MULTIVARIABLE INTEGRATION

(PLANE & CYLINDRICAL POLAR COORDINATES)

PLANE POLAR COORDINATES

Question 1

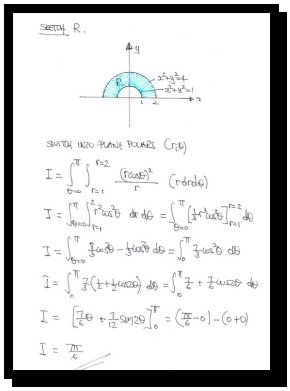
The finite region on the x-y plane satisfies

$$1 \le x^2 + y^2 \le 4 , \ y \ge 0 .$$

Find, in terms of π, the value of I.

$$I = \int_R \frac{x^2}{\sqrt{x^2 + y^2}} \, dx \, dy .$$

$\frac{7}{6}\pi$



Question 2

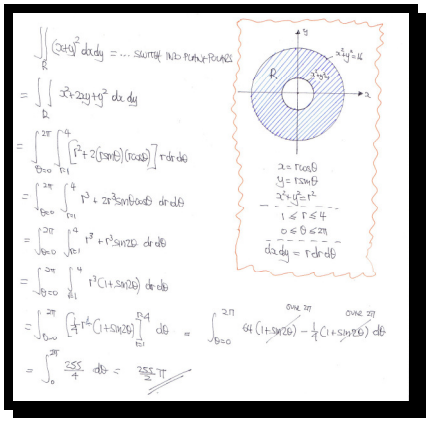
The finite region on the x-y plane satisfies

$$1 \le x^2 + y^2 \le 16, \ y \ge 0.$$

Find, in terms of π, the value of I.

$$I = \int_R (x+y)^2 \ dx\,dy.$$

$$\frac{255\pi}{2}$$

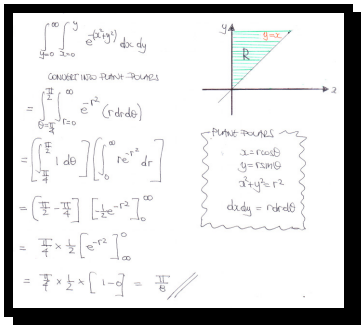


Question 3

Find the exact simplified value for the following integral.

$$\int_0^{\infty}\int_0^{y} \mathrm{e}^{-\left(x^2+y^2\right)}\, dx\, dy\,.$$

$$\frac{\pi}{8}$$

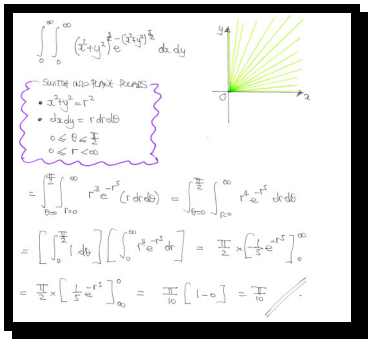


Question 4

Find the exact simplified value for the following integral.

$$\int_0^{\infty}\int_0^{\infty} \left(x^2+y^2\right)^{\frac{3}{2}} \mathrm{e}^{-\left(x^2+y^2\right)^{\frac{5}{2}}}\, dx\, dy\,.$$

$$\frac{\pi}{10}$$



Question 5

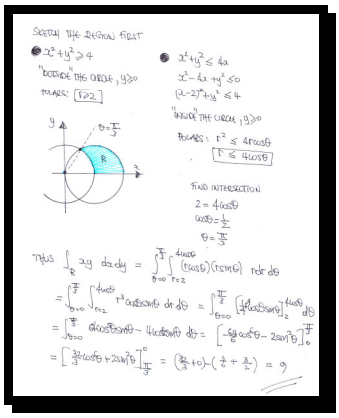
The finite region on the x-y plane satisfies

$$4 \le x^2 + y^2 \le 4x\,,\ \ y \ge 0\,.$$

Find the value of I.

$$I = \int_R xy \ \ dx\,dy\,.$$

9



Question 6

The points A and B have Cartesian coordinates $(1,0)$ and $(1,1)$, respectively.

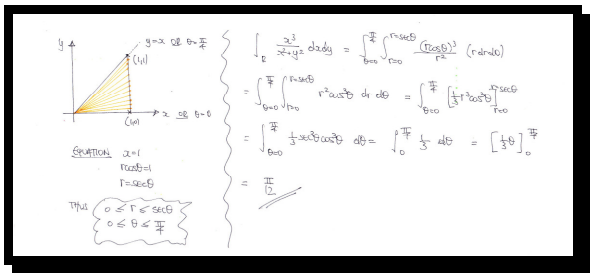
The finite region R is defined as the triangle OAB, where O is the origin.

Use plane **polar** coordinates, (r,θ) to determine the value of

$$\int_R \frac{x^3}{x^2+y^2}\,dx\,dy\,.$$

[*No credit will be given for workings in other coordinate systems.*]

$$\boxed{\frac{\pi}{12}}$$



Question 7

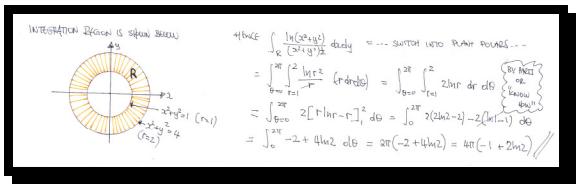
The finite region R is defined as

$$1 \le x^2 + y^2 \le 4 \, .$$

Determine an exact simplified value for

$$\int_R \frac{\ln\left(x^2+y^2\right)}{\sqrt{x^2+y^2}} \, dx\, dy \, .$$

$$\boxed{4\pi\left(-1+2\ln 2\right)}$$



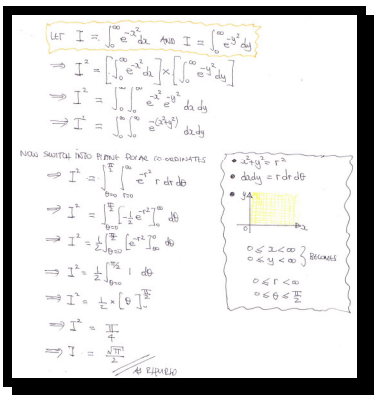
Question 8

$$I = \int_0^{\infty} \mathrm{e}^{-x^2} \, dx \quad \text{and} \quad I = \int_0^{\infty} \mathrm{e}^{-y^2} \, dy \, .$$

By considering an expression for I^2 and the use of plane polar coordinates, show clearly that

$$I = \frac{1}{2}\sqrt{\pi} \, .$$

proof



Question 9

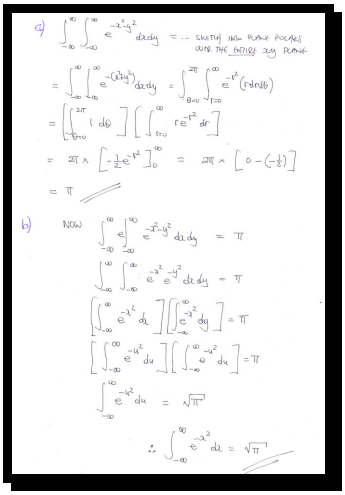
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-x^2-y^2} \, dx\,dy \, .$$

a) Use plane polar coordinates (r,θ), to find the exact simplified value of the above integral.

b) Hence evaluate

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \, dx \, .$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-x^2-y^2} \, dx\,dy = \pi \quad , \quad \int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \, dx = \sqrt{\pi}$$



Question 10

The points A and B have Cartesian coordinates $(0,1)$ and $(1,1)$, respectively.

