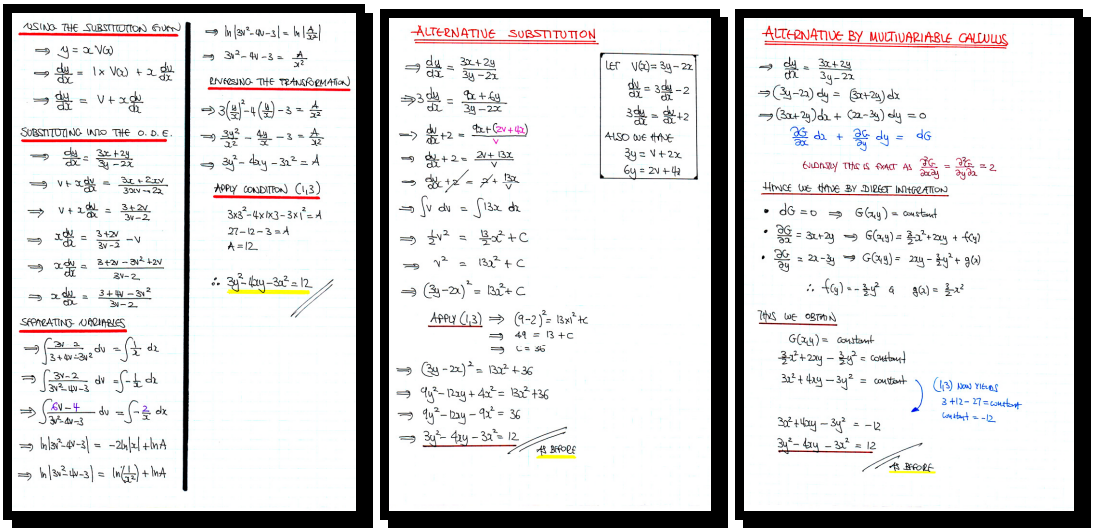
Question 12 (***)

Solve the following differential equation

$$\frac{dy}{dx}=\frac{3x+2y}{3y-2x},\ y(1)=3.$$

Give the final answer in the form $F(x,y)=12$

$\square$, $\boxed{3y^2-4xy-3x^2=12}$



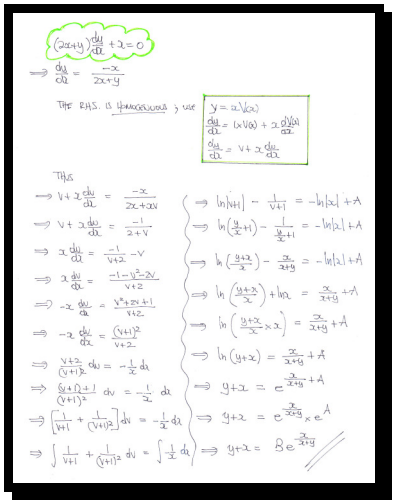
Question 13 (*+)**

Find a general solution for the following differential equation

$$(2x+y)\frac{dy}{dx}+x=0\,.$$

The final answer must not contain natural logarithms.

$$\boxed{y+x=A\,\mathrm{e}^{\frac{x}{x+y}}}$$

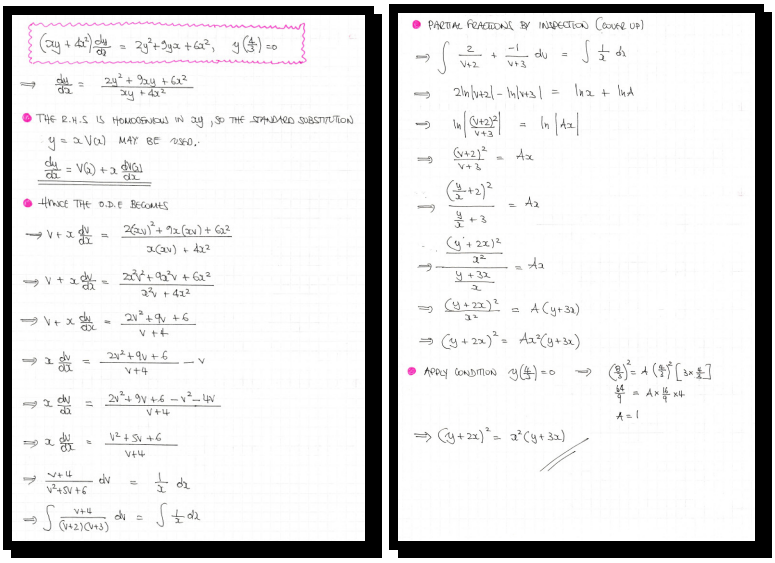


Question 14 (*+)**

Solve the following differential equation.

$$\left(xy+4x^2\right)\frac{dy}{dx}=2y^2+9xy+6x^2,\quad y\left(\frac{4}{3}\right)=0.$$

$$\boxed{\left(y+2x\right)^2=x^2\left(y+3x\right)}$$



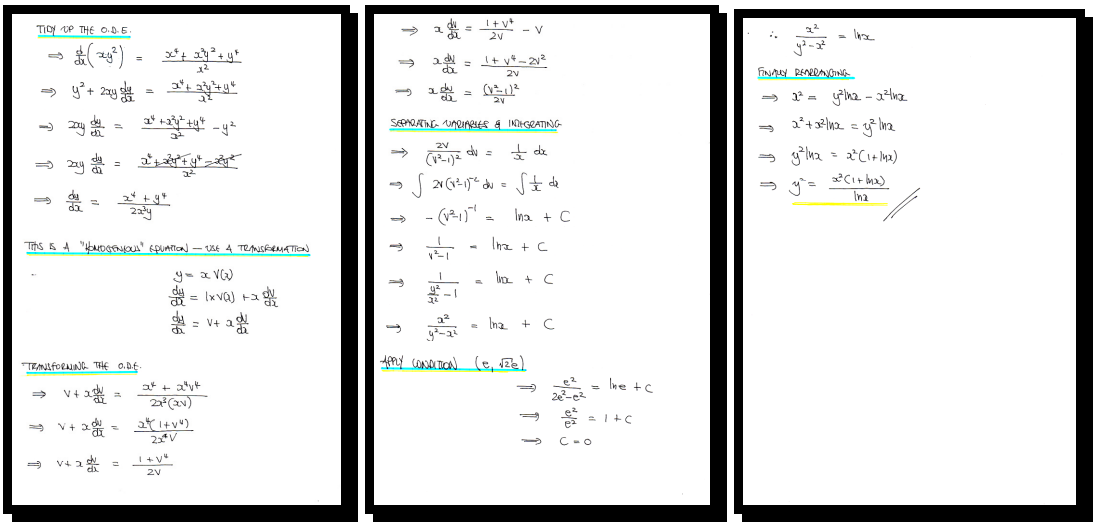
Question 15 (****)

Solve the differential equation

$$\frac{d}{dx}\left(xy^2\right)=\frac{x^4+x^2y^2+y^4}{x^2},\ y\left(\mathrm{e}\right)=\sqrt{2}\,\mathrm{e}\,.$$

Give the answer in the form $y^2=f\left(x\right)$.

$\square$, $\boxed{y^2=\dfrac{x^2\left(1+\ln x\right)}{\ln x}}$

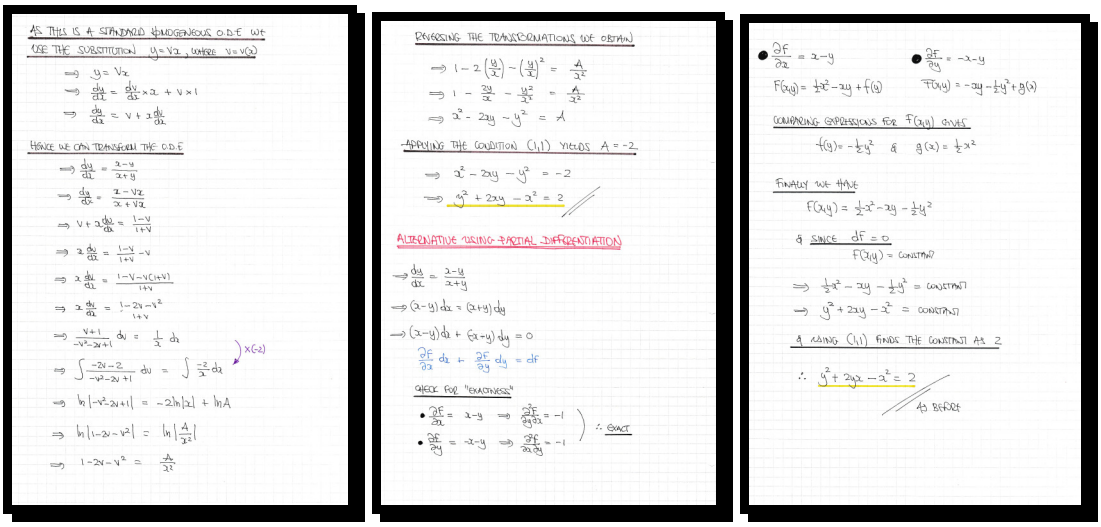


Question 16 (****)

Solve the differential equation

$$\frac{dy}{dx}=\frac{x-y}{x+y},\ y(1)=1.$$

$\boxed{}$, $\boxed{y^2+2xy-x^2=2}$



Question 17 (****)

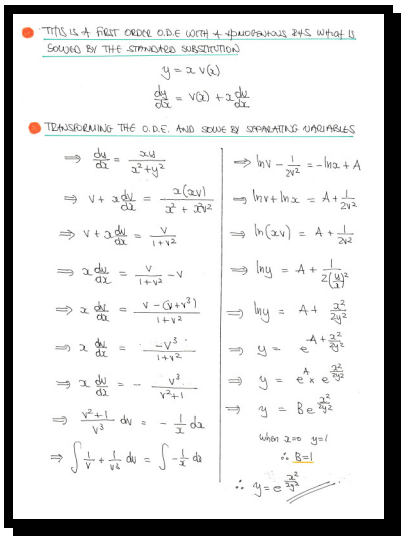
It is given that a curve with equation $f(x,y)=0$ passes through the point $(0,1)$ and satisfies the differential equation

$$\frac{dy}{dx}=\frac{xy}{x^2+y^2}.$$

By solving the differential equation, show that an equation for the curve is

$$y=\exp\left[\frac{x^2}{2y^2}\right].$$

, proof



1^{ST} ORDER BERNOULLI TYPE

Question 1 **(**+)**

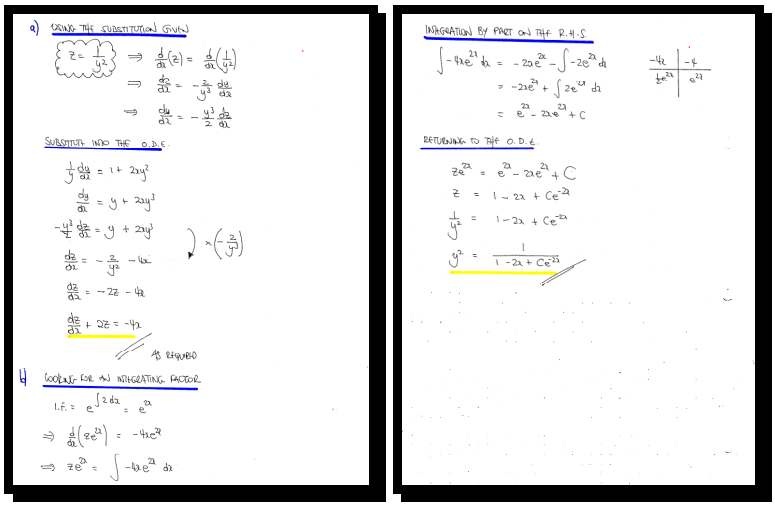
$$\frac{1}{y}\frac{dy}{dx}=1+2xy^2\,,\ \ y>0\,.$$

a) Show that the substitution $z=\dfrac{1}{y^2}$ transforms the above differential equation into the new differential equation

$$\frac{dz}{dx}+2z=-4x\,.$$

b) Hence find the general solution of the original differential equation, giving the answer in the form $y^2=f\left(x\right)$.

