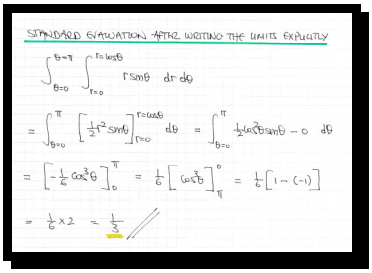
# MULTIVARIABLE INTEGRATION IN CARTESIAN

**Question 1**

Find the value of the following multiple integral

$$\int_0^{\pi}\int_0^{\cos\theta} r\sin\theta \; dr\, d\theta \, .$$

$\square$, $\boxed{\frac{1}{3}}$

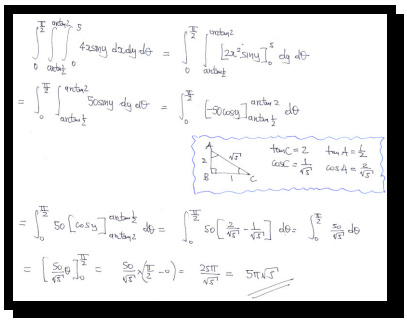



**Question 2**

Find the exact value of

$$\int_0^{\frac{\pi}{2}} \int_{\arctan\frac{1}{2}}^{\arctan 2} \int_0^5 4x\sin y \;\; dx\, dy\, d\theta \, .$$

$\boxed{5\pi\sqrt{5}}$

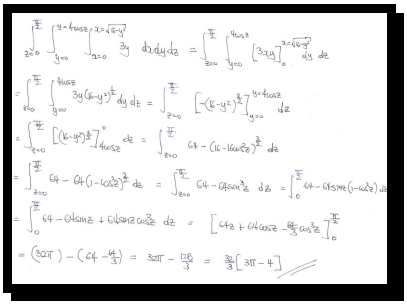


**Question 3**

Find the exact value of

$$\int_0^{\frac{\pi}{2}} \int_0^{4\cos z} \int_0^{\sqrt{16-y^2}} 3y \; dx\,dy\,dz \,.$$

$$\boxed{\frac{32}{3}\left[3\pi-4\right]}$$

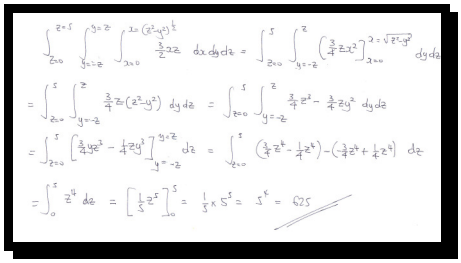


**Question 4**

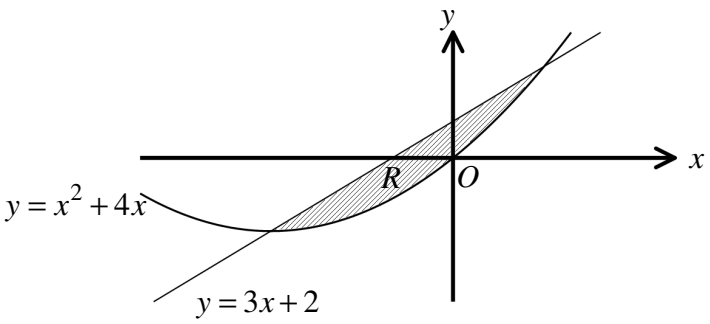
Find the value of

$$\int_0^5 \int_{-z}^{z} \int_0^{\sqrt{z^2-y^2}} \frac{3}{2}xz \; dx\,dy\,dz \,.$$

$$\boxed{625}$$



**Question 5**




The finite region $R$ in the $x$-$y$ plane, is bounded a curve and a straight line with the following equations.

$$y = x^2 + 4x \quad \text{and} \quad y = 3x + 2 .$$

Use double integration to find

$$\int_R 2 \; dx\, dy .$$

9

$\int_{-2}^{1}\int_{x^2+4x}^{3x+2} 2 \; dy\,dx = \int_{-2}^{1}\left[2y\right]_{y=x^2+4x}^{y=3x+2} dx = \int_{-2}^{1}(6x+4)-(2x^2+8x)\,dx$

$= \int_{-2}^{1} -2x^2-2x+4 \; dx = \left[-\frac{2}{3}x^3-x^2+4x\right]_{-2}^{1} = \left[\frac{2}{3}x^3+x^2-4x\right]_{1}^{-2}$