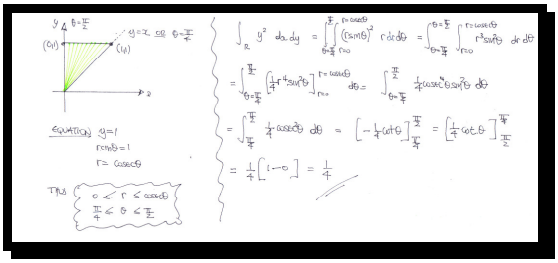
The finite region R is defined as the triangle OAB, where O is the origin.

Use plane **polar** coordinates, (r,θ) to determine the value of

$$\int_R y^2 \, dx\, dy \, .$$

[*No credit will be given for workings in other coordinate systems*.]

$\boxed{\frac{1}{4}}$



Question 11

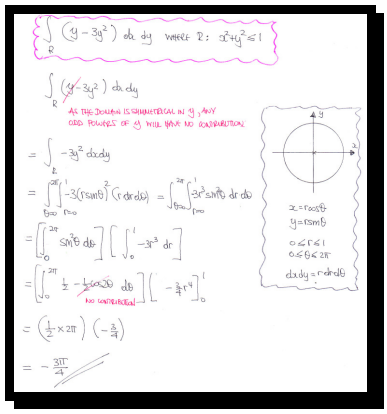
The finite region R, on the x-y plane, satisfies

$$x^2 + y^2 \leq 1 .$$

Find, in terms of π, the value of

$$\int_R \left(y - 3y^2 \right) \, dx \, dy .$$

$$\boxed{-\frac{3\pi}{4}}$$

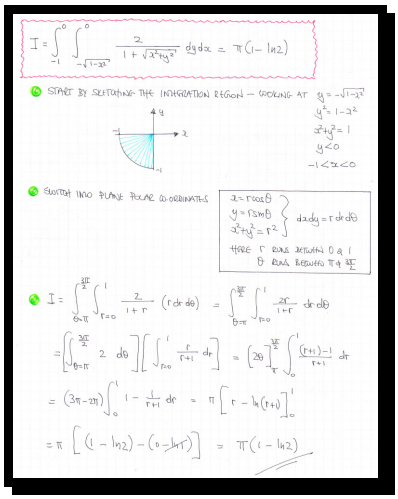


Question 12

Find the exact simplified value for the following integral.

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y}}\,dy\,dx\,.$$

$$\boxed{\pi(1-\ln 2)}$$

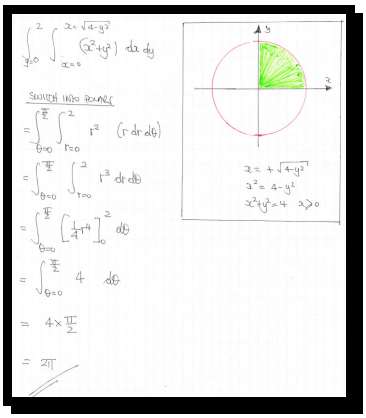


Question 13

Find the exact simplified value for the following integral.

$$\int_0^2 \int_0^{\sqrt{4-x^2}} x^2 + y^2 \quad dy\, dx\,.$$

$\boxed{2\pi}$



Question 14

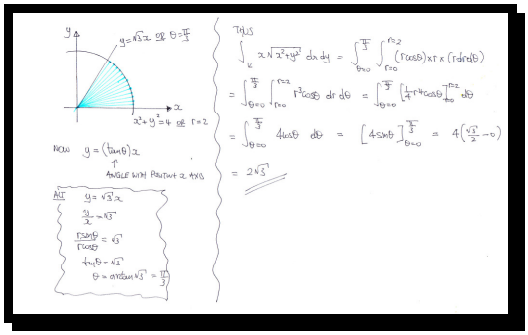
The finite region R is bounded by the straight lines and curves with the following equations.

$$y=0,\ x=0,\ x^2+y^2=4 \quad \text{and} \quad y=\sqrt{3}x.$$

Determine an exact simplified value for

$$\int_R x\sqrt{x^2+y^2}\ dx\,dy.$$

$$\boxed{2\sqrt{3}}$$

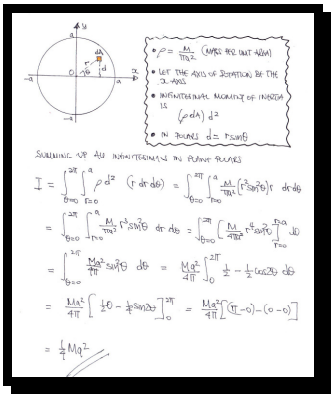


Question 15

A uniform circular lamina of mass M and radius a.

Use double integration to find the moment of inertia of the lamina, when the axis of rotation is a diameter.

$$\boxed{\frac{1}{4}Ma^2}$$



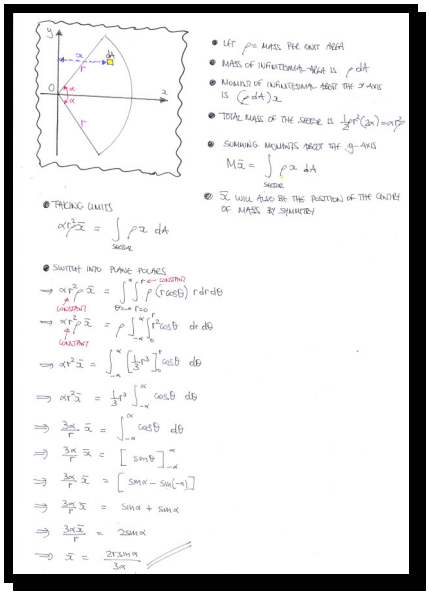
Question 16

A circular sector of radius r subtends an angle of 2α at its centre O. The position of the centre of mass of this sector lies at the point G, along its axis of symmetry.