$\square$, $\boxed{y^2=\dfrac{1}{Ae^{-2x}-2x+1}}$



Question 2 (***)

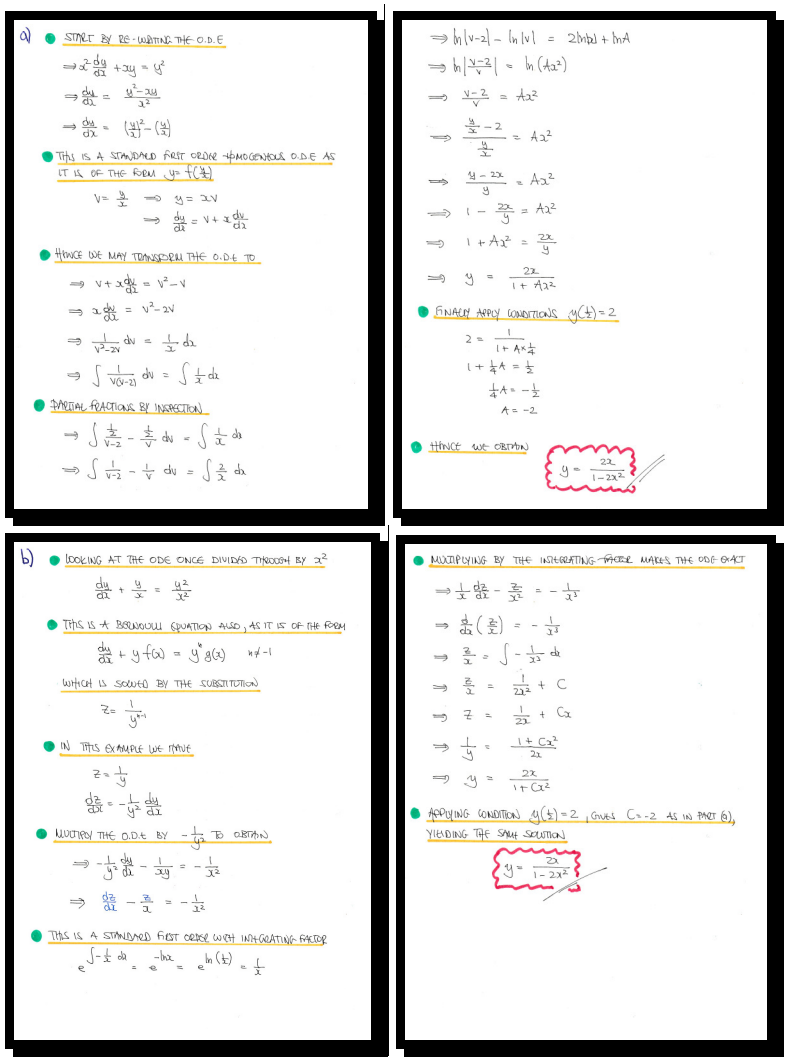
a) Use the suitable substitution to solve the differential equation

$$x^2\frac{dy}{dx}+xy=y^2\,,\qquad y\left(\frac{1}{2}\right)=2\,.$$

Give the answer in the form $y=f\left(x\right)$.

b) Verify the answer of part **(a)** by solving the above differential equation with an alternative method.

$\square$, $\boxed{y=\dfrac{2x}{1-2x^2}}$

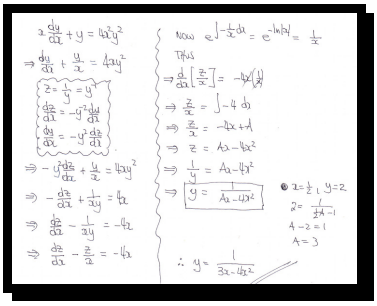


Question 3 (***)

Solve the differential equation

$$x\frac{dy}{dx}+y=4x^2y^2,\quad y\left(\frac{1}{2}\right)=2.$$

$$\boxed{y=\frac{1}{3x-4x^2}}$$



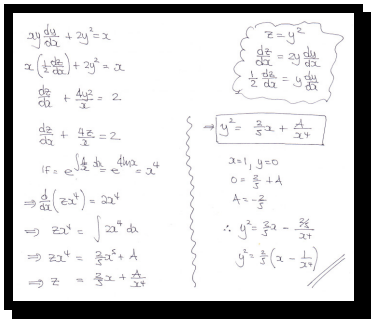
Question 4 (***)

By using a suitable substitution, solve the differential equation

$$xy\frac{dy}{dx}+2y^2=x,\ y(1)=0.$$

Give the answer in the form $y^2=f(x)$.

$$\boxed{y^2=\frac{2}{5}\left(x-\frac{1}{x^4}\right)}$$



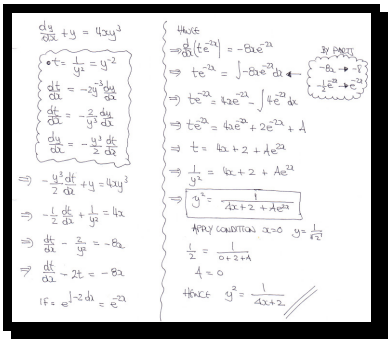
Question 5 (***)

Solve the differential equation

$$\frac{dy}{dx}+y=4xy^3,\ y(0)=\frac{1}{\sqrt{2}}.$$

Give the answer in the form $y^2=f(x)$.

$$\boxed{y^2=\frac{1}{4x+2}}$$



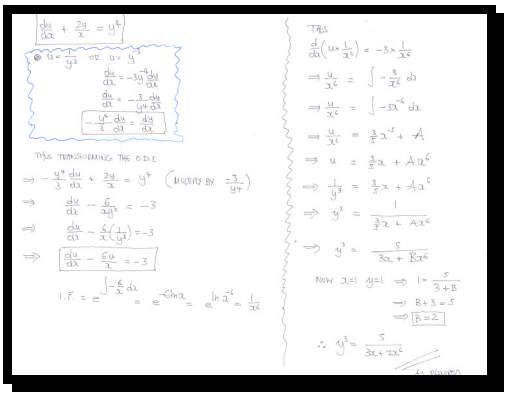
Question 6 (*+)**

$$\frac{dy}{dx}+\frac{2y}{x}=y^4,\ x>0,\ y>0.$$

Given that $y(1)=1$, show that

$$y^3=\frac{5}{3x+2x^6}.$$

proof



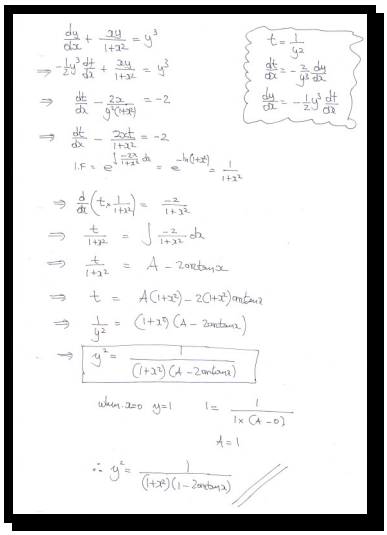
Question 7 (*+)**

Solve the differential equation

$$\frac{dy}{dx}+\frac{xy}{1+x^2}=y^3,\ y(0)=1.$$

Give the answer in the form $y^2=f(x)$.

$$\boxed{y^2=\frac{1}{\left(1+x^2\right)\left(1-2\arctan x\right)}}$$

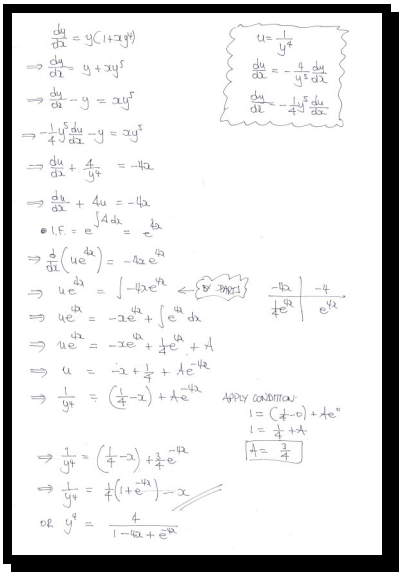


Question 8 (*+)**

Solve the differential equation

$$\frac{dy}{dx}=y\left(1+xy^4\right),\ y\left(0\right)=1.$$

$$\boxed{\frac{1}{y^4}=\frac{1}{4}\left(1+3\mathrm{e}^{-4x}\right)-x}$$



Question 9 (****)

A curve C passes through the point $(1,1)$ and satisfies the differential equation

$$\frac{dy}{dx}-\frac{y}{x}=\frac{x^3}{4y^3},\ x>0,\ y>0,$$

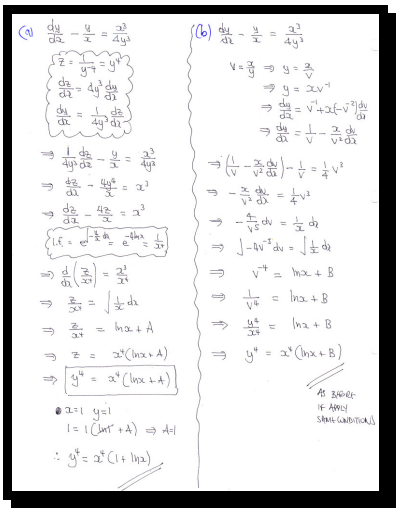
subject to the condition $y=1$ at $x=1$.

a) Find an equation of C by using the substitution $z=y^4$.

b) Find an equation of C by using the substitution $v=\frac{x}{y}$.

Give the answer in the form $y^4=f(x)$.

$$\boxed{y^4=x^4(1+\ln x)}$$



1ST ORDER BY PARTIAL DIFFERENTIATION TECHNIQUES

Question 1 (+)**

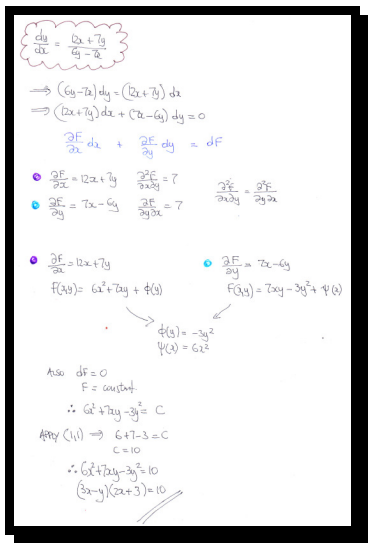
$$\frac{dy}{dx}=\frac{12x+7y}{6y-7x},\ y(1)=1.$$

Use a method involving partial differentiation to show that the solution of the above differential equation can be written as

$$(ax+by)(cx+dy)=10,$$

where a, b, c and d are integers to be found.

$$\boxed{(3x-y)(2x+3y)=k}$$



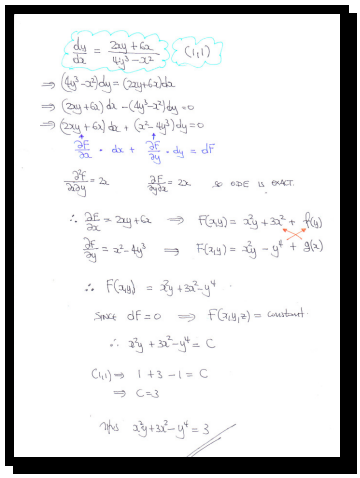
Question 2 (***)

Solve the differential equation

$$\frac{dy}{dx}=\frac{2xy+6x}{4y^3-x^2},$$

subject to the boundary condition $y=1$ at $x=1$.

$$\boxed{x^2y+3x^2-y^4=3}$$

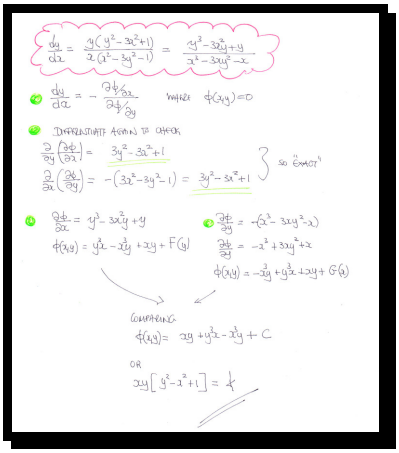


Question 3 (***)

