O.D.E.s

SERIES SOLUTIONS

LEIBNIZ METHOD

Leibniz Theorem

If $y = u(x)v(x)$ then

$$y_n = \sum_{r=1}^{n} \binom{n}{r} u_r v_{n-r} = u_n + nu_{n-1}v_1 + \frac{n(n-1)}{2!}u_{n-2}v_2 + \frac{n(n-1)(n-2)}{3!}u_{n-3}v_3 + \ldots \ ,$$

where $u_m = \dfrac{d^m u}{dx^m}$ and $v_m = \dfrac{d^m v}{dx^m}$.

nth order differential coefficients

$$\frac{d^n}{dx^n}\left(x^a\right) = y_n = \frac{a!}{(a-n)!}a^{a-n}$$

$$\frac{d^n}{dx^n}\left(\mathrm{e}^{ax}\right) = y_n = a^n \mathrm{e}^{ax}$$

$$\frac{d^n}{dx^n}(\sin ax) = y_n = a^n \sin\left[ax + \frac{n\pi}{2}\right]$$

$$\frac{d^n}{dx^n}(\cos ax) = y_n = a^n \cos\left[ax + \frac{n\pi}{2}\right]$$

$$\frac{d^n}{dx^n}(\sinh ax) = y_n = \frac{1}{2}a^n\left[\left[1-(-1)^n\right]\sinh ax + \left[1+(-1)^n\right]\cosh ax\right]$$

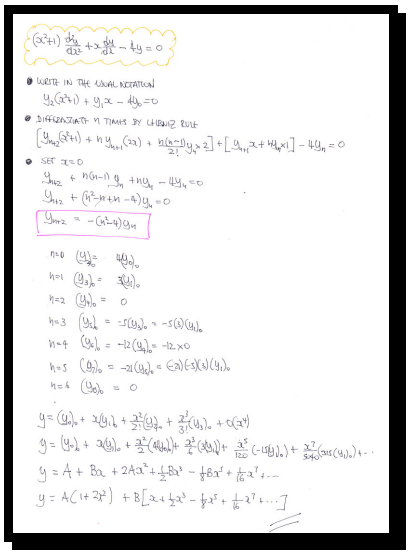
$$\frac{d^n}{dx^n}(\cosh ax) = y_n = \frac{1}{2}a^n\left[\left[1+(-1)^n\right]\sinh ax + \left[1-(-1)^n\right]\cosh ax\right]$$

Question 1

Use the Leibniz rule to find a general solution, as an infinite series, for the following differential equation

$$\left(1+x^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0\,.$$

$$\boxed{y=A\left(1+2x^2\right)+Bx\left(1+\frac{1}{2}x^2-\frac{1}{8}x^4+\frac{1}{16}x^6+...\right)}$$

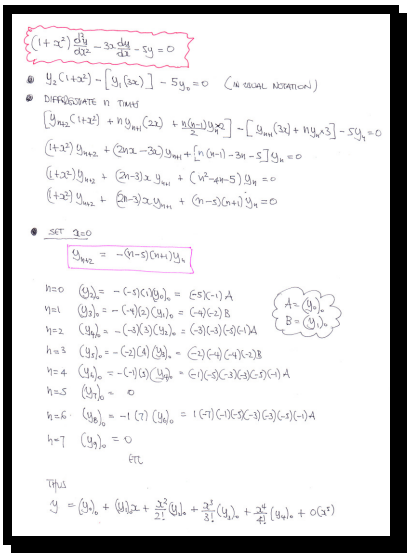


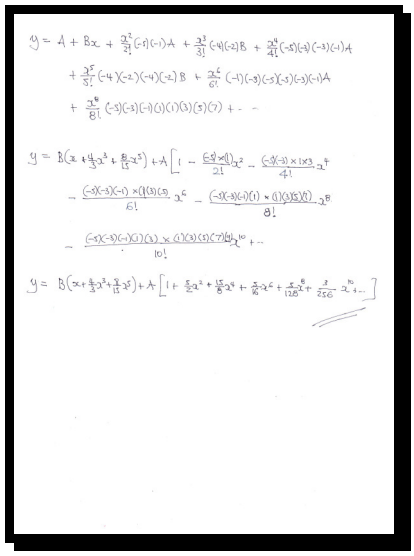
Question 2

Use the Leibniz rule to find a general solution, as an infinite series, for the following differential equation

$$\left(1+x^2\right)\frac{d^2y}{dx^2}-3\frac{dy}{dx}-5y=0.$$

$$\boxed{y=A\left(1+\frac{5}{2}x^2+\frac{15}{8}x^4+\frac{5}{16}x^6+\frac{5}{128}x^8+\frac{3}{256}x^{10}+\ldots\right)+Bx\left(1+\frac{4}{3}x^2+\frac{8}{15}x^4\right)}$$





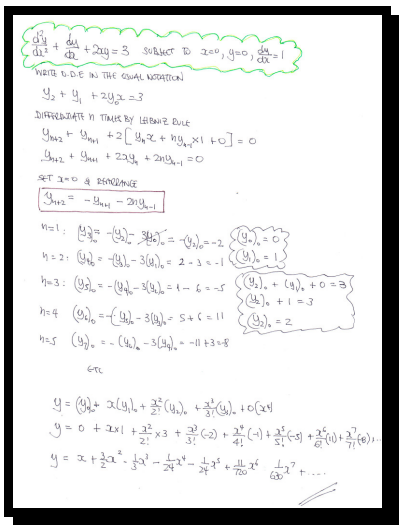
Question 3

Use the Leibniz rule to find a general solution, as an infinite series, for the following differential equation

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+2xy=3,$$

subject to the boundary conditions $y=0,\ \frac{dy}{dx}=1$ at $x=0$.

$$\boxed{y=x+\frac{3}{2}x^2-\frac{1}{3}x^3-\frac{1}{24}x^4-\frac{1}{24}x^5+\frac{11}{720}x^6-\frac{1}{630}x^7\ldots}$$



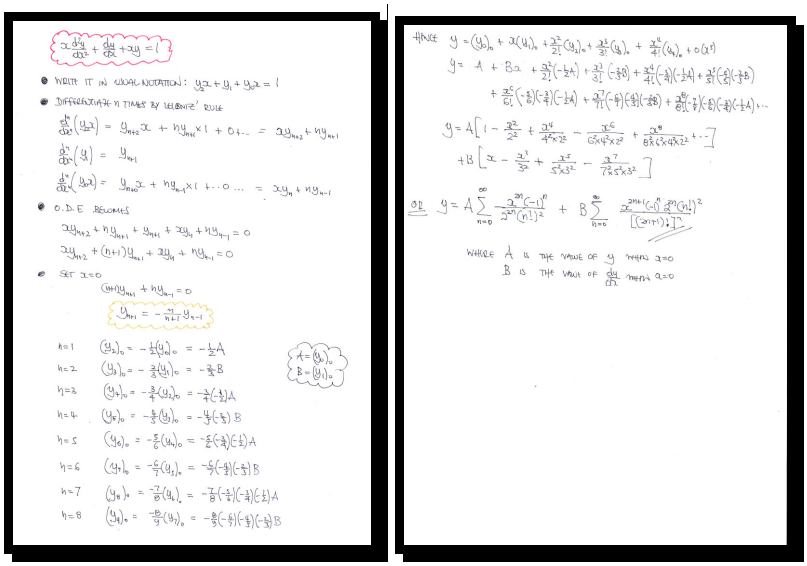
Question 4

Use the Leibniz rule to find a general solution, as an infinite series, for the following differential equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=1.$$

Give the final answer in simplified Sigma notation.

$$y=A\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2^{2n}(n!)^2}x^{2n}\right]+B\sum_{n=0}^{\infty}\left[\frac{(-1)^n\,2^{2n}(n!)^2}{[(2n+1)!]^2}x^{2n+1}\right]$$