Use calculus to show that

$$|OG| = \frac{2r\sin\alpha}{3\alpha}.$$

proof



Question 17

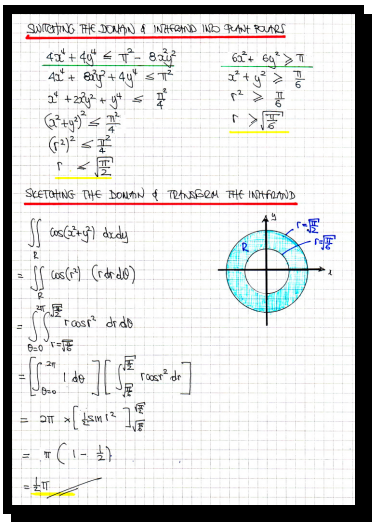
The finite region on the x-y plane satisfies

$$4x^4+4y^4\le\pi^2-8x^2y^2 \quad \text{and} \quad 6x^2+6y^2\ge\pi\,.$$

Find the value of the following integral.

$$\int_R \cos\left(x^2+y^2\right)\,dx\,dy\,.$$

$\square$, $\boxed{\frac{1}{2}\pi}$



Question 18

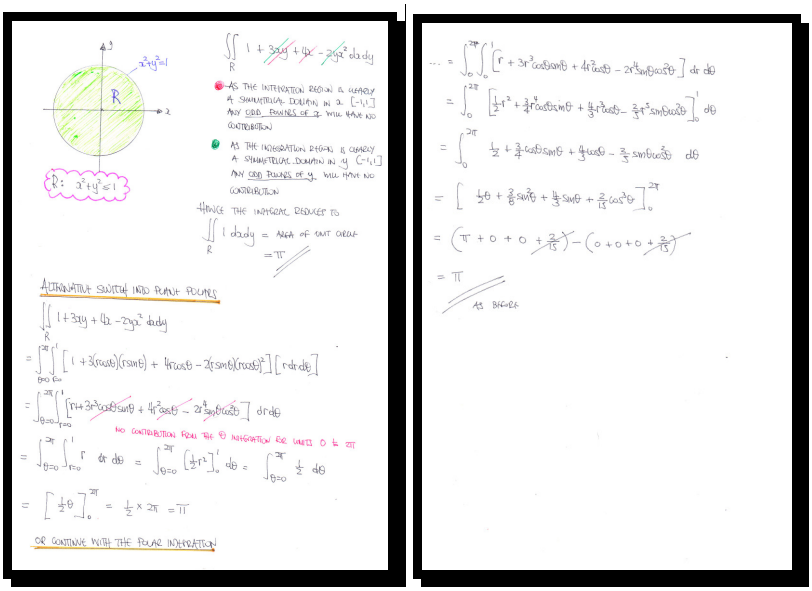
The finite region R, on the x-y plane, satisfies

$$x^2 + y^2 \leq 1 .$$

Find, in terms of π, the value of I.

$$I = \int_R \left(1 + 3xy + 4x - 2yx^2\right) \, dx\, dy .$$

π



Question 19

The finite region R is bounded by the straight lines with the following equations.

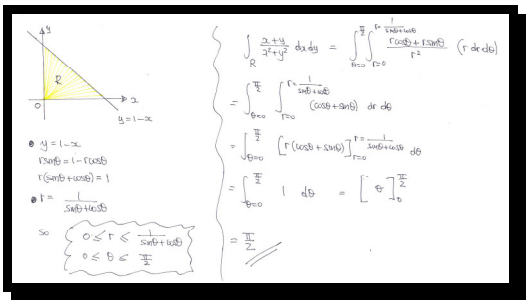
$$x=0\,, \qquad y=0 \quad \text{and} \quad y=1-x\,.$$

Use plane **polar** coordinates, (r,θ) to determine the value of

$$\int_R \frac{x+y}{x^2+y^2}\,dx\,dy\,.$$

[*No credit will be given for workings in other coordinate systems.*]

$$\boxed{\frac{\pi}{2}}$$

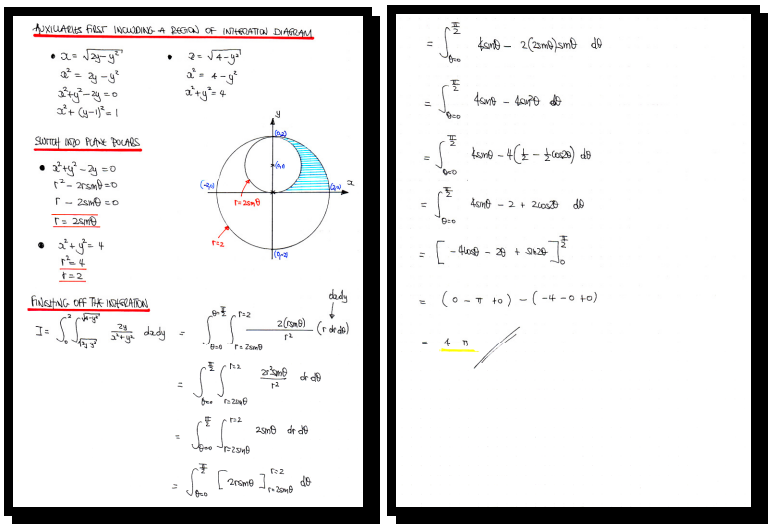


Question 20

$$I=\int_0^2\int_{\sqrt{2y-y^2}}^{\sqrt{4-y^2}}\frac{2y}{x^2+y^2}\,dx\,dy\,.$$

Use polar coordinates to find an exact simplified answer for I.

$\square$, $\boxed{4-\pi}$



Question 21

The finite region R is bounded by the straight lines with the following equations.

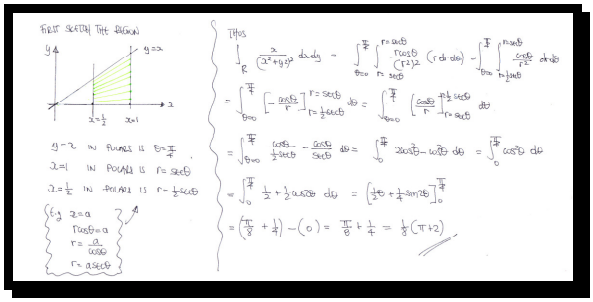
$$y=0, \quad x=\frac{1}{2}, \quad x=1 \quad \text{and} \quad y=x.$$

Use plane **polar** coordinates, (r,θ) to determine the value of

$$\int_R \frac{x}{\left(x^2+y^2\right)^2}\, dx\, dy.$$

[*No credit will be given for workings in other coordinate systems*.]