Find a general solution of the following differential equation

$$\frac{dy}{dx}=\frac{y\left(y^2-3x^2+1\right)}{x\left(x^2-3y^2-1\right)}.$$

$$\boxed{xy\left(x^2-y^2-1\right)=\text{constant}}$$



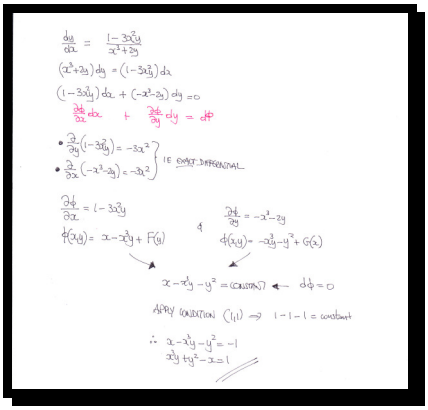
Question 4 (***)

Find the solution of the following differential equation

$$\frac{dy}{dx}=\frac{1-3x^2y}{x^3+2y},$$

subject to the boundary condition $y=1$ at $x=1$.

$$\boxed{x^3y+y^2-x=1}$$



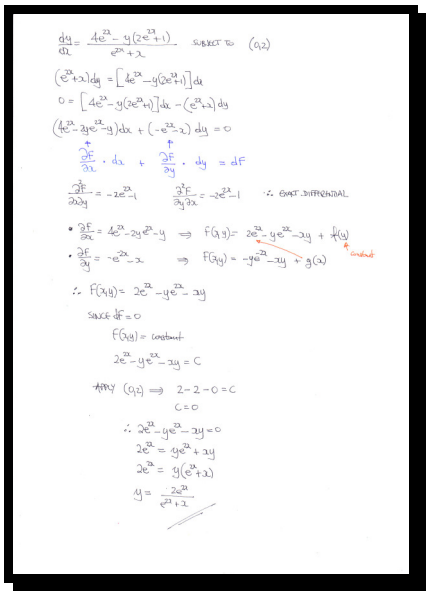
Question 5 (***)

Solve the differential equation

$$\frac{dy}{dx}=\frac{4\mathrm{e}^{2x}-y\left(2\mathrm{e}^{2x}+1\right)}{\mathrm{e}^{2x}+x},$$

subject to the boundary condition $y=2$ at $x=0$.

$$\boxed{y=\frac{2\mathrm{e}^{2x}}{\mathrm{e}^{2x}+x}}$$

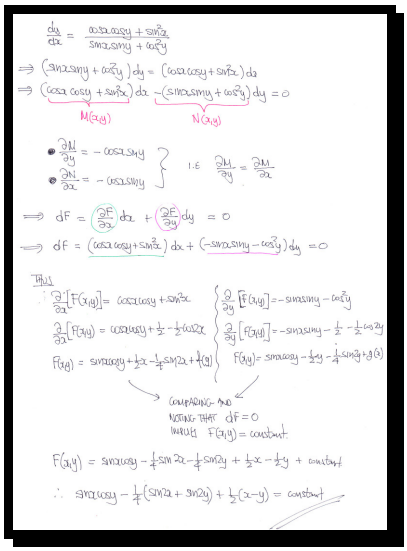


Question 6 (*+)**

Find a general solution of the following differential equation

$$\frac{dy}{dx}=\frac{\cos x\cos y+\sin^2 x}{\sin x\sin y+\cos^2 y}.$$

$$\boxed{\sin x\cos y-\frac{1}{4}\left(\sin 2x+\sin 2y\right)+\frac{1}{2}\left(x-y\right)=\text{constant}}$$

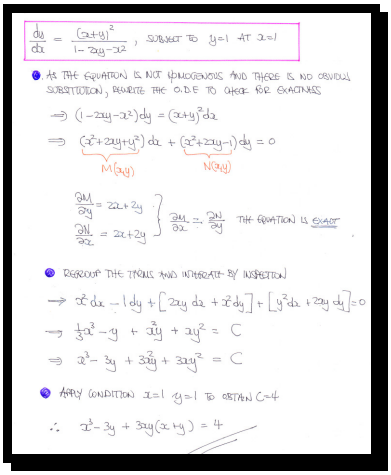


Question 7 (***)

Determine the solution of the following differential equation by looking for a suitable integrating factor.

$$\frac{dy}{dx}=\frac{(x+y)^2}{1-2xy-x^2}, \quad y(1)=1.$$

$$\boxed{x^3-3y+3xy(x+y)=4}$$

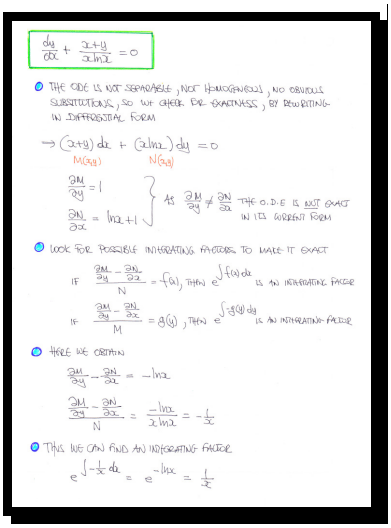


Question 8 (*+)**

Find a general solution of the following differential equation by looking for a suitable integrating factor.

$$\frac{dy}{dx}+\frac{x+y}{x\ln x}=0\,.$$

$$\boxed{x+y\ln x=C}$$



$\frac{dy}{dx}+\frac{x+y}{x\ln x}=0$

- THE ODE IS NOT SEPARABLE, NOT HOMOGENEOUS, NO OBVIOUS SUBSTITUTIONS, SO WE CHECK FOR EXACTNESS, BY REWRITING IN DIFFERENTIAL FORM

$\Rightarrow (x+y)\,dx+(x\ln x)\,dy=0$

$M(x,y)$ $N(x,y)$

$\frac{\partial M}{\partial y}=1$, $\frac{\partial N}{\partial x}=\ln x+1$ $\Rightarrow$ AS $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ THE O.D.E IS NOT EXACT IN ITS CURRENT FORM

- LOOK FOR POSSIBLE INTEGRATING FACTORS TO MAKE IT EXACT

IF $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x)$, THEN $e^{\int f(x)\,dx}$ IS AN INTEGRATING FACTOR

IF $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=g(y)$, THEN $e^{\int g(y)\,dy}$ IS AN INTEGRATING FACTOR

- HERE WE OBTAIN

$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=-\ln x$

$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{-\ln x}{x\ln x}=-\frac{1}{x}$

- THUS WE CAN FIND AN INTEGRATING FACTOR

$e^{\int -\frac{1}{x}\,dx}=e^{-\ln x}=\frac{1}{x}$

- MULTIPLY THE O.D.E BY $\frac{1}{x}$, AND INTEGRATE BY INSPECTION

$\Rightarrow \left(1+\frac{y}{x}\right)dx+(\ln x)\,dy=0$

$\Rightarrow 1\,dx+\left(\frac{y}{x}\,dx+\ln x\,dy\right)=0$

$\Rightarrow x+y\ln x=C$

ALTERNATIVE WITHOUT INSPECTION

$\left(1+\frac{y}{x}\right)dx+(\ln x)\,dy=0$

$\frac{\partial\phi}{\partial x}\cdot dx+\frac{\partial\phi}{\partial y}\cdot dy=d\phi$

- $\frac{\partial\phi}{\partial x}=1+\frac{y}{x}$, $\phi(x,y)=x+y\ln x+f(y)$
- $\frac{\partial\phi}{\partial y}=\ln x$, $\phi(x,y)=y\ln x+g(x)$

COMPARING: $g(x)=x$, $f(y)=$ CONSTANT.

SO $d\phi=0$, $\phi(x,y)=C$, $x+y\ln x=C$

Question 9 **(***+)**

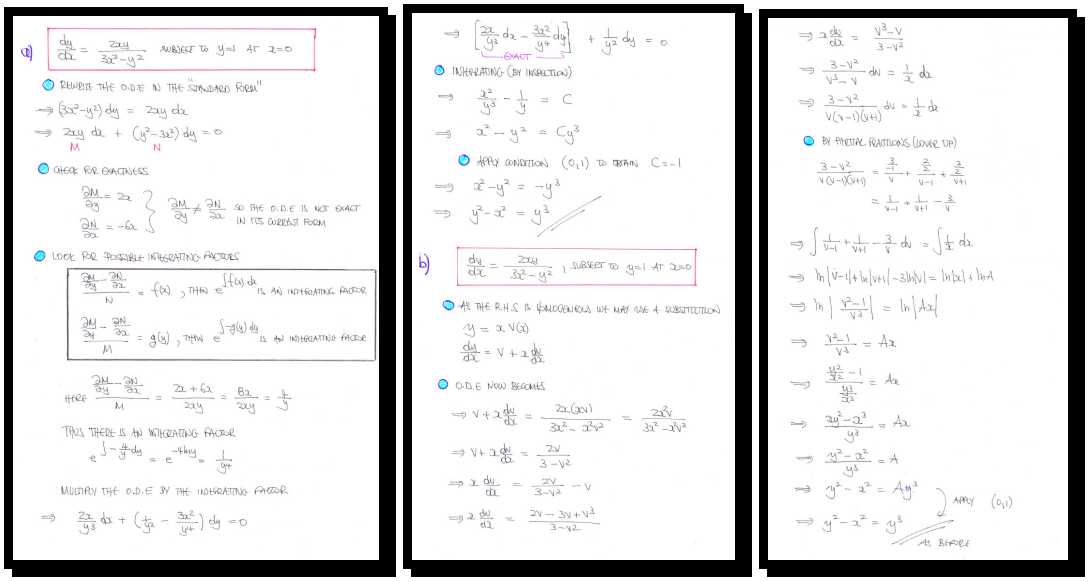
$$\frac{dy}{dx}=\frac{2xy}{3x^2-y^2},\ y(0)=1.$$

a) Find an integrating factor for the above differential equation and hence show

$$y^3=y^2-x^2.$$

b) Verify the answer of part **(a)** by a solving the differential equation by a suitable substitution.

proof

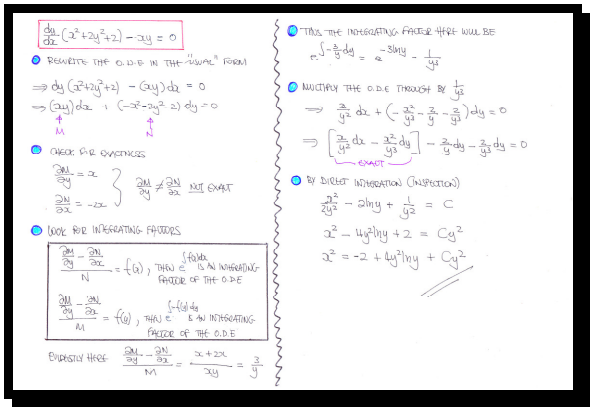


Question 10 (*+)**

Find a general solution of the following differential equation by looking for a suitable integrating factor.

$$\frac{dy}{dx}\left(x^2+2y^2+2\right)-xy=0\,.$$

$$\boxed{x^2=-2+4y^2\ln y+Cy^2}$$

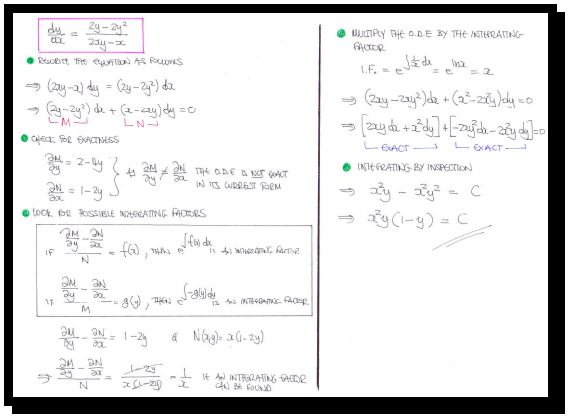


Question 11 (*+)**

Find a general solution of the following differential equation by looking for a suitable integrating factor.