Question 5

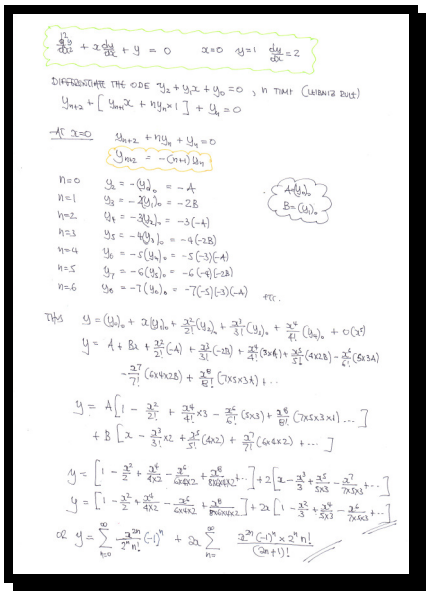
Use the Leibniz rule to find a general solution, as an infinite series, for the following differential equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=1,$$

subject to the boundary conditions $y=1,\ \dfrac{dy}{dx}=2$ at $x=0$

Give the final answer in simplified Sigma notation.

$$y=\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2^n n!}x^{2n}\right]+2x\sum_{n=0}^{\infty}\left[\frac{(-2)^n n!}{(2n+1)!}x^{2n}\right]$$



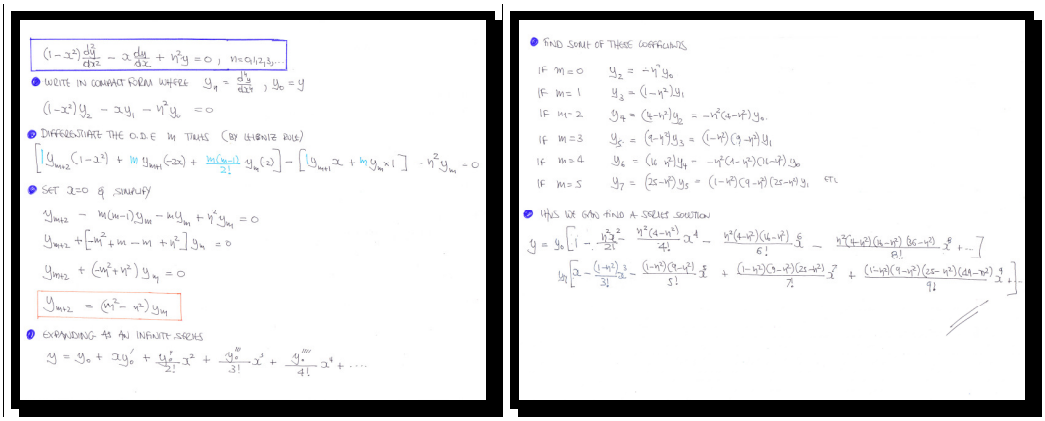
Question 6

Chebyshev's equation is shown below

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0\,,\ n=0,\,1,\,2,\,3,\,...$$

Find a series solution for Chebyshev's equation, by using the Leibniz method

$$y=$$
$$A\left[x-\frac{\left(1-n^2\right)}{3!}x^3-\frac{\left(1-n^2\right)\left(9-n^2\right)}{5!}x^5-\frac{\left(1-n^2\right)\left(9-n^2\right)\left(25-n^2\right)}{7!}x^7-...\right]$$
$$+$$
$$B\left[1-\frac{n^2}{2!}x^2-\frac{n^2\left(4-n^2\right)}{4!}x^4-\frac{n^2\left(4-n^2\right)\left(16-n^2\right)}{6!}x^6-\frac{n^2\left(4-n^2\right)\left(16-n^2\right)\left(36-n^2\right)}{8!}x^8-...\right]$$



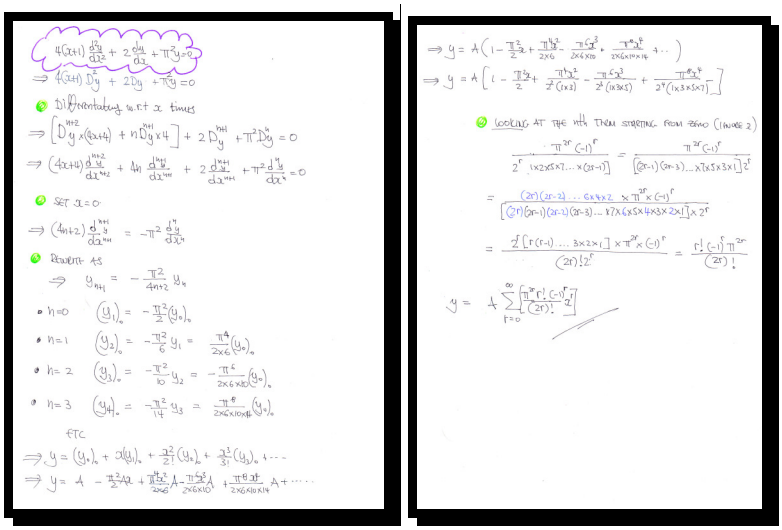
Question 7

Use Leibniz rule to find a solution, as an infinite series, for the following differential equation

$$4(x+1)\frac{d^2y}{dx^2}+2\frac{dy}{dx}+\pi^2 y=0\,.$$

Give the final answer in simplified Sigma notation.

$$y=A\sum_{r=0}^{\infty}\left[\frac{(-1)^r\, r!\,\pi^{2r}x^r}{(2r)!}\right]$$



FROBENIUS METHOD

[analytic at $x = 0$]

Question 1

$$(x+1)\frac{dy}{dx}-(x+2)y=0,\ \ y(0)=1.$$

a) Find the solution of the above differential equation, by separation of variables.

b) Show that the solution can be written as

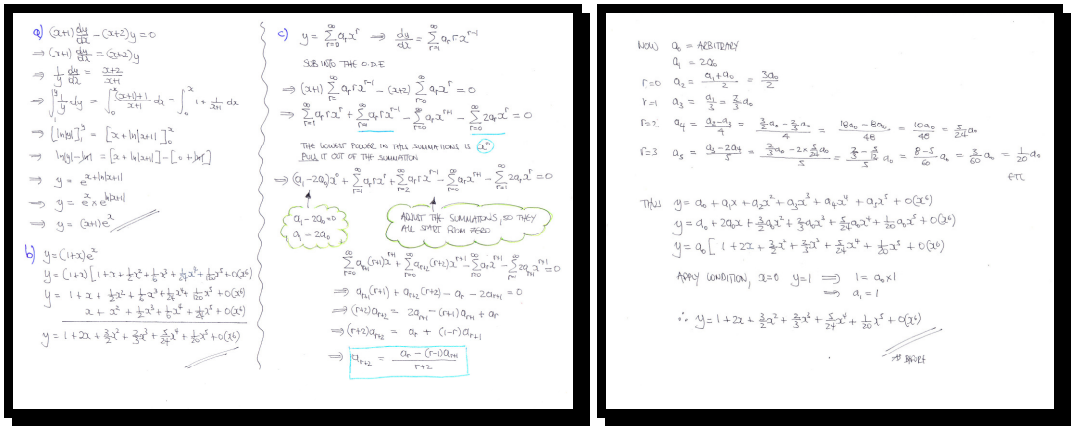
$$y=1+2x+\frac{3}{2}x^2+\frac{2}{3}x^3+\frac{5}{24}x^4+\frac{1}{20}x^5+O\left(x^6\right).$$

c) Assuming a solution of the form

$$y=\sum_{r=1}^{\infty}a_r x^r,$$

use the Frobenius method to verify the answer of part **(b).**

$$\boxed{y=(x+1)\mathrm{e}^x}$$



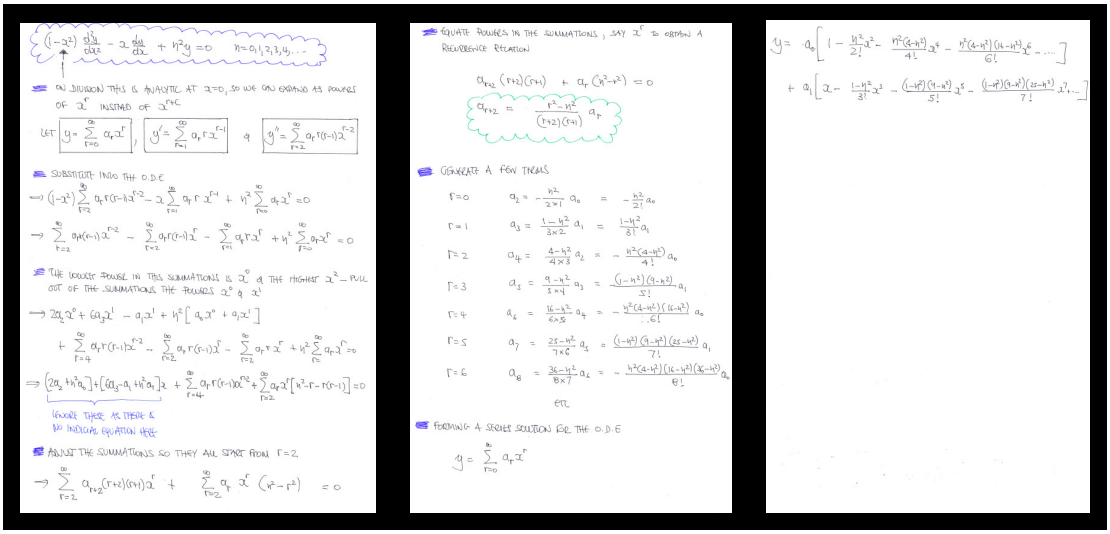
Question 2

Chebyshev's equation is shown below

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0\,,\ n=0,\,1,\,2,\,3,\,...$$