$$\boxed{\frac{1}{8}(\pi+2)}$$



Question 22

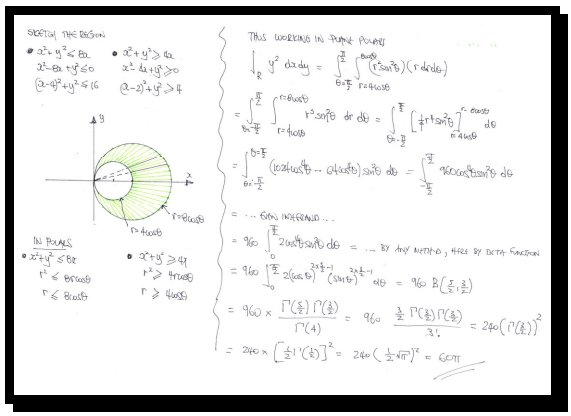
The finite region R is defined as

$$4x \le x^2 + y^2 \le 8x \, .$$

Determine the value of

$$\int_R y^2 \, dx \, dy \, .$$

$\boxed{60\pi}$

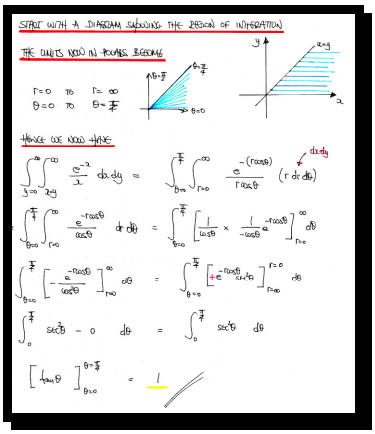


Question 23

Use plane **polar** coordinates, (r,θ) to determine the value of

$$\int_{y=0}^{\infty}\int_{x=y}^{\infty}\frac{\mathrm{e}^{-x}}{x}\,dxdy\,.$$

$\square$, $\boxed{1}$



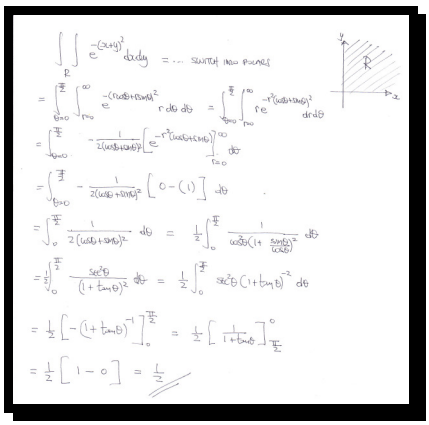
Question 24

Use plane **polar** coordinates, (r,θ) to determine the value of

$$\int_R \mathrm{e}^{-(x+y)^2}\, dx\, dy\,,$$

where R is the region in the first quadrant in a standard Cartesian coordinate system.

$$\frac{1}{2}$$

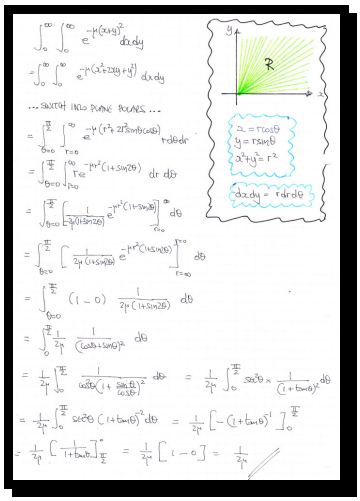


Question 25

Given that μ is a positive constant determine the value of

$$\int_0^{\infty}\int_0^{\infty} \mathrm{e}^{-\mu(x+y)^2}\ dx\,dy\ ,$$

$$\boxed{\frac{1}{2\mu}}$$



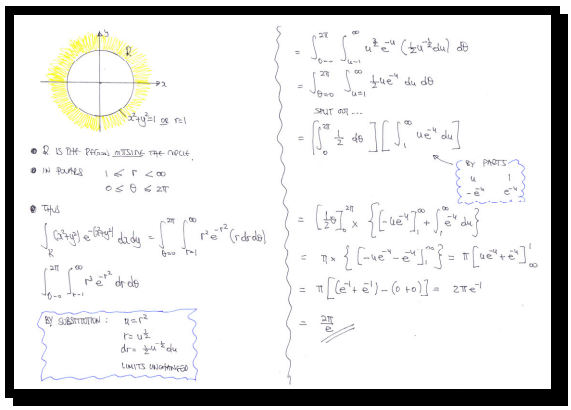
Question 26

Determine an exact simplified value for

$$\int_R \left(x^2+y^2\right)\mathrm{e}^{-\left(x^2+y^2\right)}\,dx\,dy,$$

where R is the region $x^2+y^2>1$

$$\boxed{\frac{2\pi}{\mathrm{e}}}$$

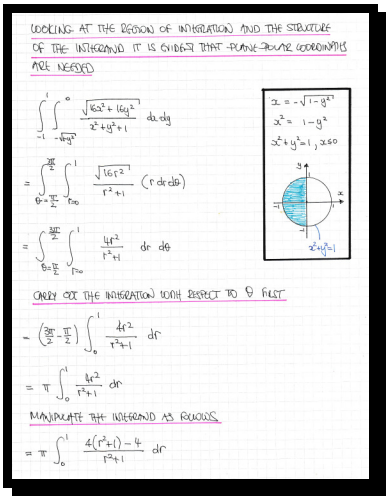


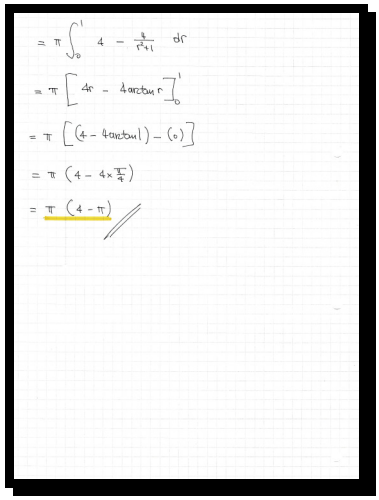
Question 27

Find the exact simplified value for the following integral.

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0} \frac{\sqrt{16x^2+16y^2}}{x^2+y^2+1}\,dx\,dy\,.$$

$\boxed{}$, $\boxed{\pi(4-\pi)}$





Question 28

The finite region R is bounded by the straight lines with the following equations.

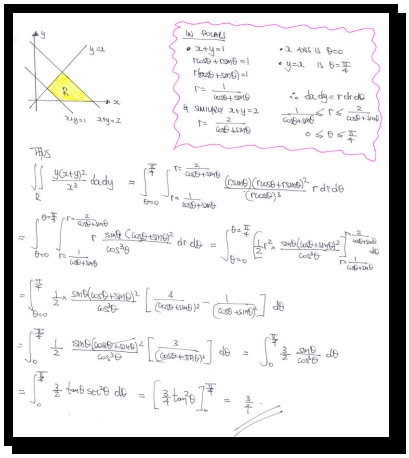
$$x+y=1, \quad x+y=2, \quad y=x \quad \text{and} \quad y=0.$$

Use plane polar coordinates (r,θ) to find the value of

$$\iint_R \frac{y(x+y)^2}{x^3}\,dx\,dy\,.$$

[*No credit will be given for workings in other coordinate systems*.]

$\boxed{\frac{3}{4}}$



Question 29

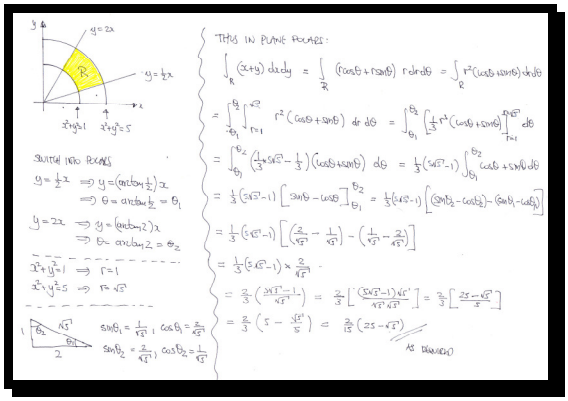
The region R on the x-y plane is defined by the inequalities

$$1 \le x^2 + y^2 \le 5 \qquad \text{and} \qquad \frac{1}{2}x \le y \le 2x .$$

Show clearly that

$$\int_R (x+y)\, dx\, dy = \frac{2}{15}\left(25-\sqrt{5}\right).$$

proof



Question 30

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

This implies that $\phi^2 - \phi - 1 = 0$, $\phi = \frac{1}{2}\left(1+\sqrt{5}\right) \approx 1.62$.