$$\frac{dy}{dx}=\frac{2y-2y^2}{2xy-x}\,.$$

$$\boxed{x^2y(1-y)=C}$$

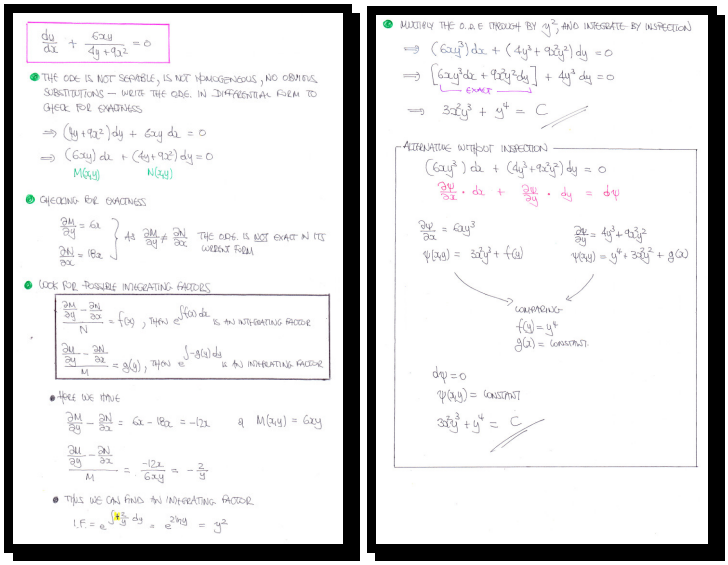


Question 12 (*+)**

Find a general solution of the following differential equation by looking for a suitable integrating factor.

$$\frac{dy}{dx}=\frac{6xy}{4y+9x^2}.$$

$$\boxed{3x^2y^3+y^4=C}$$

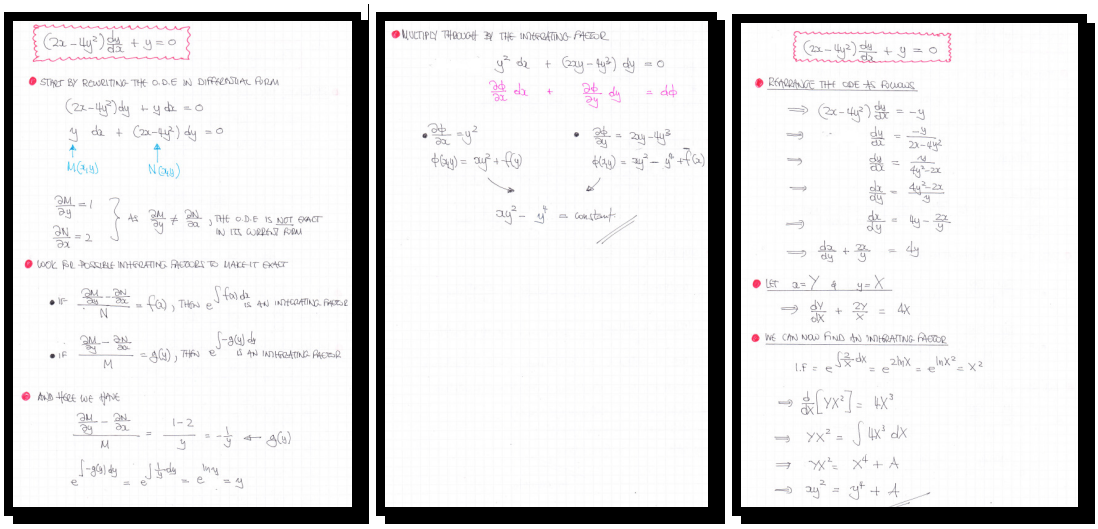


Question 13 **(***+)**

$$\left(2x-4y^2\right)\frac{dy}{dx}+y=0\,.$$

By finding a suitable integrating factor for the above differential equation determine its general solution.

$$\boxed{xy^2-y^4=C}$$

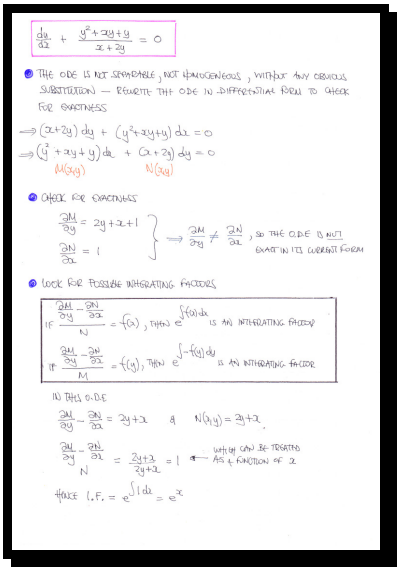


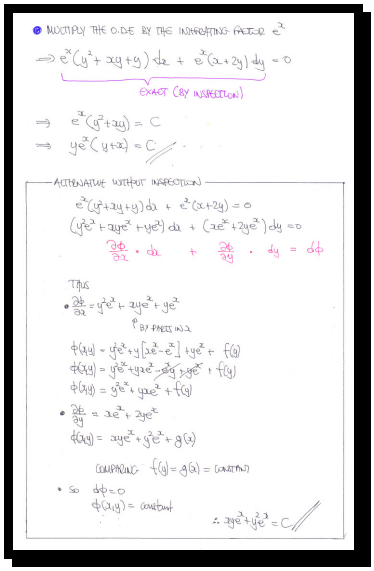
Question 14 (*+)**

Find a general solution of the following differential equation by looking for a suitable integrating factor.

$$\frac{dy}{dx}=\frac{y^2+xy+y}{x+2y}.$$

$$\boxed{y\mathrm{e}^x\left(y+x\right)=C}$$



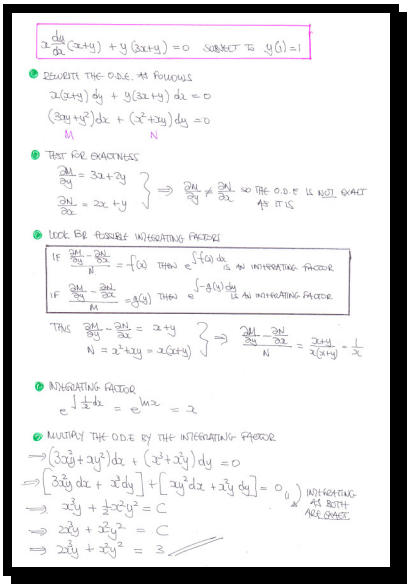


Question 15 (*+)**

Determine the solution of the following differential equation by looking for a suitable integrating factor.

$$x\frac{dy}{dx}(x+y)+y(3x+y)=0, \quad y(1)=1.$$

$$\boxed{2yx^3+x^2y^2=3}$$

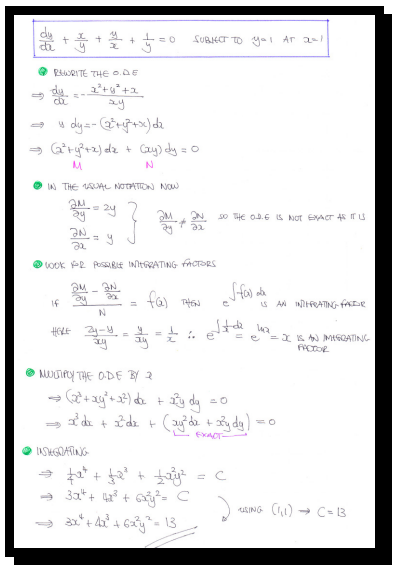


Question 16 (*+)**

Determine the solution of the following differential equation by looking for a suitable integrating factor.

$$\frac{dy}{dx}+\frac{x}{y}+\frac{y}{x}+\frac{1}{y}=0\,,\quad y(1)=1\,.$$

$$\boxed{3x^4+4x^3+6x^2y^2=13}$$

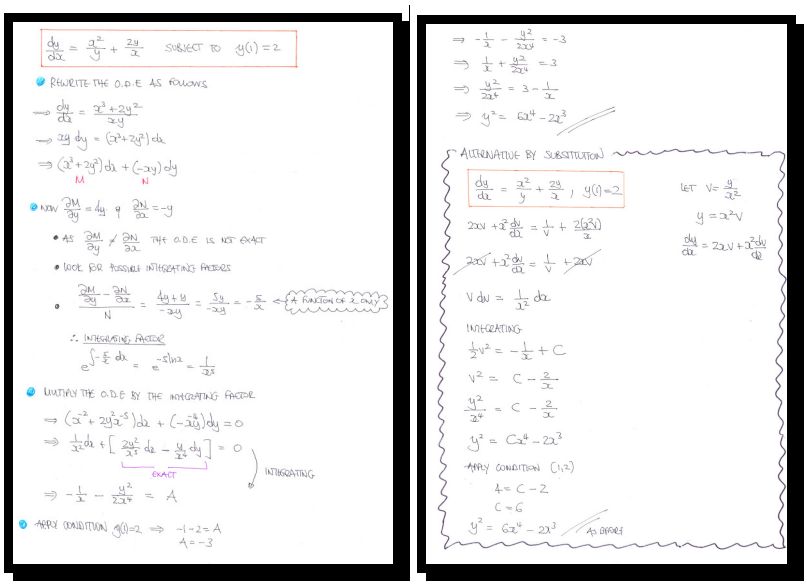


Question 17 (*+)**

Determine the solution of the following differential equation.

$$\frac{dy}{dx}=\frac{x^2}{y}+\frac{2y}{x}, \quad y(1)=2.$$

$$\boxed{y^2=6x^4-2x^3}$$

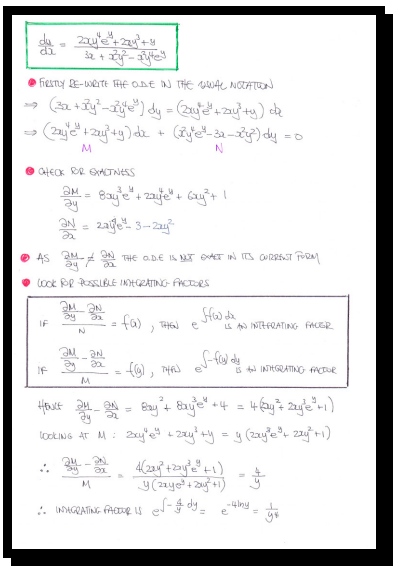


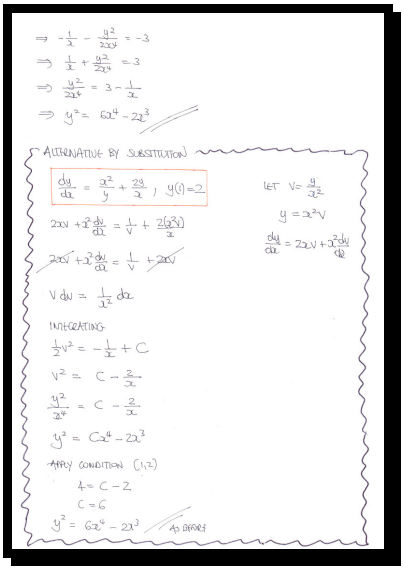
Question 18 (****)

Determine a general solution of the following differential equation by looking for a suitable integrating factor.

$$\frac{dy}{dx}=\frac{2xy^4\,\mathrm{e}^y+2xy^3+y}{3x+x^2y^2-x^2y^4\,\mathrm{e}^y}.$$

$$\boxed{x^2\,\mathrm{e}^y+\frac{x^2}{y}+\frac{x}{y^3}=C}$$





1^{ST} ORDER BY VARIOUS TECHNIQUES

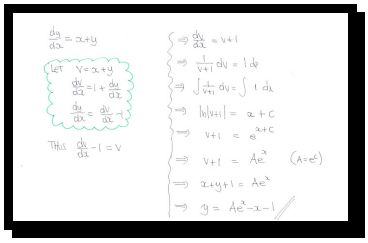
Question 1 (**)

By using a suitable substitution find a general solution of the differential equation

$$\frac{dy}{dx}=x+y\,,$$

giving the answer in the form $y=f\left(x\right)$.

$$\boxed{y=A\mathrm{e}^{x}-x-1}$$



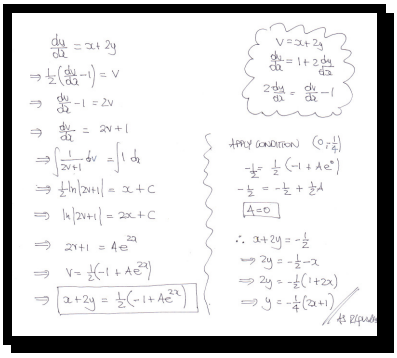
Question 2 (**)

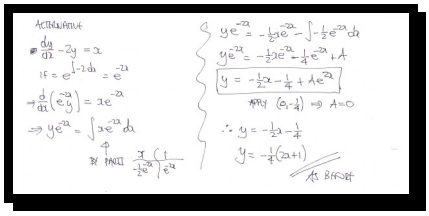
$$\frac{dy}{dx}=x+2y\,,\text{ with }\ y=-\frac{1}{4}\ \text{ at }\ x=0\,.$$

By using a suitable substitution, show that the solution of the differential equation is

$$y=-\frac{1}{4}\left(2x+1\right).$$

proof





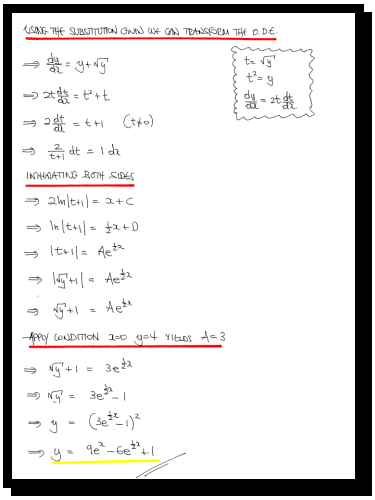
Question 3 (**)

Use the substitution $t=\sqrt{y}$ to solve the following differential equation.

$$\frac{dy}{dx}=y+\sqrt{y}\,,\quad y>0\,,\quad y(0)=4\,.$$

Given the answer in the form $y=f(x)$.