Find a series solution for Chebyshev's equation, by using the Frobenius method.

$$y=$$
$$A\left[x-\frac{\left(1-n^2\right)}{3!}x^3-\frac{\left(1-n^2\right)\left(9-n^2\right)}{5!}x^5-\frac{\left(1-n^2\right)\left(9-n^2\right)\left(25-n^2\right)}{7!}x^7-...\right]$$
$$+$$
$$B\left[1-\frac{n^2}{2!}x^2-\frac{n^2\left(4-n^2\right)}{4!}x^4-\frac{n^2\left(4-n^2\right)\left(16-n^2\right)}{6!}x^6-\frac{n^2\left(4-n^2\right)\left(16-n^2\right)\left(36-n^2\right)}{8!}x^8-...\right]$$



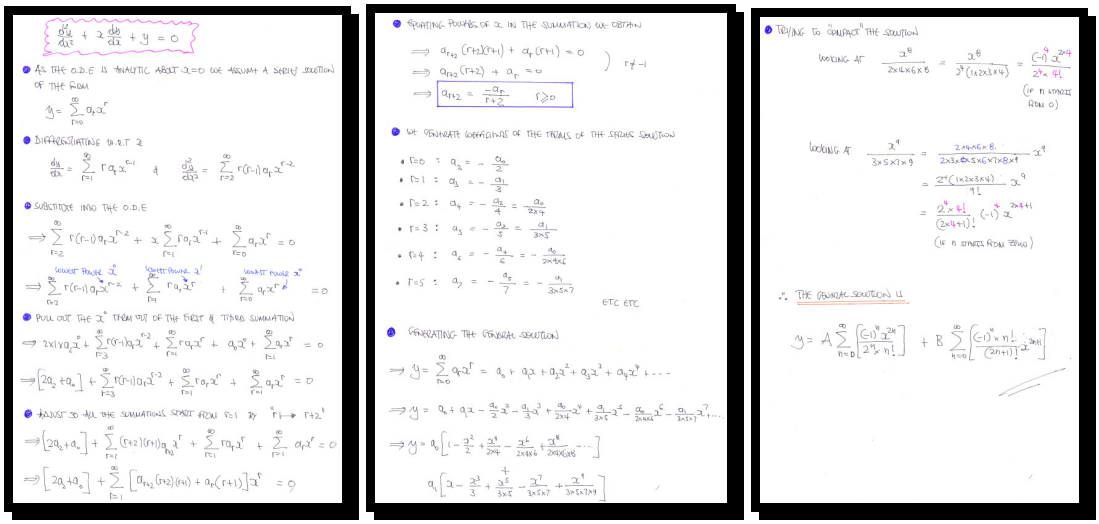
Question 3

Find the two independent solutions of the following differential equation

$$\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0\,.$$

Give the final answer in simplified Sigma notation.

$$\boxed{y=A\sum_{n=0}^{\infty}\left[\frac{(-1)^n\,x^{2n}}{2^n\,n!}\right]+B\sum_{n=0}^{\infty}\left[\frac{(-1)^n\,n!\,x^{2n+1}}{(2n+1)!}\right]}$$



Question 4

Find the two independent solutions of the following differential equation

$$\left(x^2-1\right)\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=0\ ,\quad |x|<1.$$

Give the final answer in simplified form without involving infinite sums.

$\square$, $\boxed{y=\dfrac{A+Bx}{1-x^2}}$



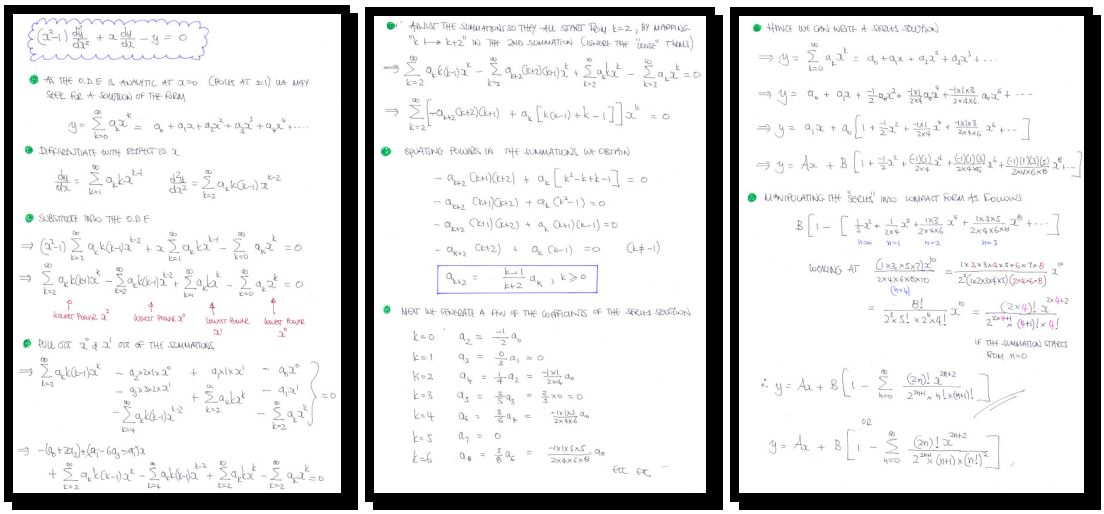
Question 5

Find the two independent solutions of the following differential equation

$$\left(x^2-1\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0\,.$$

Give the final answer in simplified Sigma notation.

$$\boxed{y=Ax+B\left[1-\sum_{n=0}^{\infty}\left[\frac{(2n)!\,x^{2n+2}}{2^{2n+1}\,n!\,(n+1)!}\right]\right]}$$



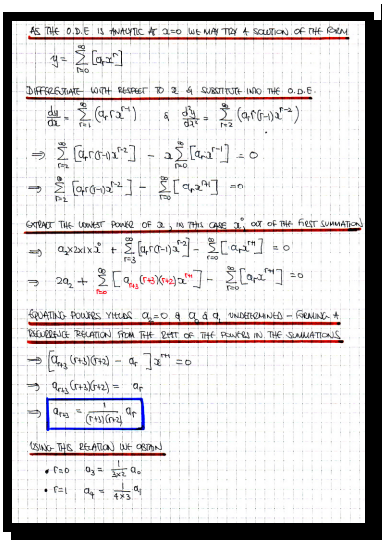
Question 6

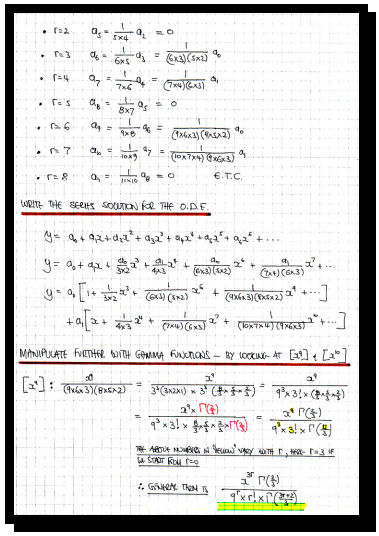
Use the Frobenius method to find a general solution, as an infinite series, for Airy's differential equation