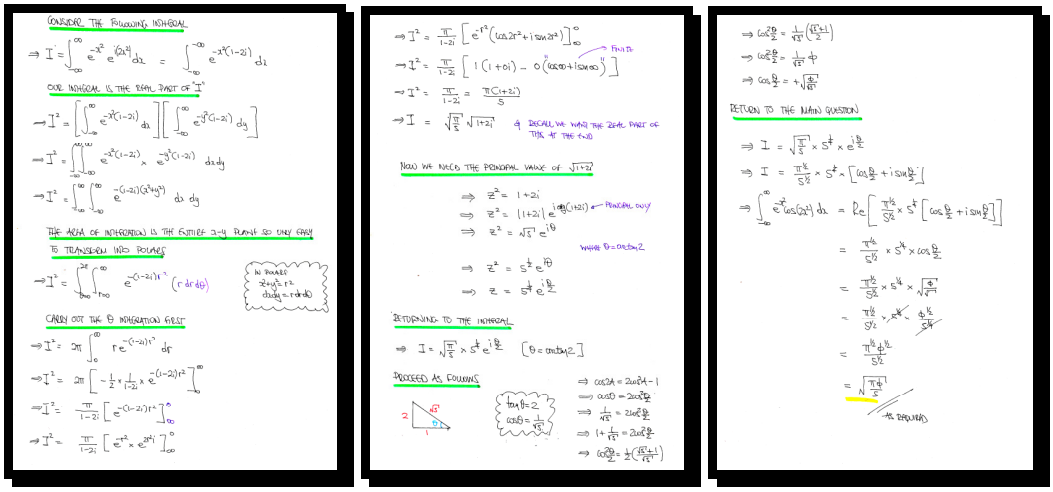
It is asserted that

$$I = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \cos\left(2x^2\right)\, dx = \sqrt{\frac{\pi\phi}{5}}\,.$$

By considering the real part of a suitable function, use double integration in plane polar coordinates to prove the validity of the above result.

You may assume the principal value in any required complex evaluation.

, proof



CYLINDRICAL COORDINATES

Question 1

Find the value of

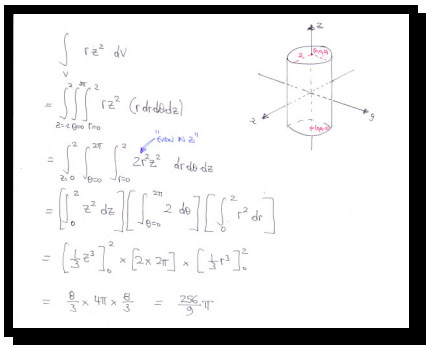
$$\int_{\Omega} rz^2 \ dV \,,$$

where Ω is the region inside the cylinder with equation

$$x^2 + y^2 = 4 \,, \ -2 \le z \le 2$$

In this question use cylindrical polar coordinates (r, θ, z).

$$\boxed{\frac{256\pi}{9}}$$



Question 2

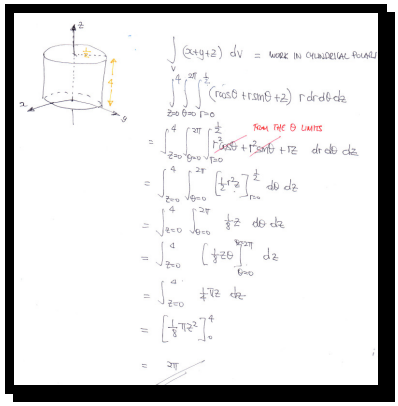
Find the value of

$$\int_V (x+y+z)\ dx\,dydz\,,$$

where V is the region inside the cylinder with equation

$$x^2+y^2=\frac{1}{4},\ 0\le z\le 4\,.$$

$\boxed{2\pi}$



Question 3

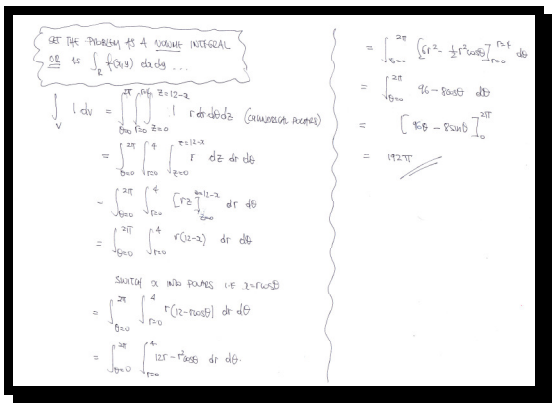
Find in exact form the volume enclosed by the cylinder with equation

$$x^2 + y^2 = 16\,,\ z \geq 0\,,$$

and the plane with equation

$$z = 12 - x\,.$$

$\boxed{192\pi}$



Question 4

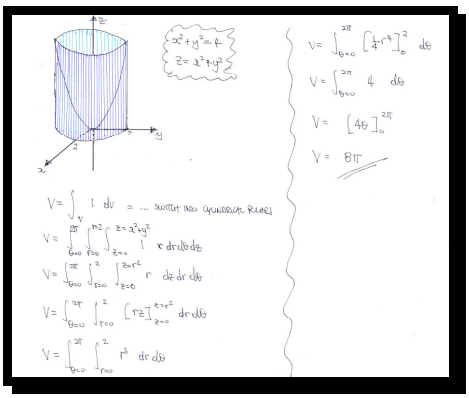
Find the volume of the region bounded by the cylinder with equation

$$x^2 + y^2 = 4\,,$$

and the surfaces with equations

$$z = x^2 + y^2 \quad \text{and} \quad z = 0\,.$$

$\boxed{8\pi}$

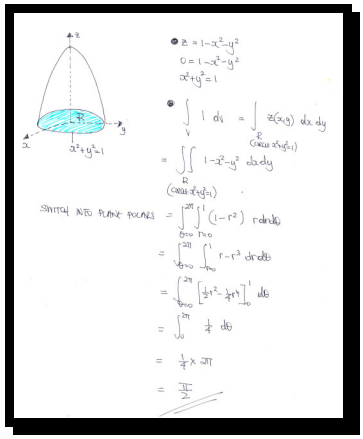


Question 5

Find the volume of the paraboloid with equation

$$z = 1 - x^2 - y^2 , \ z \geq 0 .$$

$$\boxed{\frac{\pi}{2}}$$



Question 6

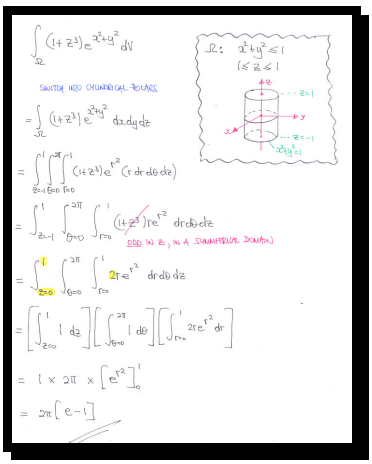
The finite region Ω is enclosed by the cylinder with Cartesian equation