$\boxed{}$, $\boxed{y=9\mathrm{e}^{x}-6\mathrm{e}^{\frac{1}{2}x}+1}$



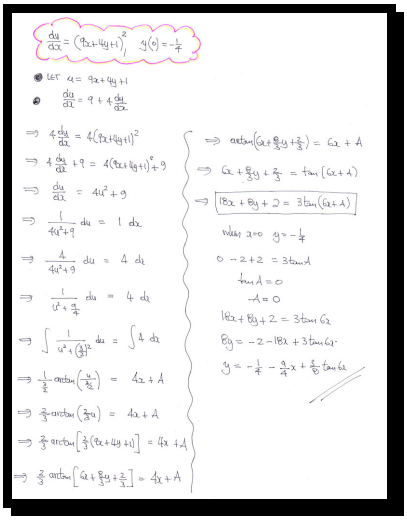
Question 4 (***)

Solve the differential equation

$$\frac{dy}{dx}=(9x+4y+1)^2,\quad y(0)=-\frac{1}{4}.$$

Give the answer in the form $y=f(x)$.

$$\boxed{y=-\frac{1}{4}-\frac{9}{4}x+\frac{3}{8}\tan 6x}$$

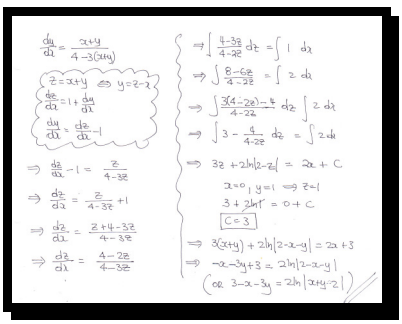


Question 5 (***)

Use a suitable substitution to solve the differential equation

$$\frac{dy}{dx}=\frac{x+y}{4-3(x+y)},\quad y(0)=1.$$

$$\boxed{2\ln|x+y-2|=3-x-3y}$$

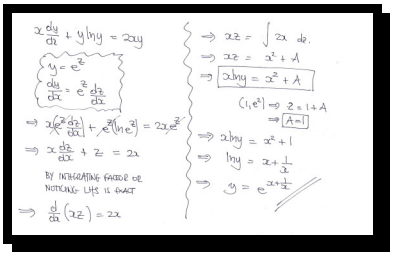


Question 6 (***)

Use the substitution $y = \mathrm{e}^z$ to solve the differential equation

$$x\frac{dy}{dx} + y\ln y = 2xy, \quad y(1) = \mathrm{e}^2.$$

$$\boxed{y = \mathrm{e}^{x+\frac{1}{x}}}$$



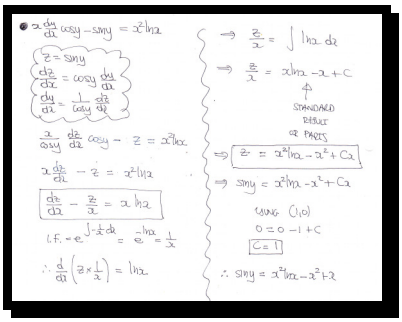
Question 7 (***)

Use the substitution $z = \sin y$ to solve the differential equation

$$x\frac{dy}{dx}\cos y - \sin y = x^2\ln x, \quad y(1) = 0$$

subject to the condition $y = 0$ at $x = 1$.

$$\boxed{\sin y = x^2\ln x - x^2 + x}$$

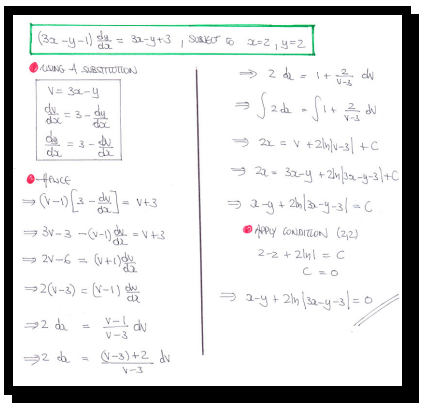


Question 8 **(***+)**

Use a suitable substitution to find the solution of the following differential equation.

$$(3x-y-1)\frac{dy}{dx}=3x-y+3, \quad y(2)=2.$$

$$\boxed{x-y+2\ln|3x-y-3|=0}$$



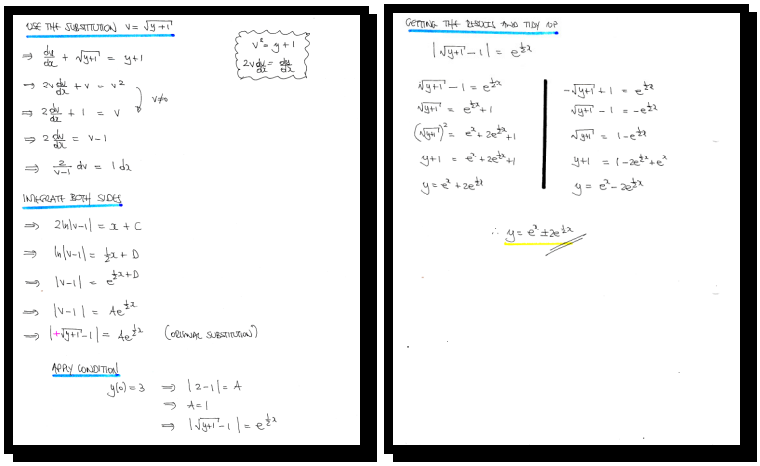
Question 9 **(***+)**

Use a suitable substitution to solve the following differential equation.

$$\frac{dy}{dx}+\sqrt{y+1}=y+1, \quad y>-1, \quad y(0)=3.$$

Given the answer in the form $y=f(x)$.

$\square$, $\boxed{y=e^{x}\pm 2e^{\frac{1}{2}x}}$



Question 10 (*+)**

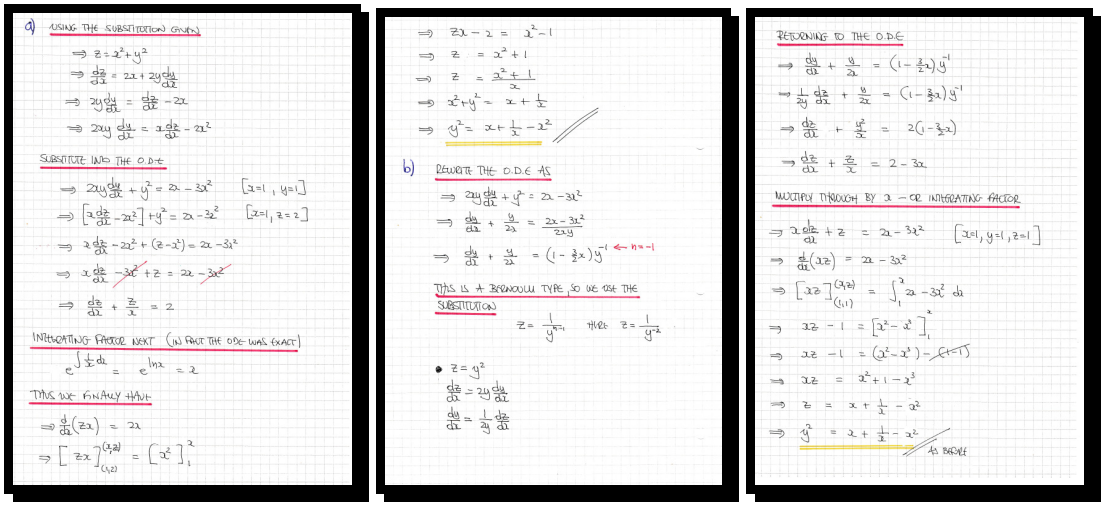
a) By using the substitution $z = x^2 + y^2$, solve the following differential equation

$$2xy\frac{dy}{dx} + y^2 = 2x - 3x^2,$$

subject to the condition $y = 1$ at $x = 1$.

b) Verify the answer to part **(a)** by using the substitution $z = y^2$ to solve the same differential equation and subject to the same condition.

$\square$, $\boxed{y^2 = x - x^2 + \frac{1}{x}}$



Question 11 (*+)**

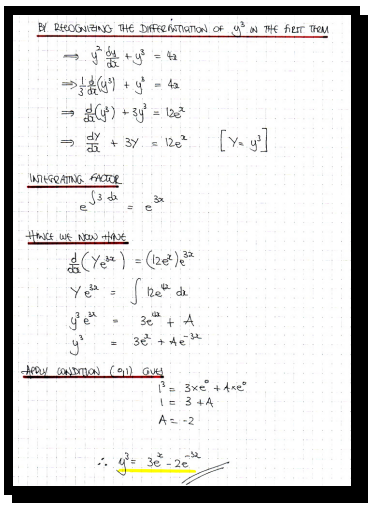
A curve with equation $y=f(x)$ passes through the point with coordinates $(0,1)$ and satisfies the differential equation

$$y^2\frac{dy}{dx}+y^3=4\mathrm{e}^x.$$

By finding a suitable integrating factor, solve the differential equation to show that

$$y^3=3\mathrm{e}^x-2\mathrm{e}^{-3x}.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 12 (*+)**

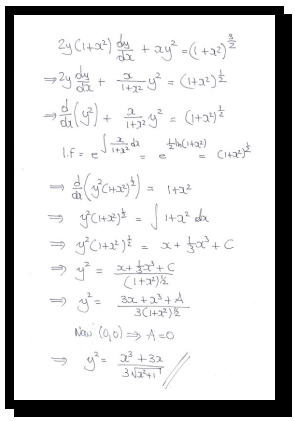
A curve with equation $y=f(x)$ passes through the origin and satisfies the differential equation

$$2y\left(1+x^2\right)\frac{dy}{dx}+xy^2=\left(1+x^2\right)^{\frac{3}{2}}.$$

By finding a suitable integrating factor, or otherwise, show that

$$y^2=\frac{x^3+3x}{3\sqrt{x^2+1}}.$$

[ ], [proof]



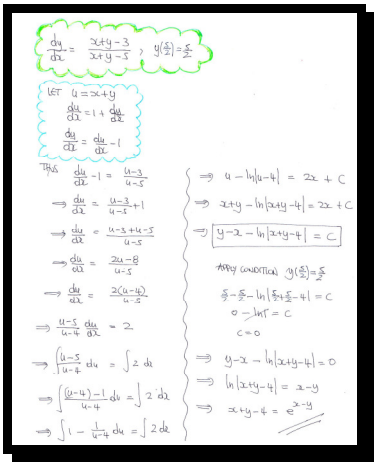
Question 13 (****)