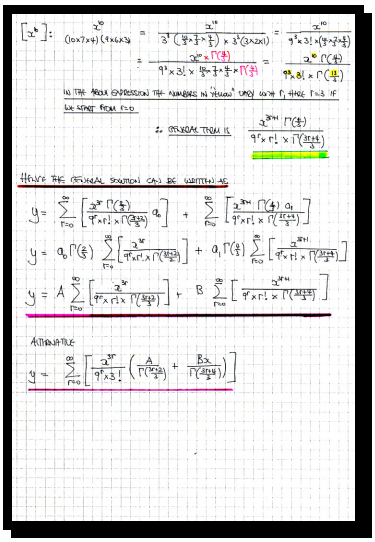
$$\frac{d^2y}{dx^2} - xy = 0 .$$

Give the final answer in simplified Sigma notation.

$$\boxed{}, \quad \boxed{y = \sum_{r=0}^{\infty} \left[\frac{x^{3r}}{9^r \times r!} \left[\frac{A}{\Gamma\left(\frac{3r+2}{3}\right)} + \frac{Bx}{\Gamma\left(\frac{3r+4}{3}\right)} \right] \right]}$$







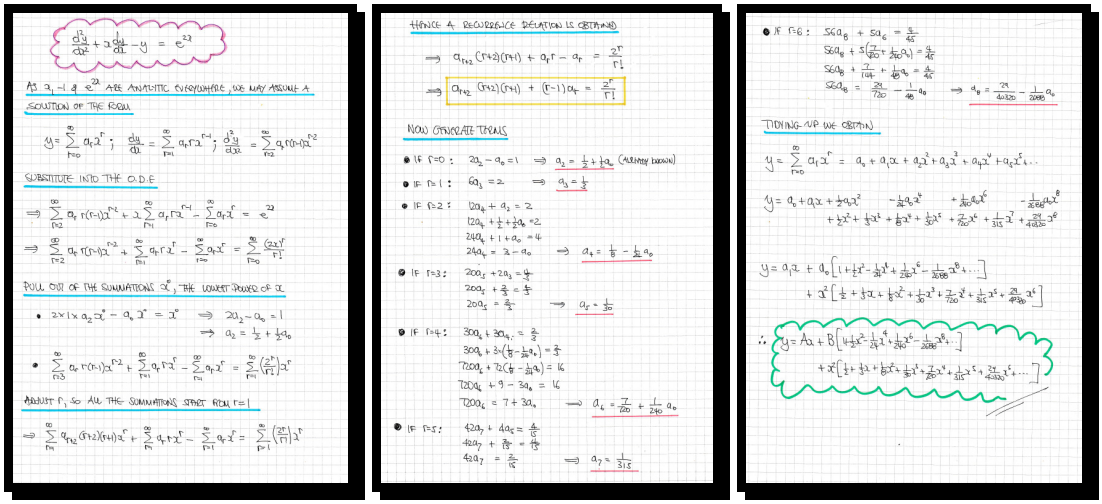
Question 7

Find, as a series, a solution of the following differential equation

$$\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=\mathrm{e}^{2x} \ .$$

Give the final answer in simplified form up and including the term in x^8.

$$\boxed{}, \quad \boxed{\begin{aligned} y &= Ax \\ &+ B\left[1+\frac{1}{2}x^2-\frac{1}{24}x^4+\frac{1}{240}x^6-\frac{1}{2688}x^8+\ldots\right] \\ &+ x^2\left[\frac{1}{2}+\frac{1}{3}x+\frac{1}{8}x^2+\frac{1}{30}x^3+\frac{7}{720}x^4+\frac{1}{315}x^5+\frac{29}{40320}x^6+\ldots\ldots\right] \end{aligned}}$$



Question 8

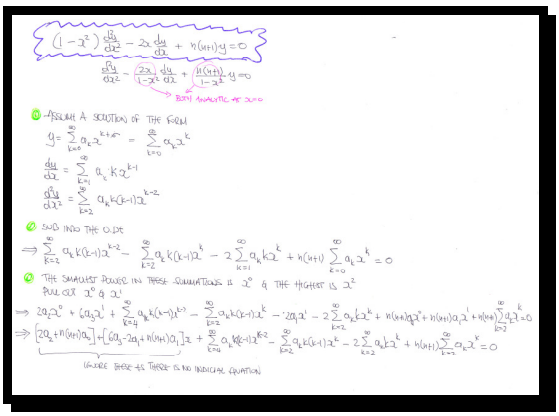
Find the two independent solutions of Legendre's equation

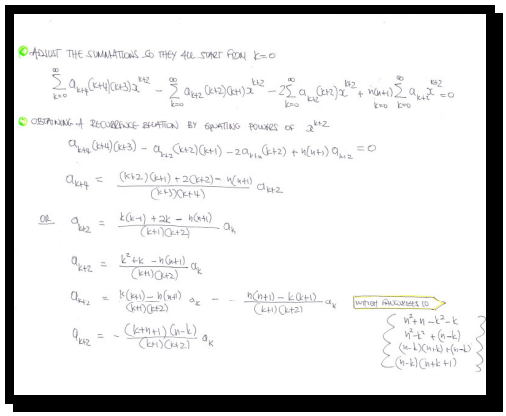
$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0\,,\ n\in\mathbb{R}\,.$$

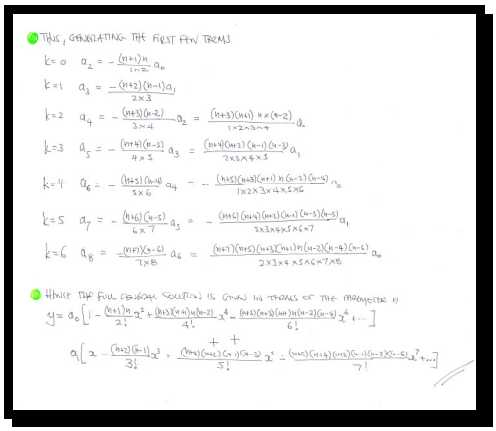
$$y=A\left[1-\frac{(n+1)n}{2!}x^2+\frac{(n+3)(n+1)n(n-2)}{4!}x^4-\frac{(n+5)(n+3)(n+1)n(n-2)(n-4)}{6!}x^6+\ldots\right]$$

$$+$$

$$B\left[x-\frac{(n+2)(n-1)}{3!}x^3+\frac{(n+4)(n+2)(n-1)(n-3)}{3!}x^5-\frac{(n+6)(n+4)(n+2)(n-1)(n-3)(n-5)}{7!}x^7+\ldots\right]$$







Question 9

Find one series solution for the Legendre's equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0\,,\ n\in\mathbb{R}\,,$$

about $x=1$.

$$y=A\sum_{r=0}^{\infty}\left[\frac{(n+r)!}{(n-r)!}\times\frac{1}{(r!)^2}\times\left(\frac{x-1}{2}\right)^2\right]$$



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FROBENIUS METHOD

[2nd order O.D.E.s, where the roots of the indicial equation do not differ by an integer]

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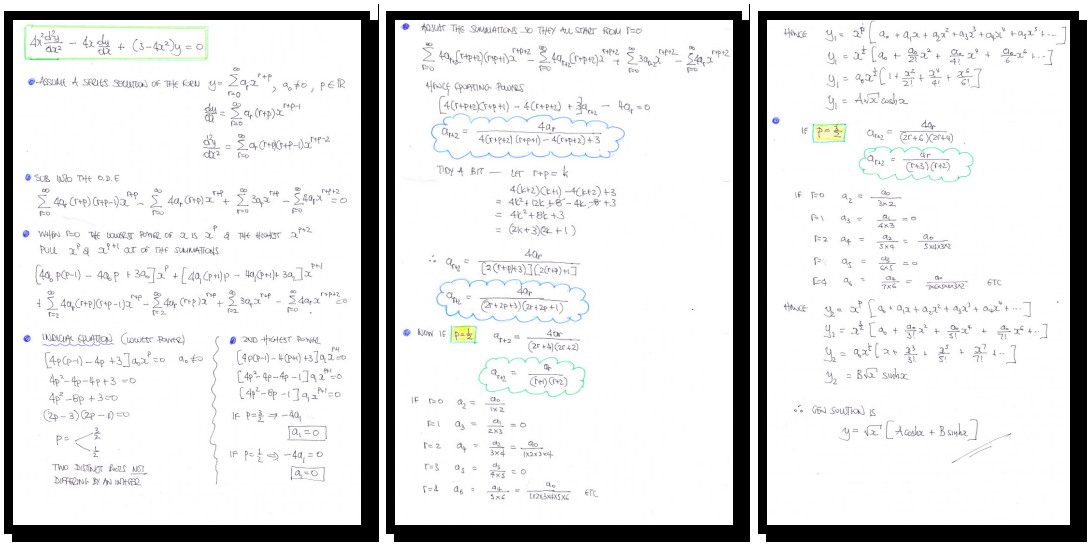
Question 1