$$x^2+y^2=1,\ \ -1\le z\le 1.$$

Determine an exact simplified value for

$$\int_{\Omega}\left(1+z^3\right)\mathrm{e}^{x^2+y^2}\ dx\,dy\,dz\,.$$

$$\boxed{2\pi\left[\mathrm{e}-1\right]}$$



Question 7

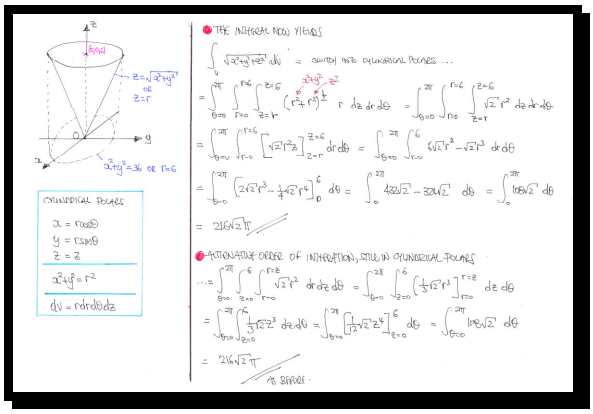
The finite region V is enclosed by the cone with Cartesian equation

$$z=\sqrt{x^2+y^2}\ ,\quad 0\leq z\leq 6\ .$$

Determine an exact simplified value for

$$\int_V \sqrt{x^2+y^2+z^2}\ \ dx\,dy\,dz\ .$$

$$\boxed{216\sqrt{2}\,\pi}$$



Question 8

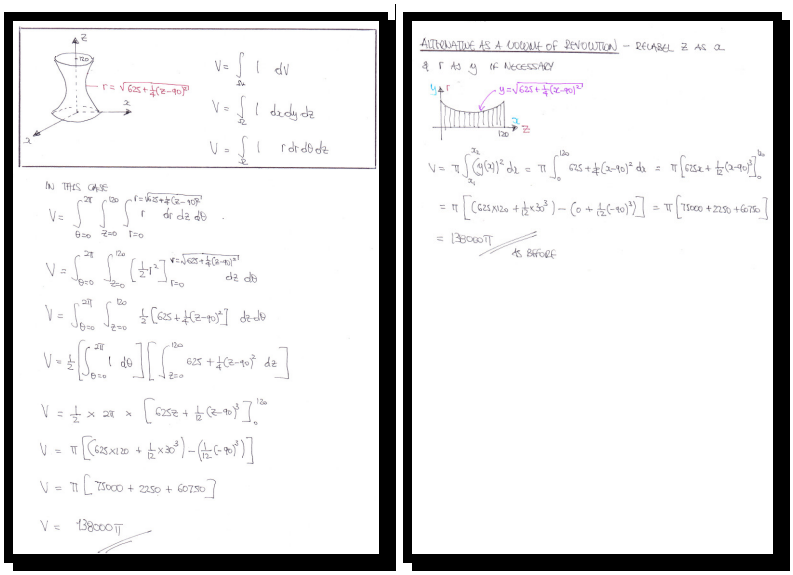
The height z, of a cooling tower, is 120 m.

The radius r m, of any of the circular cross sections of the cooling tower is given by the equation

$$r=\sqrt{625+\frac{1}{4}(z-90)}\,.$$

Use cylindrical polar coordinates (r,θ,z), to find the volume of the tower.

$$\boxed{138000\pi}$$

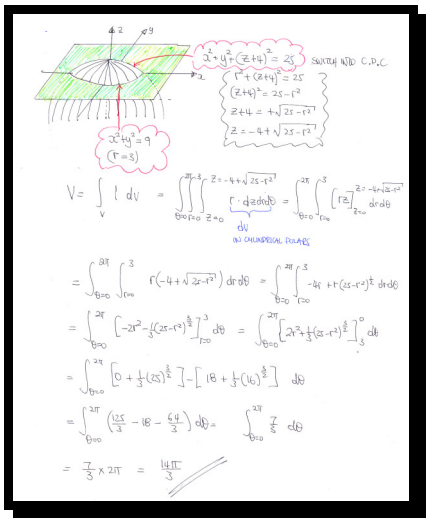


Question 9

Use cylindrical polar coordinates (r,θ,z) to find the volume of the region defined as

$$x^2+y^2+(z+4)^2\le 25,\ z\ge 0.$$

$$\boxed{\frac{14}{3}\pi}$$



Question 10

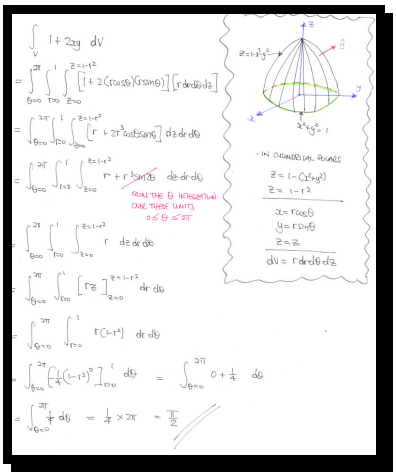
Find the value of

$$\int_V (1+2xy)\ dV\,,$$

where V is the finite region enclosed by the surface with Cartesian equation

$$z=1-x^2-y^2\,,\ z\geq 0\,.$$

$$\boxed{\frac{\pi}{2}}$$

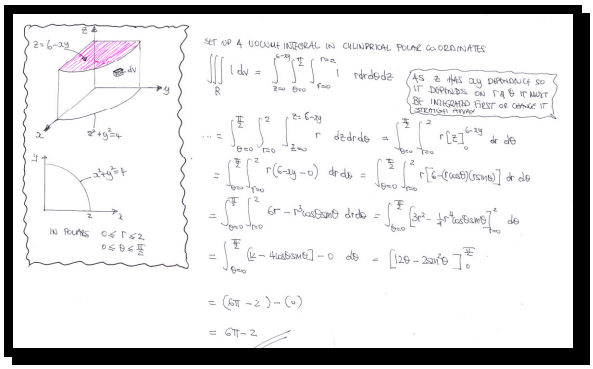


Question 11

Find in exact form the volume of the solid defined by the inequalities

$$x^2+y^2\le 4\,,\quad x\ge 0\,,\quad y\ge 0\quad \text{and}\quad 0\le z\le 6-xy\,.$$

$$\boxed{6\pi-2}$$

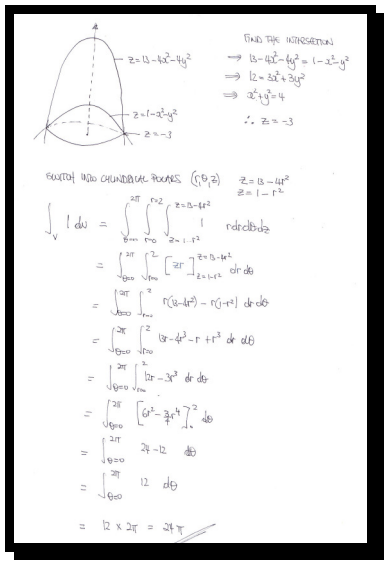


Question 12

Find the volume of the finite region bounded by the surfaces with Cartesian equations