Solve the differential equation

$$\frac{dy}{dx}=\frac{x+y-3}{x+y-5},$$

subject to the condition $y=\frac{5}{2}$ at $x=\frac{5}{2}$.

$$\boxed{x+y-4=\mathrm{e}^{x-y}}$$

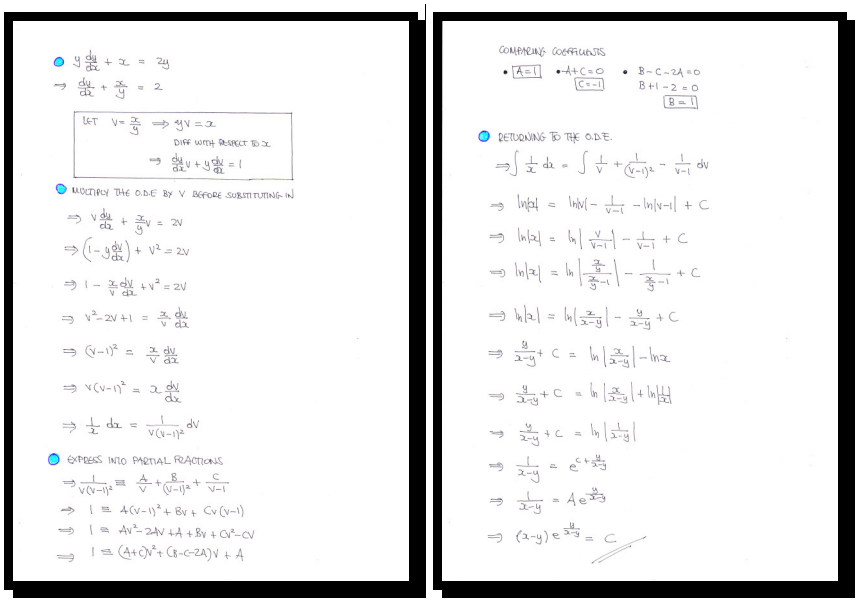


Question 14 (****)

Find a general solution of the following differential equation

$$y\frac{dy}{dx}+x=2y\,.$$

$$\boxed{(x-y)\mathrm{e}^{\frac{y}{x-y}}=C}$$



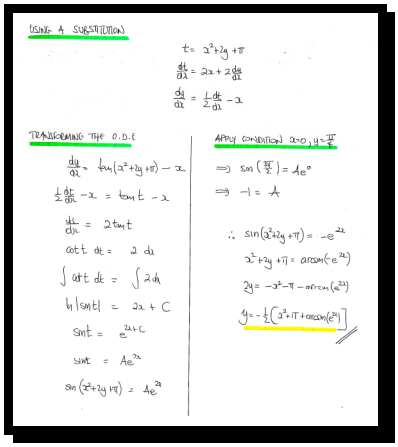
Question 15 **(****)**

$$\frac{dy}{dx}=\tan\left(x^2+2y+\pi\right)\ -\ x\,,\quad y(0)=\frac{1}{4}\pi\ .$$

Solve the above differential equation to show that

$$y=-\frac{1}{2}\left[x^2+\pi+\arcsin\left(\mathrm{e}^{2x}\right)\right].$$

[ ], [proof]



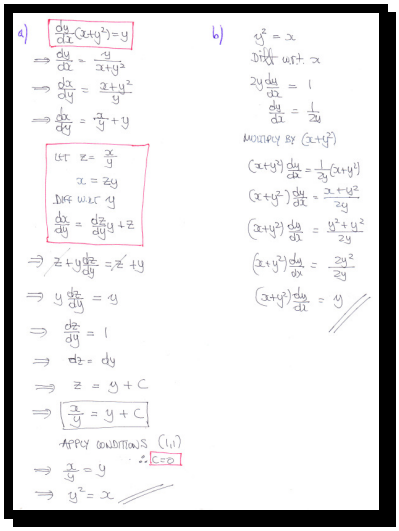
Question 16 **(****)**

$$\frac{dy}{dx}\left(x+y^2\right)=y\,.$$

a) Solve the above differential equation, subject to $y=1$ at $x=1$ by considering $\dfrac{dx}{dy}$, followed by a suitable substitution..

b) Verify the validity of the answer obtained in part **(a)**.

$$\boxed{y^2=x}$$



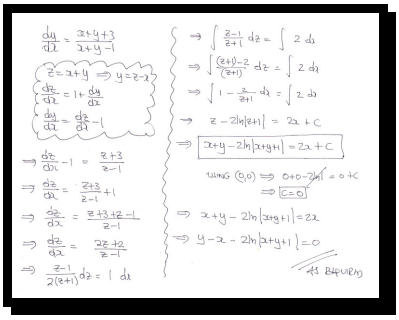
Question 17 (****)

$$\frac{dy}{dx}=\frac{x+y+3}{x+y-1},\ y(0)=0.$$

Show that the solution of the above differential equation is

$$y-x-2\ln(x+y+1)=0.$$

proof



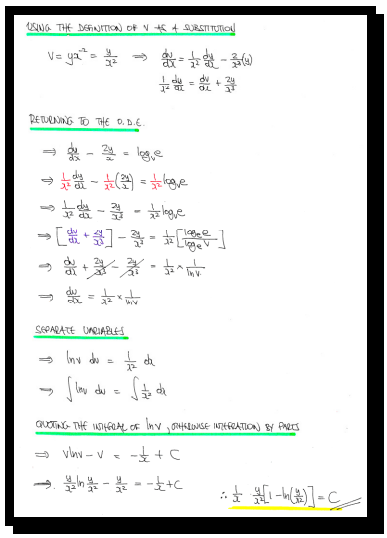
Question 18 **(****)**

Given that $v = yx^{-2}$ find a general solution for the following differential equation.

$$\frac{dy}{dx} - \frac{2y}{x} = \log_v \mathrm{e}, \quad u > 0, \quad u \neq 1.$$

Given the answer in the form $f(x, y) = \text{constant}$.

$\square$, $\boxed{\dfrac{1}{x} - \dfrac{y}{x^2}\left[1 - \ln\left(yx^{-2}\right)\right] = \text{constant}}$



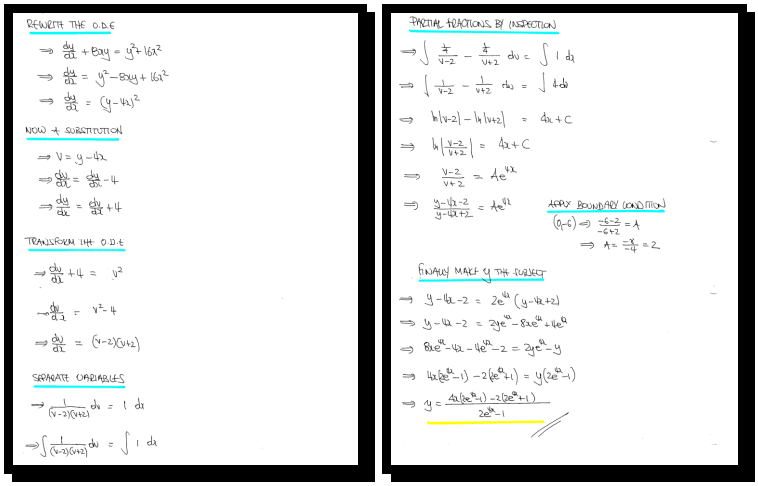
Question 19 **(****)**

Use a suitable substitution to solve the following differential equation.

$$\frac{dy}{dx}+8xy=y^2+16x^2\,,\quad y(0)=-6\,.$$

Given the answer in the form $y=f(x)$.

$$\boxed{}\,,\quad \boxed{y=\frac{4x\left(2\mathrm{e}^{4x}-1\right)-2\left(2\mathrm{e}^{4x}+1\right)}{2\mathrm{e}^{4x}-1}}$$

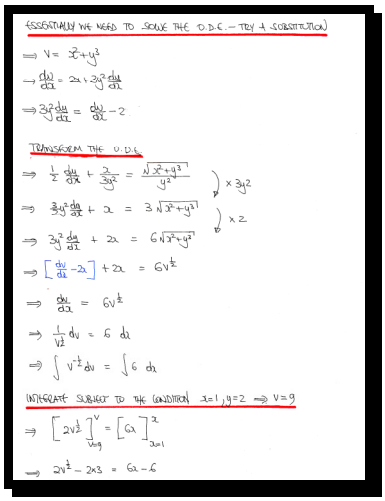


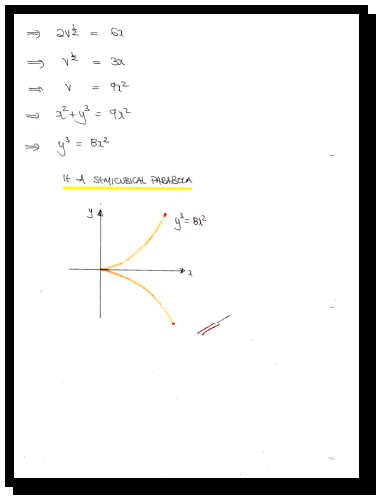
Question 20 **(****)**

Sketch the curve which passes through the point with coordinates $(1,2)$ and satisfies

$$\frac{1}{2}\frac{dy}{dx}+\frac{x}{3y^2}=\frac{\sqrt{x^2+y^3}}{y^2}.$$

[ ], graph





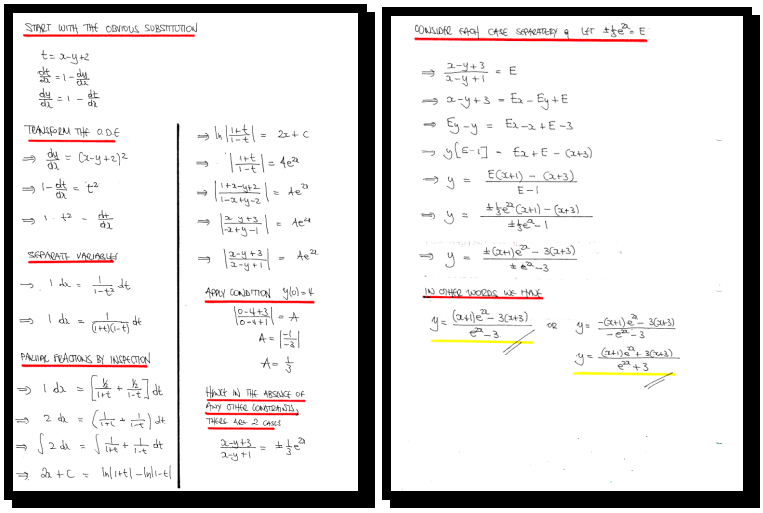
Question 21 (****)

Use a suitable substitution to solve the following differential equation.

$$\frac{dy}{dx}=(x-y+2)^2, \quad y(0)=4.$$

Given the answer in the form $y=f(x)$.