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$4x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+\left(3-4x^2\right)y=0\,.$$

Give the final answer in terms of elementary function.

$$\boxed{y=\sqrt{x}\left(A\cosh x+B\sinh x\right)}$$



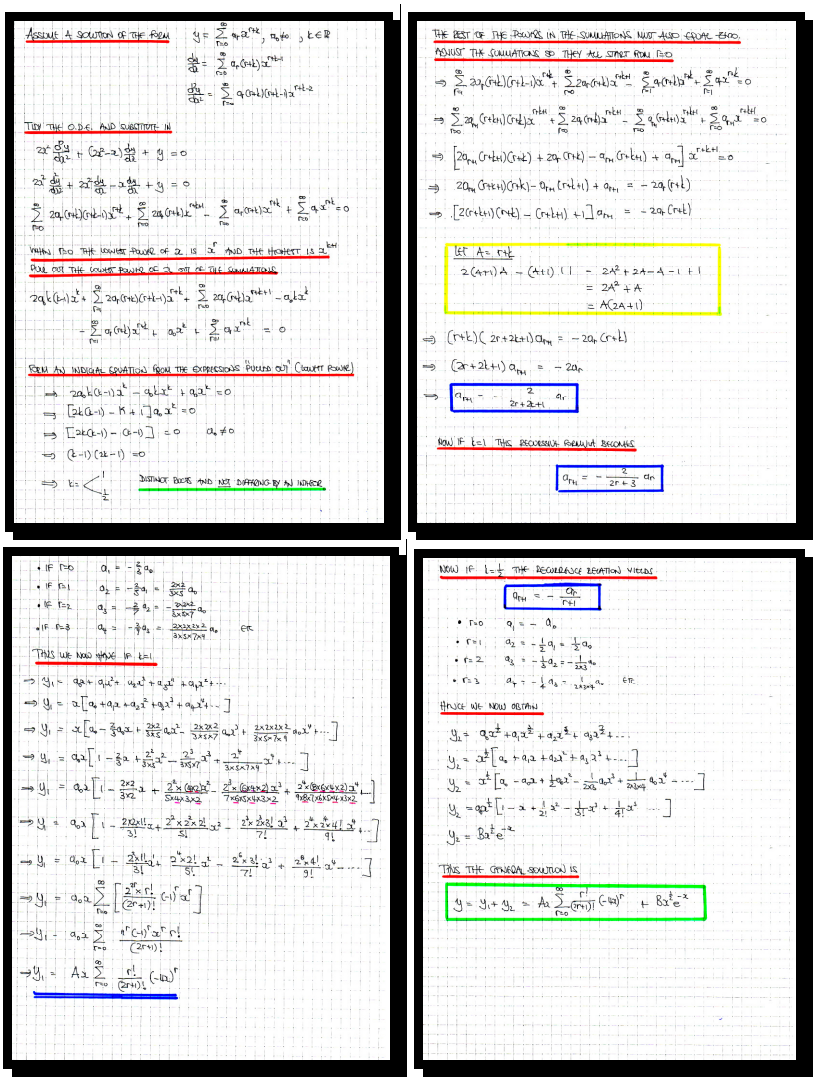
Question 2

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$\frac{d^2y}{dx^2}+\left[1-\frac{1}{2x}\right]\frac{dy}{dx}+\frac{y}{2x^2}=0.$$

Give the final answer in simplified Sigma notation.

$\square$, $\boxed{y=Ax\sum_{r=0}^{\infty}\left[\frac{r!}{(2r+1)!}(-4x)^r\right]+Bx^{\frac{1}{2}}\mathrm{e}^{-x}}$



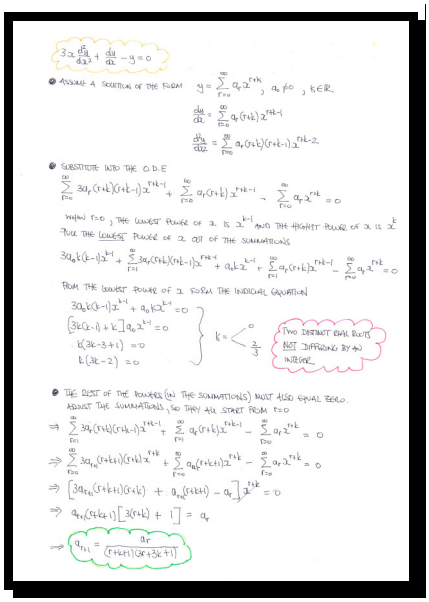
Question 3

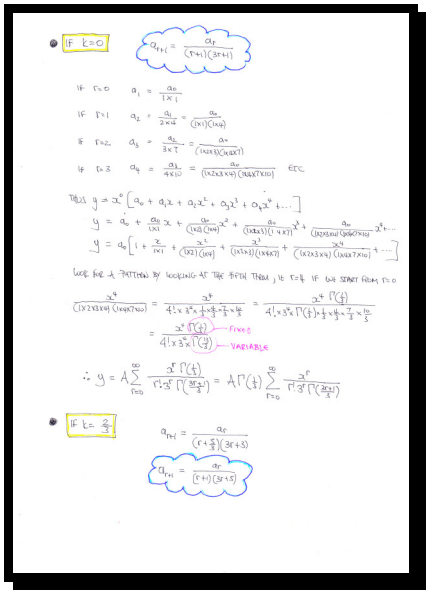
Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

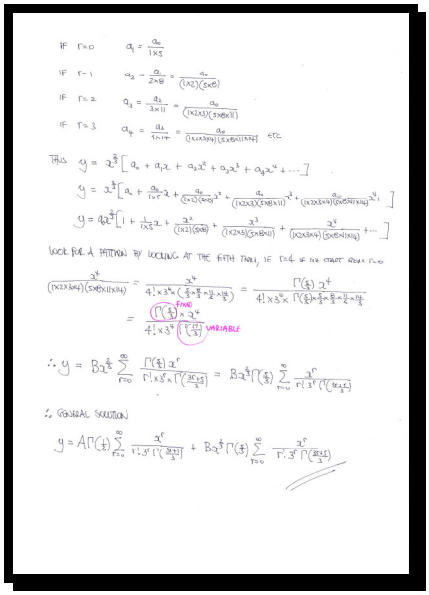
$$3x\frac{d^2y}{dx^2}+\frac{dy}{dx}-y=0.$$

Give the final answer in simplified Sigma notation.

$$y=A\times\Gamma\left(\frac{1}{3}\right)\sum_{r=0}^{\infty}\left[\frac{x^r}{r!\times 3^r\times\Gamma\left(\frac{3r+1}{3}\right)}\right]+B\times x^{\frac{2}{3}}\times\Gamma\left(\frac{5}{3}\right)\sum_{r=0}^{\infty}\left[\frac{x^r}{r!\times 3^r\times\Gamma\left(\frac{3r+5}{3}\right)}\right]$$







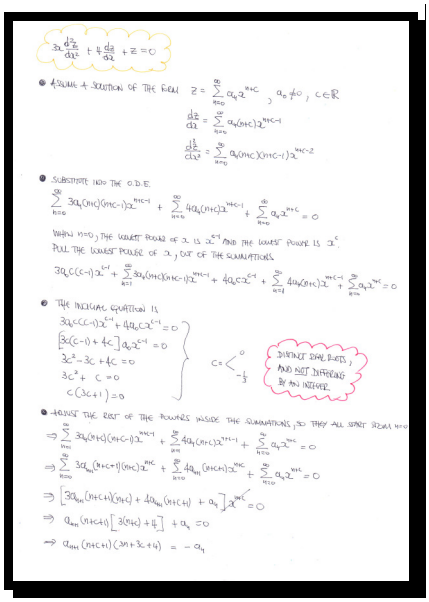
Question 4

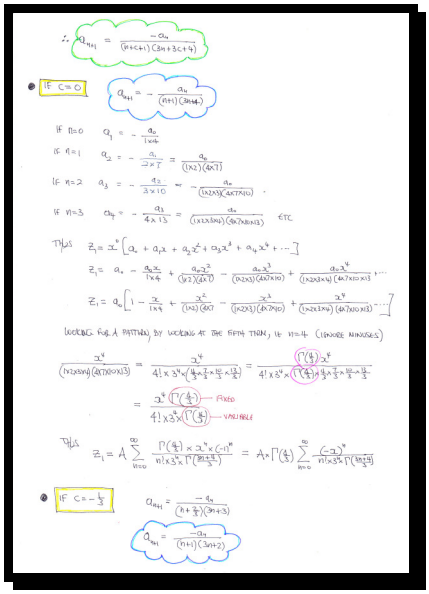
Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