$$z=13-4x^2-4y^2\quad \text{and}\quad z=1-x^2-y^2$$

$$\boxed{V=24\pi}$$



Question 13

A scalar field F exist inside the cylinder with equation

$$x^2+y^2=1,\ \ 0\le z\le 4\,.$$

Given further that

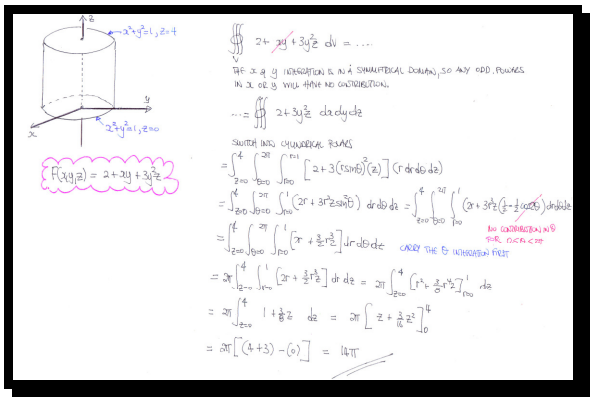
$$F(x,y,z)\equiv 2+xy+3yz^2,$$

evaluate the integral

$$\iiint_V F\ dV\,,$$

where V denotes the region enclosed by the cylinder

$$\boxed{14\pi}$$



Question 14

Use cylindrical polar coordinates (r,θ,z) to evaluate

$$\int_V \frac{5yz^2}{\sqrt{x^2+y^2}}\,dxdydz,$$

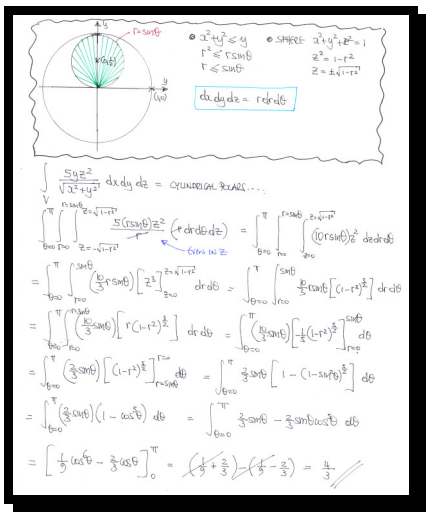
where V is the region defined as

$$x^2+y^2 \le y,$$

contained within the sphere with equation

$$x^2+y^2+z^2=1.$$

$$\boxed{\frac{4}{3}}$$



Question 15

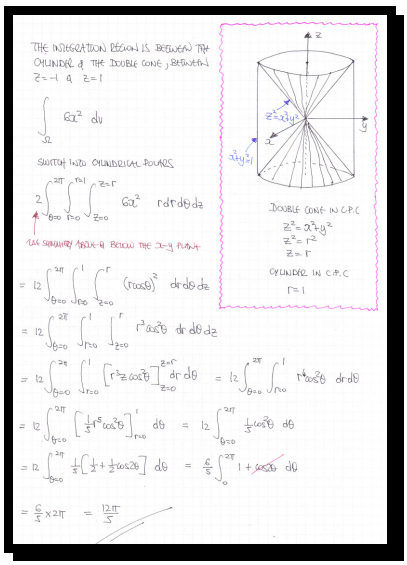
The finite region Ω is defined by the inequalities

$$x^2 + y^2 \leq 1 \qquad \text{and} \qquad |z| \leq \sqrt{x^2 + y^2} \ .$$

Use cylindrical polar coordinates to evaluate

$$\int_{\Omega} 6x^2 \ dx\,dy\,dz \ .$$

$$\frac{12\pi}{5}$$



Question 16

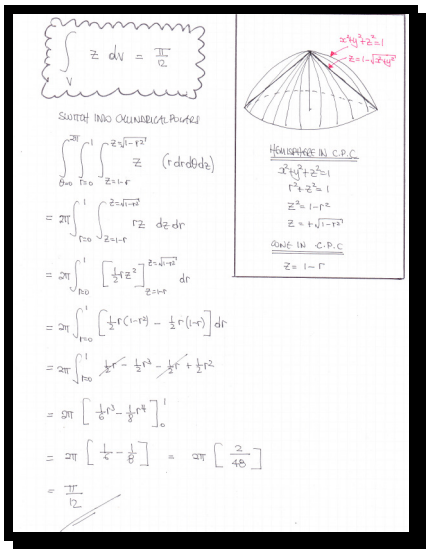
The finite region V is defined by the inequalities

$$x^2+y^2+z^2 \le 1 \qquad \text{and} \qquad z \ge 1-\sqrt{x^2+y^2}\,.$$

Use cylindrical polar coordinates to evaluate

$$\int_V z \;\; dx\,dy\,dz\,.$$

$$\frac{\pi}{12}$$

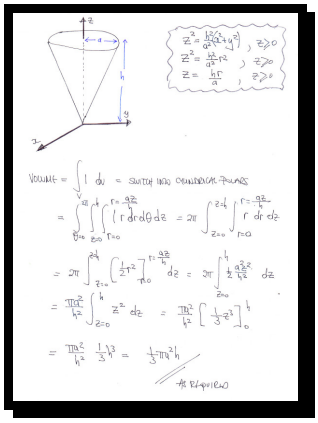


Question 17

Use cylindrical polar coordinates (r,θ,z) to show that the volume of a right circular cone of height h and base radius a is

$$\frac{1}{3}\pi a^2 h\,.$$

proof



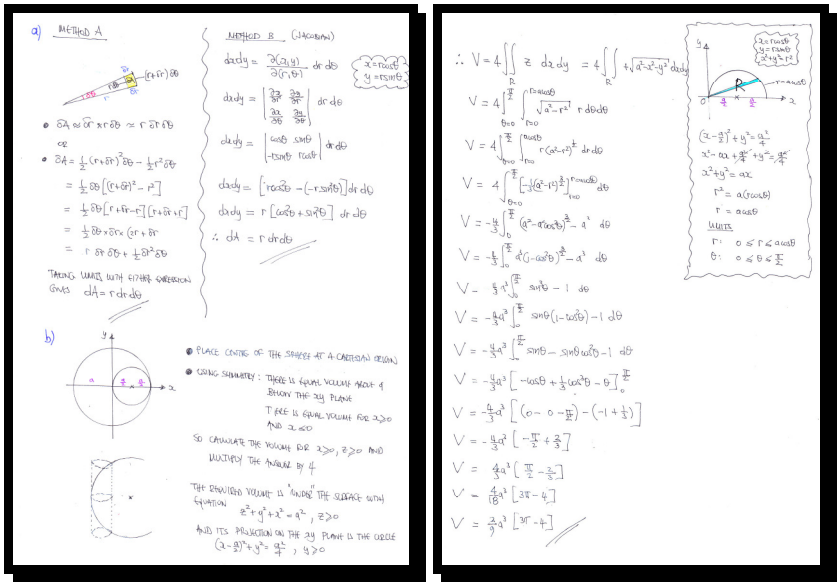
Question 18

a) Determine with the aid of a diagram or a Jacobian matrix an expression for the area element in plane polar coordinates, (r,θ).