$\square$, $\boxed{y=\dfrac{(x+1)e^{2x}\pm 3(x+3)}{e^{2x}\pm 3}=\dfrac{\pm(x+1)e^{2x}-3(x+3)}{\pm e^{2x}-3}}$

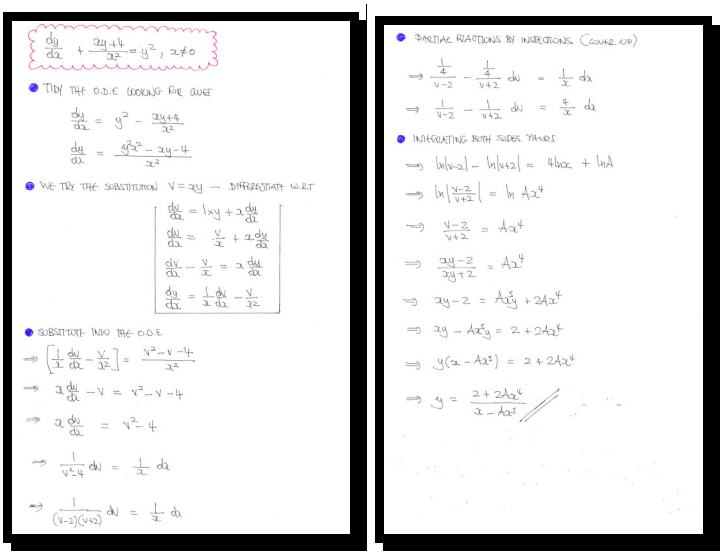


Question 22 (**+)**

$$\frac{d^2y}{dx^2}+\frac{xy+4}{x^2}=y^2\,,\ x\neq 0\,.$$

Find a general solution for the above differential equation.

$$\boxed{y=\frac{2+2Ax^4}{x-Ax^5}}$$



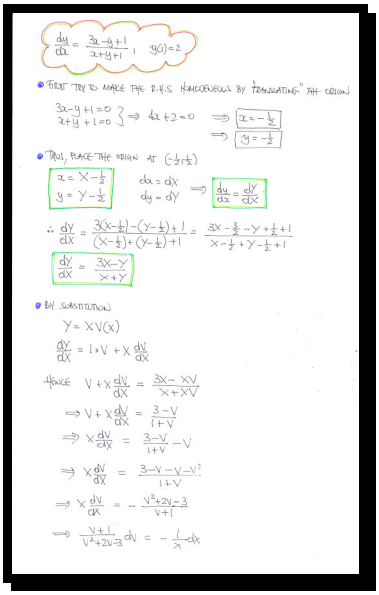
Question 23 (**+)**

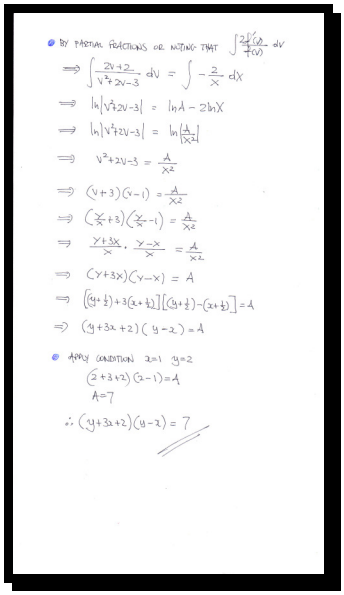
$$\frac{dy}{dx}=\frac{3x-y+1}{x+y+1},\ y(1)=2.$$

Solve the differential equation to show that

$$(y-x)(y+3x+2)=7.$$

proof





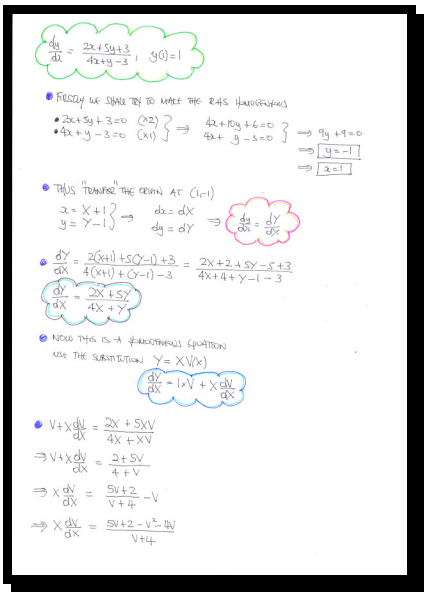
Question 24 **(****+)**

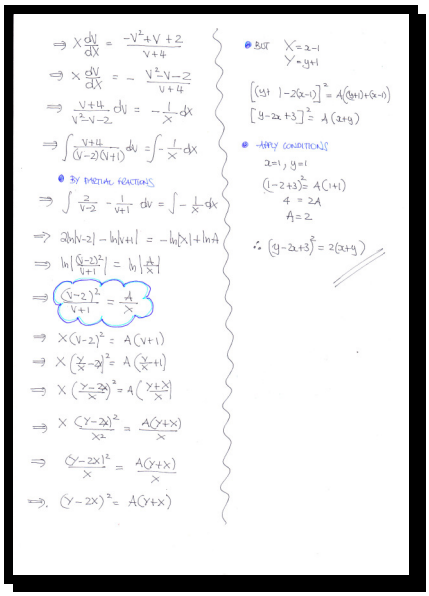
$$\frac{dy}{dx}=\frac{2x+5y+3}{4x+y-3},\ y(1)=1.$$

Solve the differential equation to show that

$$(y-2x+3)^2=2(x+y).$$

proof





Question 25 (**+)**

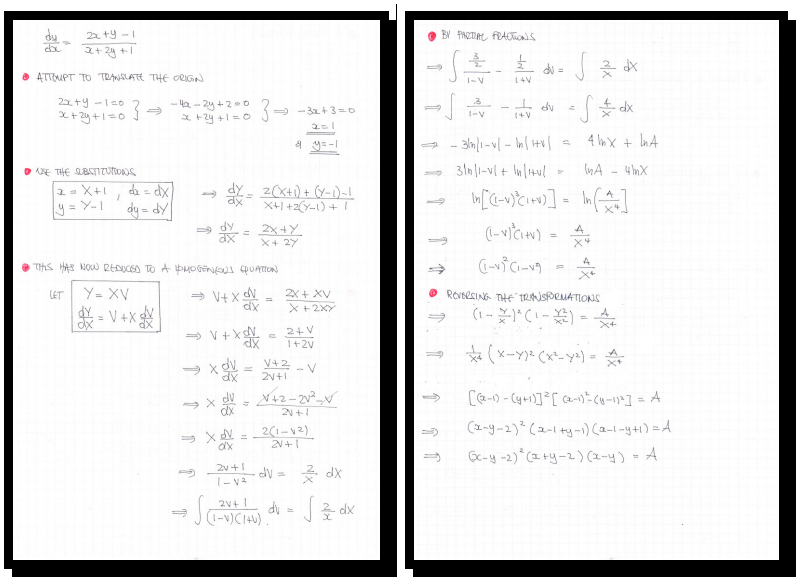
Solve the following differential equation

$$\frac{dy}{dx}=\frac{2x+y-1}{x+2y+1},$$

to show that

$$(x-y)(x+y-2)(x-y-2)^2=\text{constant}.$$

proof



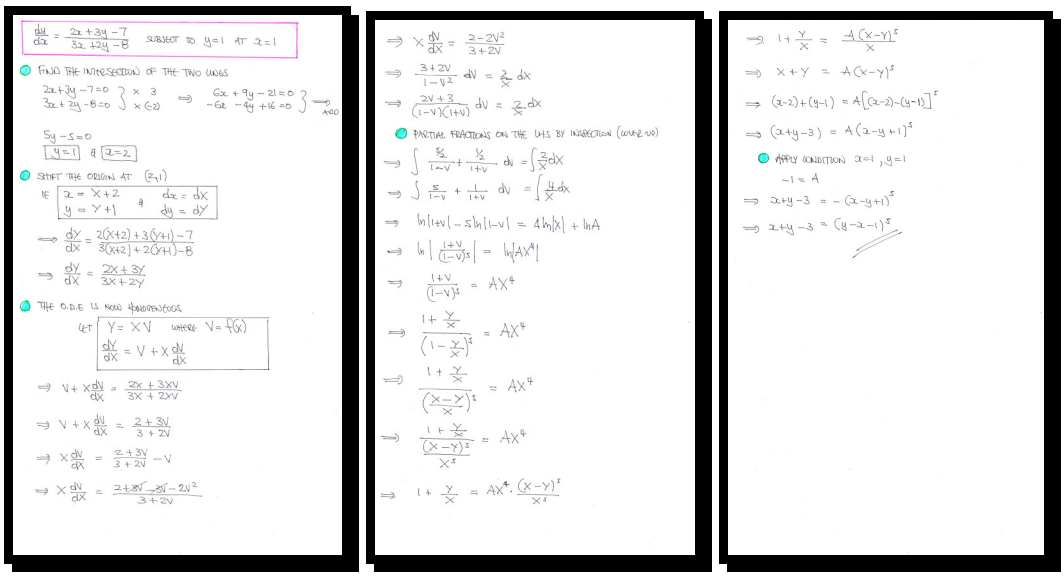
Question 26 (**+)**

Solve the following differential equation

$$\frac{dy}{dx}=\frac{2x+3y-7}{3x+2y-8},\quad y(1)=1$$

Give the answer in the form $(y-x-1)^5=f(x,y)$, where $f(x,y)$ is a function to be found.

$$\boxed{(y-x-1)^5=y+x-3}$$



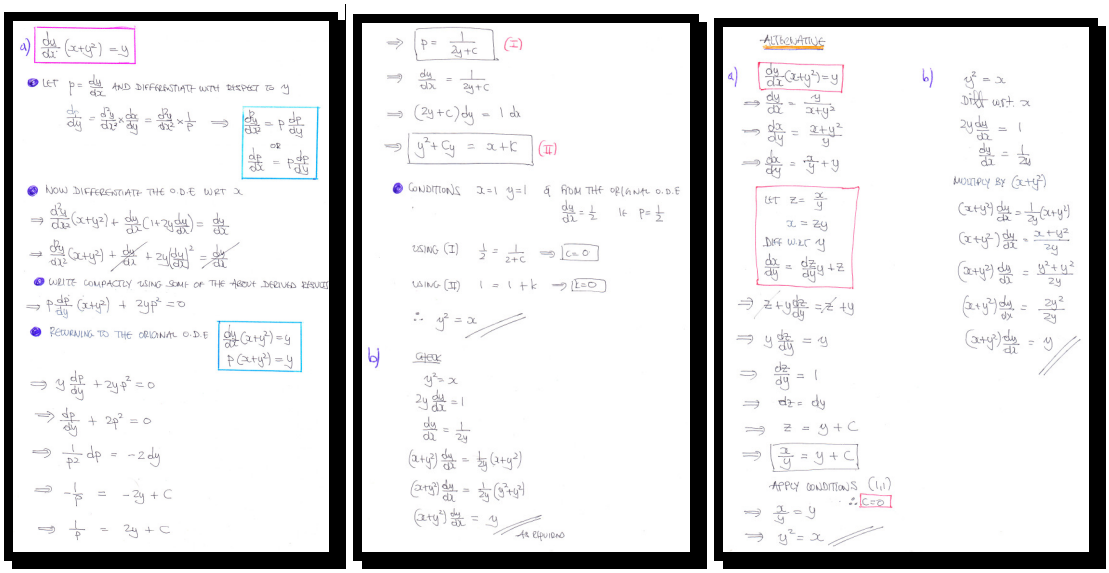
Question 27 **(****+)**

$$\frac{dy}{dx}\left(x+y^2\right)=y\,.$$

a) Solve the above differential equation, subject to $y=1$ at $x=1$.

b) Verify the validity of the answer obtained in part **(a)**.