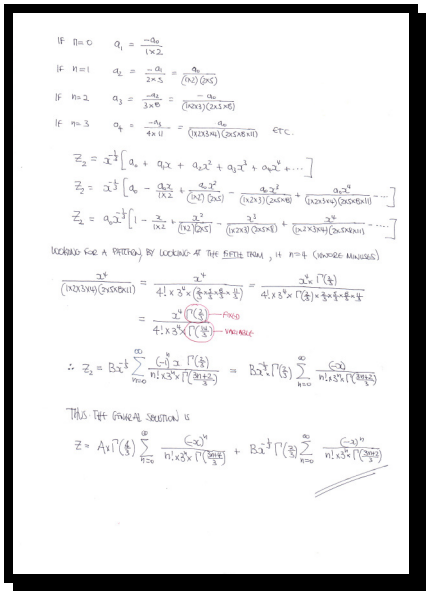
$$3x\frac{d^2z}{dx^2}+4\frac{dz}{dx}+z=0\,.$$

Give the final answer in simplified Sigma notation.

$$z=A\,\Gamma\left(\frac{4}{3}\right)\sum_{n=0}^{\infty}\left[\frac{(-x)^n}{n!\times 3^n\times\Gamma\left(\frac{3n+4}{3}\right)}\right]+Bx^{-\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)\sum_{n=0}^{\infty}\left[\frac{x^n}{n!\times 3^n\times\Gamma\left(\frac{3n+2}{3}\right)}\right]$$







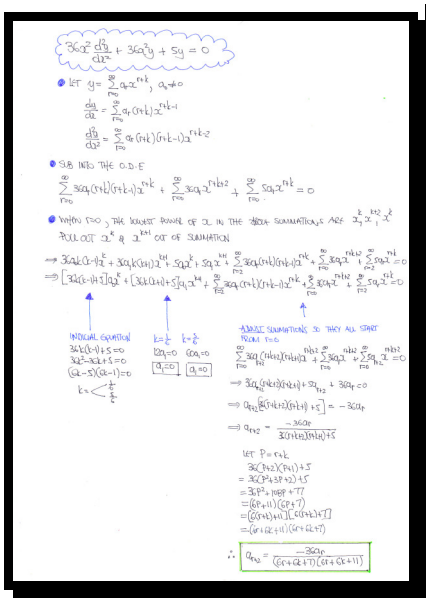
Question 5

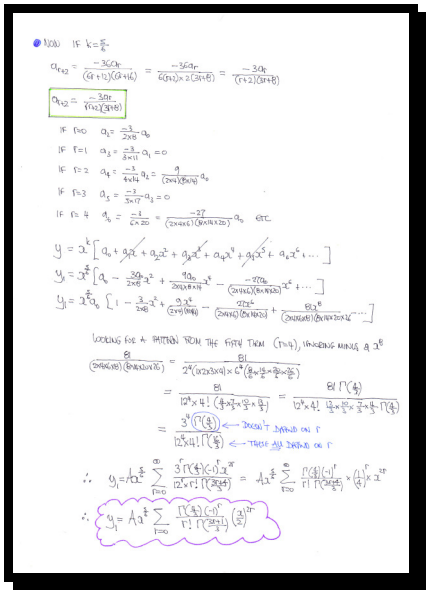
Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

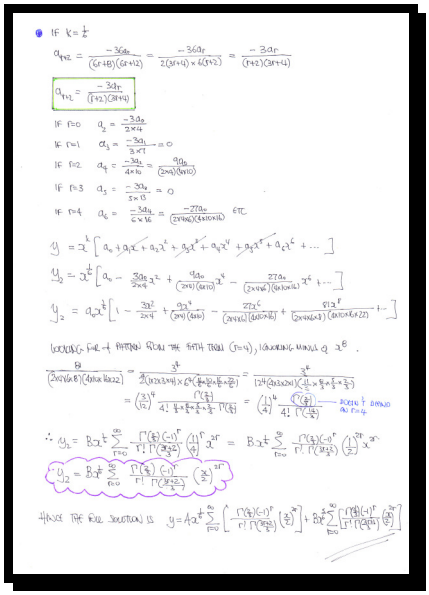
$$36x^2\frac{d^2y}{dx^2}+36x^2y+5y=0\,.$$

Give the final answer in simplified Sigma notation.

$$z=Ax^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!\times\Gamma\left(\frac{3r+2}{3}\right)}\left(\frac{x}{2}\right)^{2r}\right]+Bx^{\frac{5}{6}}\Gamma\left(\frac{4}{3}\right)\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!\times\Gamma\left(\frac{3r+1}{3}\right)}\left(\frac{x}{2}\right)^{2r}\right]$$







Question 6

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$2x\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0.$$

Give the final answer in simplified Sigma notation.

$$y=\sum_{r=0}^{\infty}\left[\frac{(-2x)^r}{(2r)!}\left(A+\frac{B\sqrt{x}}{2r+1}\right)\right]$$



Question 7

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$3x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y-xy=0\,.$$

Give the final answer in simplified Sigma notation.

$$y=Ax\Gamma\left(\frac{5}{3}\right)\sum_{r=0}^{\infty}\left[\frac{1}{r!\Gamma\left(\frac{3r+5}{3}\right)}\left(\frac{1}{3}x\right)^r\right]+Bx^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)\sum_{r=0}^{\infty}\left[\frac{1}{r!\Gamma\left(\frac{3r+1}{3}\right)}\left(\frac{1}{3}x\right)^r\right]$$



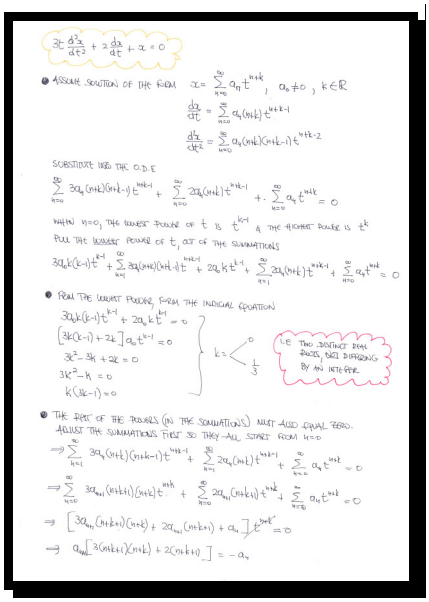
Question 8

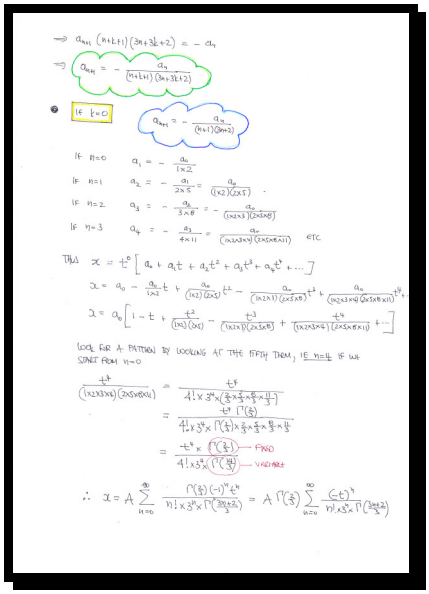
Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

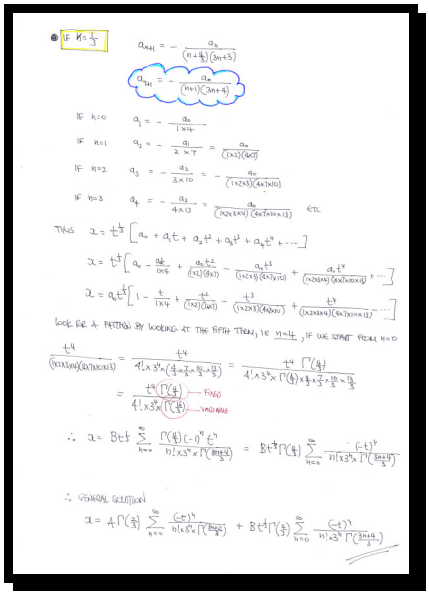
$$3t\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0\,.$$