A cylinder of radius $\frac{1}{2}a$ is cut out of a sphere of radius a.

b) Find a simplified expression for the volume of the cylinder, given that one of its generators passes through the centre of the sphere

$$\boxed{dxdy = r\,dr\,d\theta}\,, \qquad \boxed{V = \frac{2}{9}\left[3\pi - 4\right]}$$



Question 19

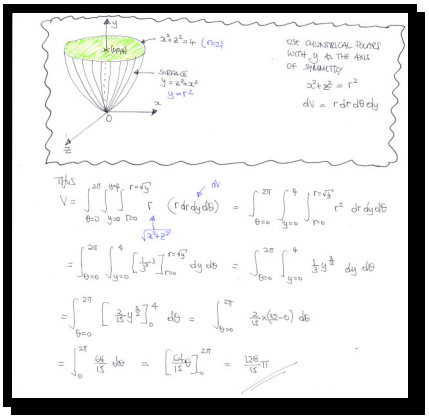
The region V is contained by the paraboloid with Cartesian equation

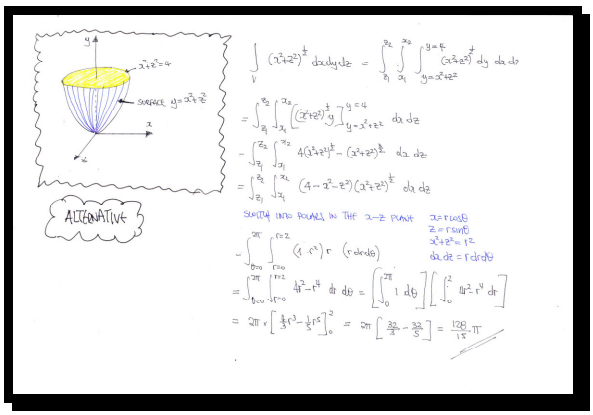
$$y = x^2 + z^2, \quad 0 \le y \le 4 .$$

Determine an exact simplified value for

$$\int_V \sqrt{x^2 + z^2} \, dx\, dy dz .$$

$$\frac{128}{15}\pi$$



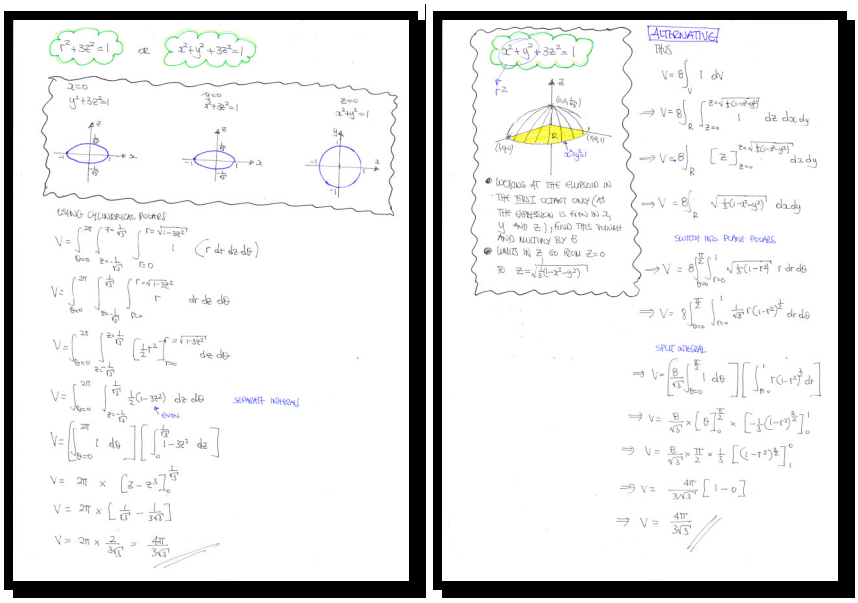


Question 20

Use cylindrical polar coordinates, (r,θ,z), to find the exact volume of the ellipsoid with Cartesian equation

$$x^2+y^2+3z^2=1.$$

$$\boxed{V=\frac{4\pi}{3\sqrt{3}}}$$



Question 21

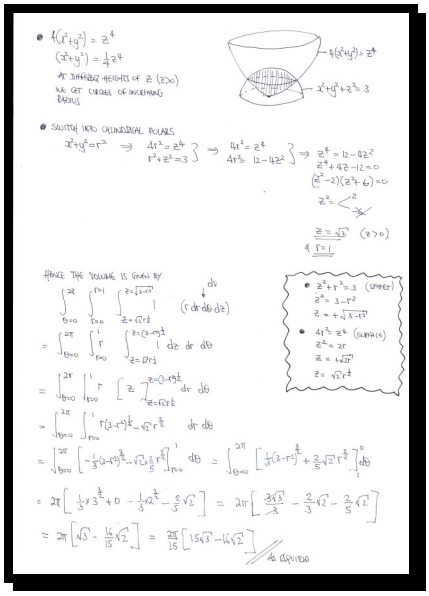
The finite region V is bounded by surfaces with Cartesian equations

$$z^4 = 4\left(x^2 + y^2\right),\ z \geq 0 \qquad \text{and} \qquad x^2 + y^2 + z^2 = 3\,,\ z \geq 0\,.$$

Use cylindrical polar coordinates (r, θ, z) to show that the volume of V is

$$\frac{2\pi}{15}\left(15\sqrt{3} - 16\sqrt{2}\right).$$

proof

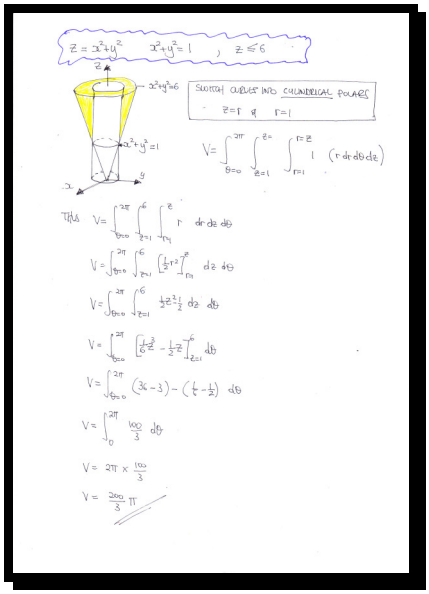


Question 22

Use cylindrical polar coordinates, (r,θ,z), to find the exact volume of the region defined by the following Cartesian inequalities

$$z \le x^2 + y^2, \quad x^2 + y^2 \ge 1 \quad \text{and} \quad z \le 6.$$

$$\boxed{V = \frac{200\pi}{3}}$$



Question 23

Use cylindrical polar coordinates, (r,θ,z), to find the volume of the region defined by the following Cartesian inequalities

$$z \geq 4-x^2-y^2, \quad z \leq 4+x^2+y^2 \quad \text{and} \quad x^2+y^2 \leq 4.$$

$$\boxed{V=16\pi}$$