$$\boxed{y^2=x}$$

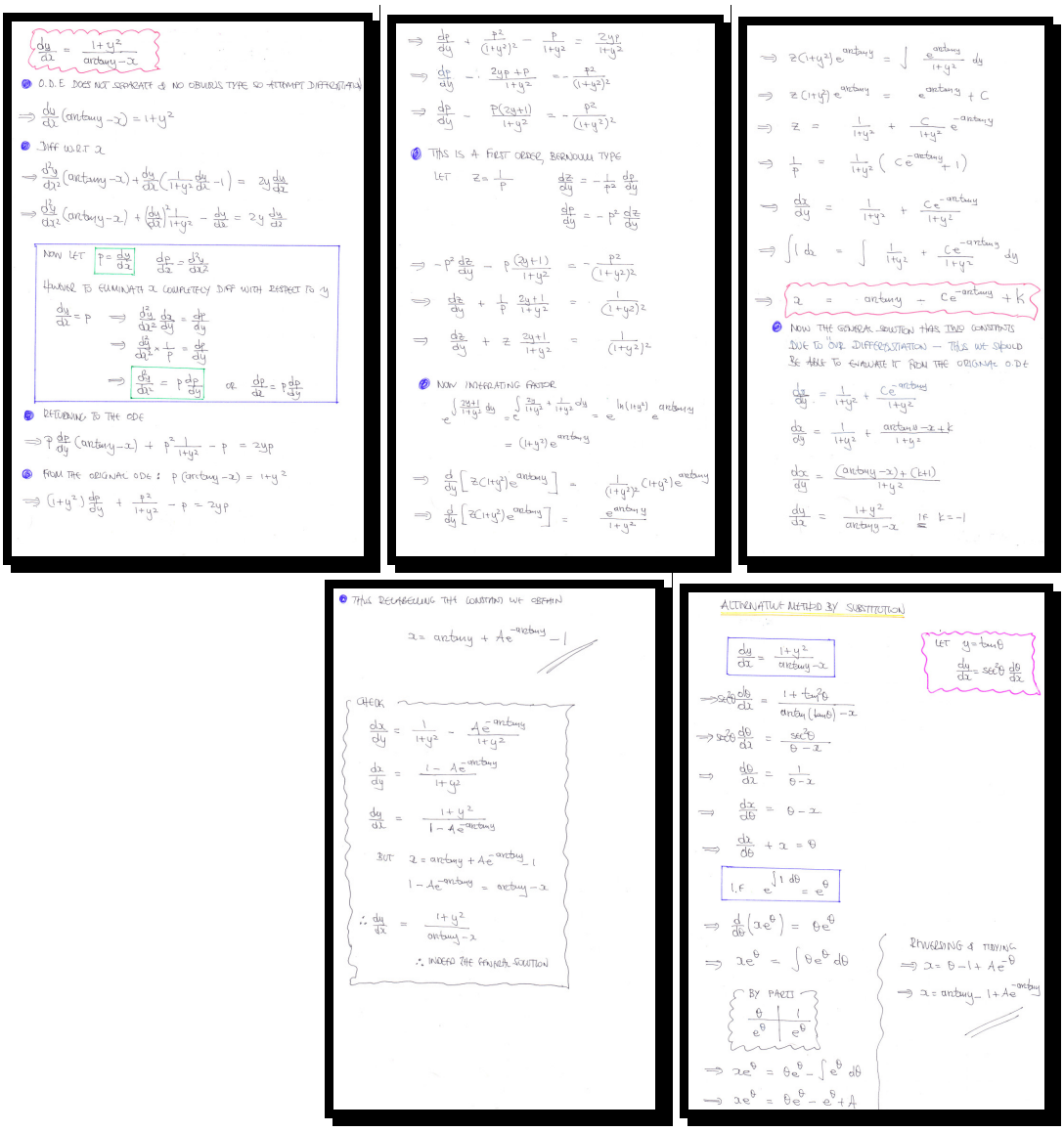


Question 28 (**+)**

Find a general solution for the following differential equation

$$\frac{dy}{dx}=\frac{1+y^2}{-x+\arctan y}.$$

$$\boxed{x=-1+\arctan y+A\mathrm{e}^{-\arctan y}}$$

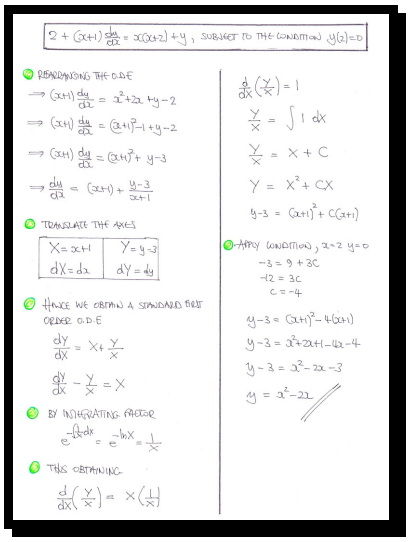


Question 29 (**+)**

$$2+(x+1)\frac{dy}{dx}=x(x+2)+y\,.$$

Solve the above differential equation, subject to $y(2)=0$.

$$\boxed{y=x^2-2x}$$



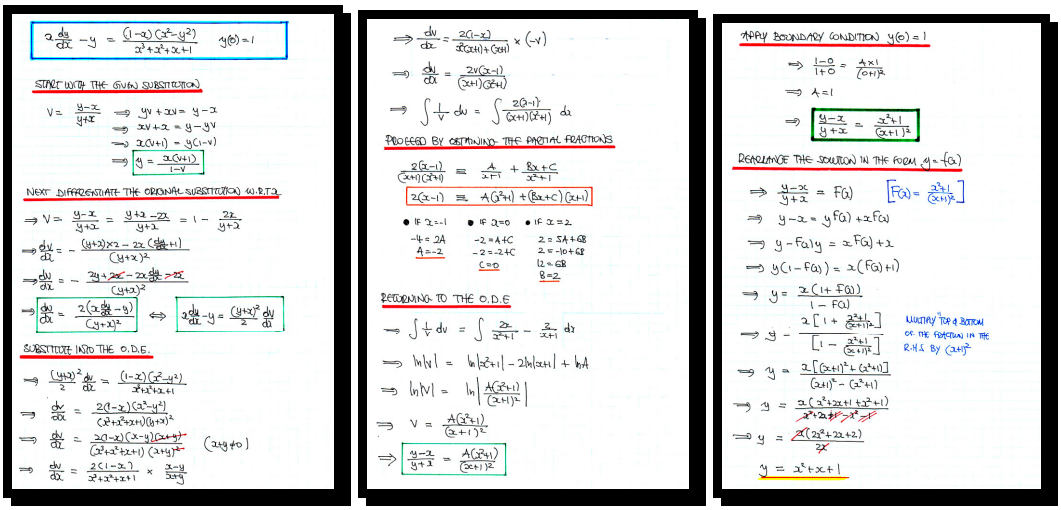
Question 30 (**+)**

Use the substitution $v=\dfrac{y-x}{y+x}$, $y+x\neq 0$, to solve the following differential equation

$$x\frac{dy}{dx}-y=\frac{(1-x)\left(x^2-y^2\right)}{x^3+x^2+x+1},\quad y(0)=1.$$

Give the answer in the form $y=f(x)$.

$\square$, $\boxed{y=x^2+x+1}$



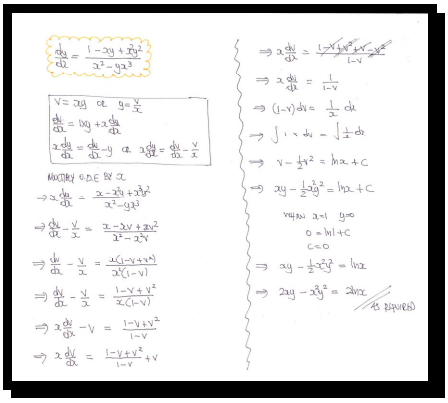
Question 31 (*****)

Solve the differential equation

$$\frac{dy}{dx}=\frac{1-xy+x^2y^2}{x^2-yx^3},\ x>0,$$

subject to the condition $y(1)=0$.

$$\boxed{2xy-x^2y^2=2\ln x}$$

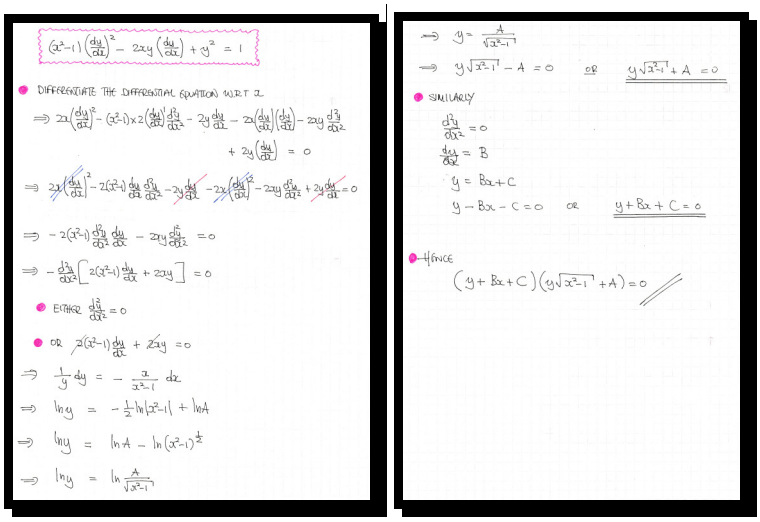


Question 32 (*****)

Find a simplified general solution for the following differential equation.

$$\left(x^2-1\right)\left(\frac{dy}{dx}\right)^2-2xy\left(\frac{dy}{dx}\right)+y^2=1\,.$$

$$\boxed{\left(A+y\sqrt{x^2-1}\right)\left(y+Bx+C\right)=0}$$

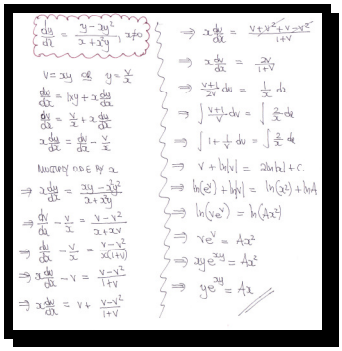


Question 33 (*****)

Find a general solution for the differential equation

$$\frac{dy}{dx}=\frac{y-xy^2}{x+yx^2}\,,\ x\neq 0\,.$$

$$\boxed{y\,\mathrm{e}^{xy}=Cx}$$



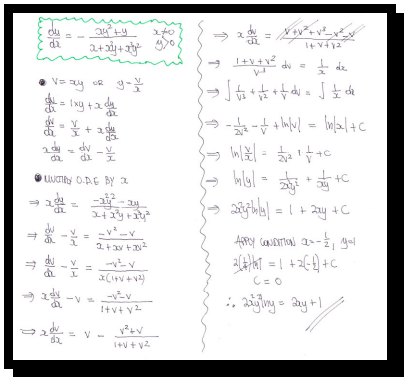
Question 34 **(*****)**

Solve the differential equation

$$\frac{dy}{dx}=-\frac{xy^2+y}{x+yx^2+x^3y^2},\ x\neq 0,\ y>0,$$

subject to the condition $y\left(\frac{1}{2}\right)=1$.

$$\boxed{2x^2y^2\ln y=2xy+1}$$



Question 35 **(*****)**

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

a) Show, with a detailed method, that $F(x) = f(\phi) x^{g(\phi)}$ is a solution of the differential equation,

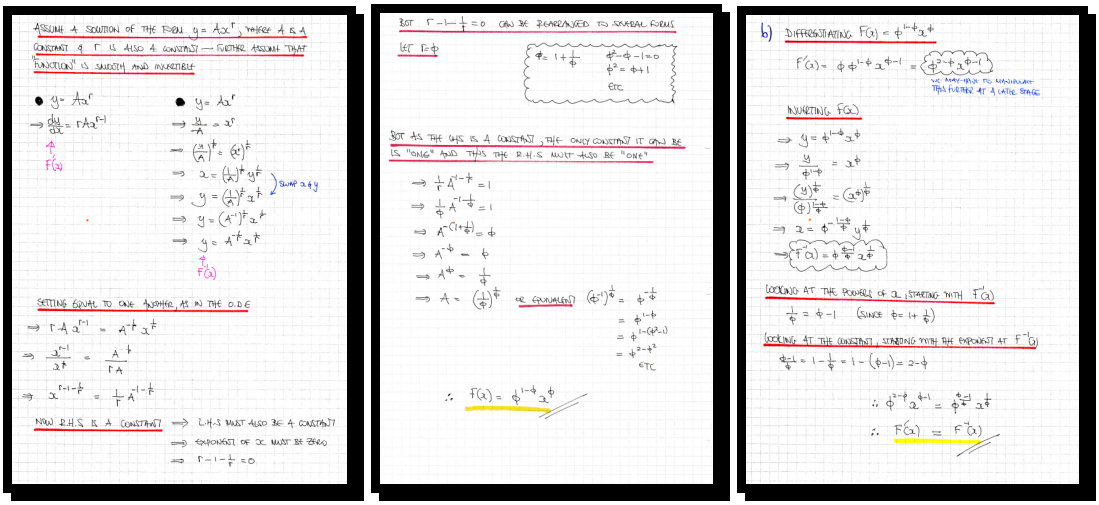
$$F'(x) = F^{-1}(x),$$

where f and g are constant expressions of ϕ, to be found in simplified form.

b) Verify the answer obtained in part **(a)** satisfies the differential equation, by differentiation and function inversion.

[You may assume that $F(x)$ is differentiable and invertible]

V, , $\boxed{F(x) = \left(\frac{1}{\phi}\right)^{\frac{1}{\phi}} x^{\phi} = \phi^{1-\phi} x^{\phi}}$



Created by T. Madas