Give the final answer in simplified Sigma notation.

$$y=A\times\Gamma\left(\frac{2}{3}\right)\sum_{n=0}^{\infty}\left[\frac{(-t)^n}{n!\times 3^n\times\Gamma\left(\frac{3n+2}{3}\right)}\right]+B\times t^{\frac{1}{3}}\times\Gamma\left(\frac{4}{3}\right)\sum_{n=0}^{\infty}\left[\frac{(-t)^n}{n!\times 3^n\times\Gamma\left(\frac{3n+4}{3}\right)}\right]$$







Question 9

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$2x\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}-2y=0\,.$$

Give the final answer in simplified Sigma notation.

$$y=A\left(1+2x+\frac{1}{3}x^2\right)+Bx^{\frac{1}{2}}\left[1+\frac{1}{2}x+\frac{1}{2}x^2-3\sum_{n=0}^{\infty}\left[\frac{(-1)^n\,(2n+1)!\;x^{n+3}}{2^n\;n!\,(2n+7)!}\right]\right]$$



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FROBENIUS METHOD

[2nd order O.D.E.s, where the roots of the indicial equation differ by an integer but one of the coefficients is undetermined]

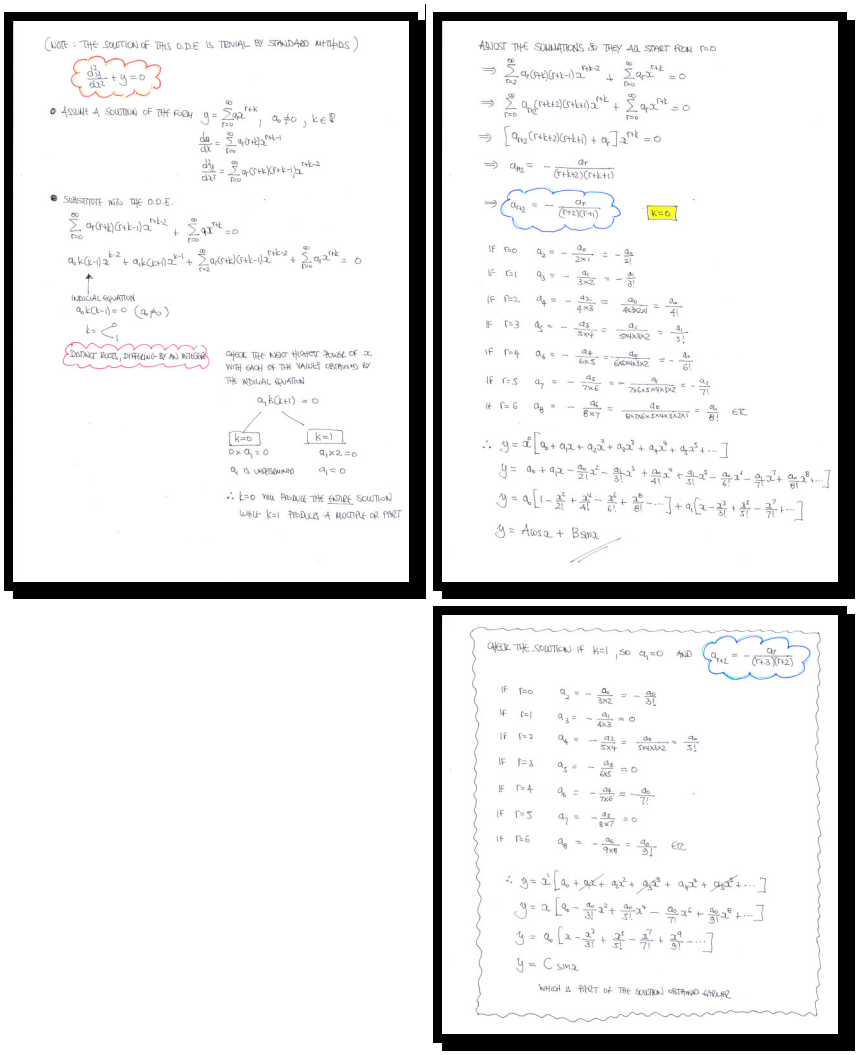
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Question 1

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$\frac{d^2y}{dx^2}+y=0.$$

$$\boxed{y=A\cos x+B\sin x}$$



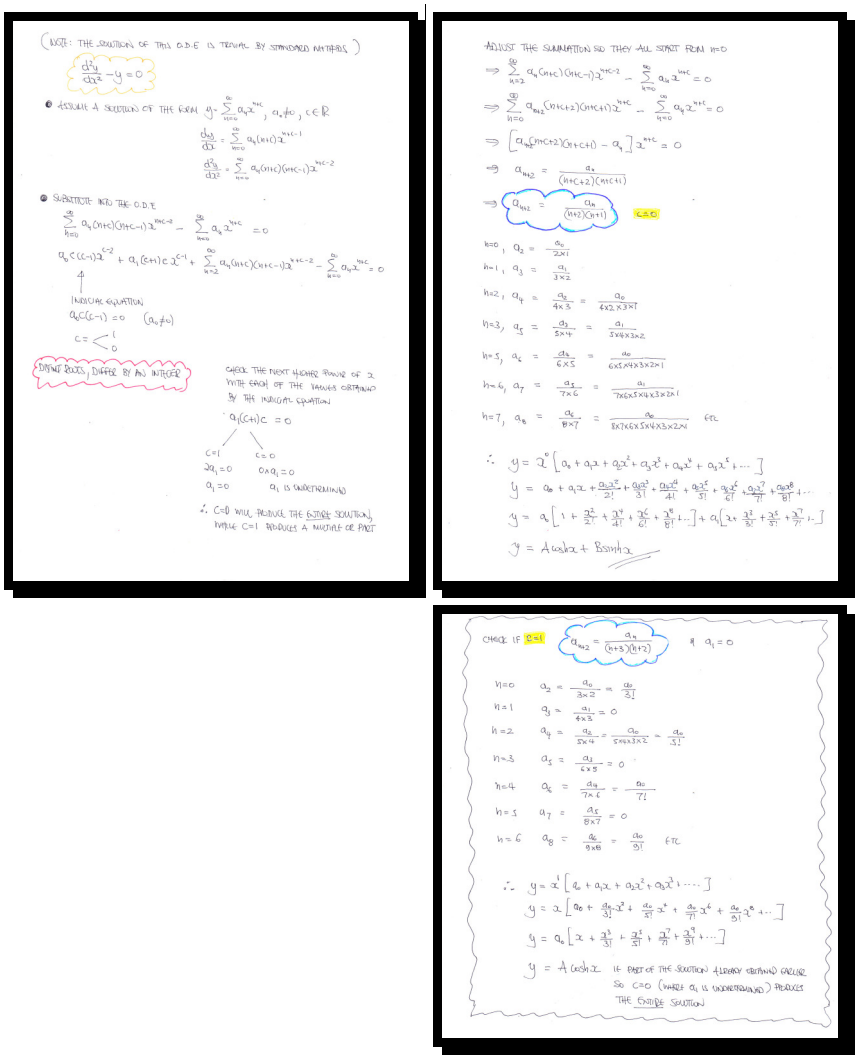
Question 2

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$\frac{d^2y}{dy^2} - y = 0 .$$

Give the final answer in a simplified form.

$$\boxed{y = A\sinh x + B\cosh x}$$



Question 3

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$\frac{d^2y}{dx^2} - yx^2 = 0 .$$

Give the final answer in simplified Sigma notation.

$$\boxed{y = A\Gamma\left(\frac{3}{4}\right)\sum_{r=0}^{\infty}\left[\frac{(-1)^r x^{4r}}{2^{4r}\times r!\times\Gamma\left(\frac{4r+3}{4}\right)}\right] + B\Gamma\left(\frac{5}{4}\right)\sum_{r=0}^{\infty}\left[\frac{(-1)^r x^{4r+1}}{2^{4r}\times r!\times\Gamma\left(\frac{4r+5}{4}\right)}\right]}$$



Question 4

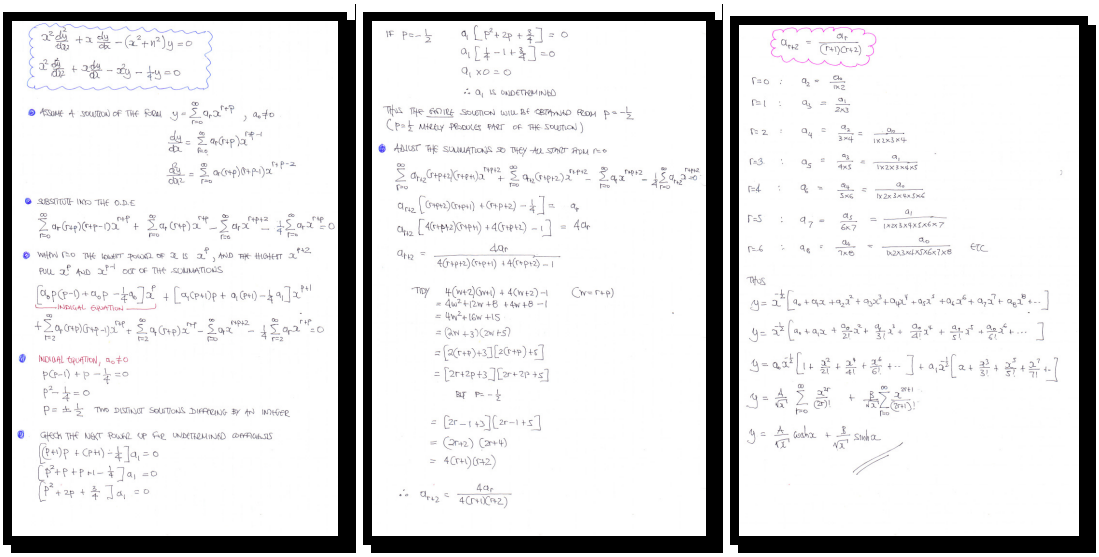
$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-\left(x^2+n^2\right)y=0\,.$$

The above differential equation is known as modified Bessel's Equation.

Use the Frobenius method to show that the general solution of this differential equation, for $n=\frac{1}{2}$, is

$$y=x^{-\frac{1}{2}}\left[A\cosh x+B\sinh x\right].$$

proof



Question 5

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$\frac{d^2y}{dx^2} - xy = 0 .$$

Give the final answer in simplified Sigma notation.

$$\boxed{}, \quad \boxed{y = \sum_{r=0}^{\infty} \left\{ \frac{x^{3r}}{9^r \times r!} \left[\frac{A\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{3r+2}{3}\right)} + \frac{Bx\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{3r+4}{3}\right)} \right] \right\}}$$



Question 6

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$\frac{d^2x}{dt^2}-t\frac{dx}{dt}+x=0\,.$$

Give the final answer in simplified Sigma notation.

$$x=At+\frac{B}{\sqrt{\pi}}\sum_{n=0}^{\infty}\left[\frac{2^{n-1}\Gamma\left(n-\frac{1}{2}\right)t^{2n}}{(2n)!}\right]$$



FROBENIUS METHOD

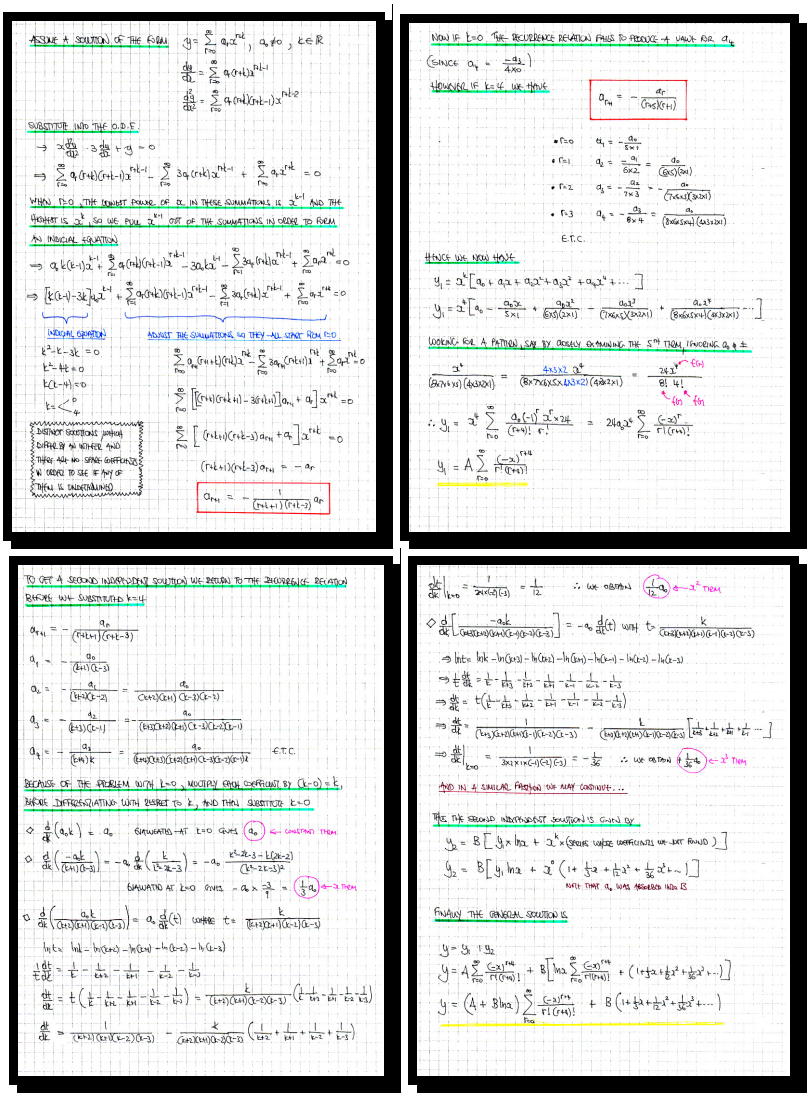
[2nd order O.D.E.s, where the roots of the indicial equation differ by an integer but no coefficient is undetermined]

Question 1

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$x\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0\,.$$

, $\boxed{y=\left(A+B\ln x\right)\sum_{r=0}^{\infty}\left[\frac{(-x)^{r+4}}{r!(r+4)!}\right]+B\left[1+\frac{1}{3}x+\frac{1}{12}x^2+\frac{1}{36}x^3+O\left(x^4\right)\right]}$



Question 2

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$x\frac{d^2y}{dx^2}+(x+2)\frac{dy}{dx}-2y=0\,.$$

$$\boxed{y=A\left[1+x+\frac{1}{6}x^2\right]+B\left[\left(1+x+\frac{1}{6}x^2\right)\ln x+\frac{1}{x}\left[1-4x-10x^2-\frac{31}{12}x^3+O\left(x^4\right)\right]\right]}$$



Question 3

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$x\frac{d^2y}{dx^2}+3\frac{dy}{dx}-y=0\,.$$

$$y=A\left[\sum_{r=0}^{\infty}\frac{x^r}{r!(r+2)!}\right]+B\left[\ln x\sum_{r=0}^{\infty}\frac{x^r}{r!(r+2)!}+\frac{1}{x^2}\left[1-x+\frac{1}{4}x^2+\frac{11}{36}x^3+O\left(x^4\right)\right]\right]$$



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FROBENIUS METHOD

[2nd order O.D.E.s, where the indicial equation has repeated roots]

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Question 1

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0\,.$$

$$y=A\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(n!)^2}\left(\frac{1}{2}x\right)^{2n}\right]+B\left[\ln x\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(n!)^2}\left(\frac{1}{2}x\right)^{2n}\right]+\sum_{n=1}^{\infty}\sum_{m=1}^{n}\left[\frac{(-1)^n}{m(n!)^2}\left(\frac{1}{2}x\right)^{2n}\right]\right]$$



Question 2

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}-3y=0.$$

$$y=A\sum_{r=0}^{\infty}\left[\frac{(3x)^r}{(r!)^2}\right]+B\left[\ln x\sum_{r=0}^{\infty}\left[\frac{(3x)^r}{(r!)^2}\right]+\sum_{n=1}^{\infty}\sum_{m=1}^{n}\left[\frac{(3x)^r}{m(n!)^2}\right]\right]$$