$= \left(\frac{15}{3}+4+8\right)-\left(\frac{2}{3}+1-4\right) = -\frac{15}{3}+12-\frac{2}{3}+3 = 9$

**Question 6**

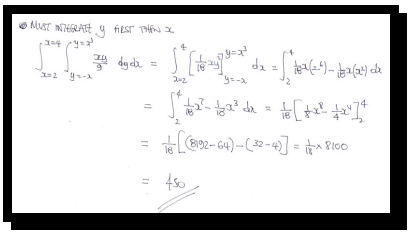
The finite region $R$ in the $x$-$y$ plane, is defined by the inequalities

$$y \le x^3 \ , \quad y \ge -x \quad \text{and} \quad 2 \le x \le 4 \ .$$

Determine the value of

$$\int_R \frac{xy}{9} \, dx \, dy \ .$$

$\boxed{450}$

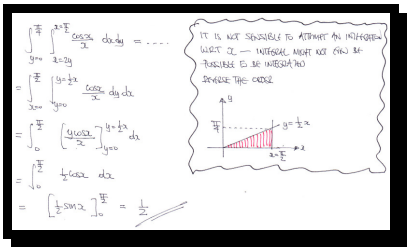


**Question 7**

Find the value of the following multiple integral

$$\int_0^{\frac{\pi}{4}} \int_{2y}^{\frac{\pi}{2}} \frac{\cos x}{x} \, dx dy \ .$$

$\boxed{\frac{1}{2}}$



**Question 8**

The finite region $R$ in the $x$-$y$ plane, is bounded a curve and a straight line with the following equations.

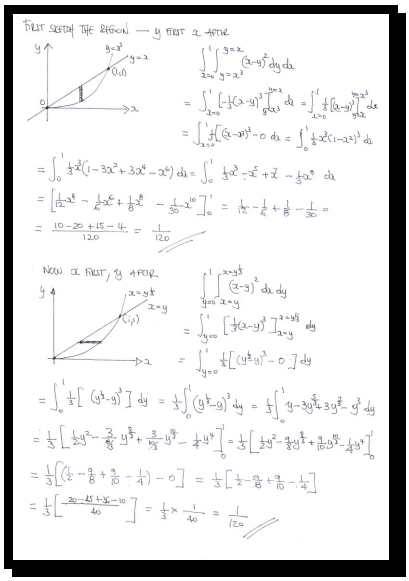
$$y = x^3 \ , \ x \geq 0 \qquad \text{and} \qquad y = x \, , \ x \geq 0 \, .$$

Determine the value for both

$$\iint_R (x-y)^2 \ dy\,dx \qquad \text{and} \qquad \iint_R (x-y)^2 \ dx\,dy \, ,$$

showing clearly that they have the same value.

$$\boxed{\frac{1}{120}}$$



**Question 9**

The finite region $R$ is bounded by the planes with equations

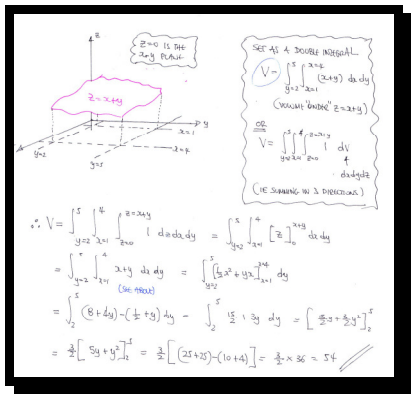
$$z=0, \quad x=1, \quad x=4, \quad y=2, \quad y=5,$$

and the surface with equation

$$z=x+y.$$

Determine the volume of $R$.

54



**Question 10**

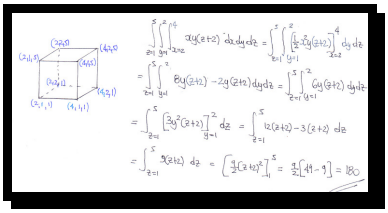
Integrate the function

$$f(x,y,z)\equiv xy(z+2)$$

over the finite region bounded by the cuboid with coordinates at

$$(2,1,1), \ (4,1,1), \ (4,2,1), \ (2,2,1), \ (2,1,5), \ (4,1,5), \ (4,2,5) \text{ and } (2,2,5).$$

180



**Question 11**

The finite region $R$ in the $x$-$y$ plane, is bounded by the straight lines with equations

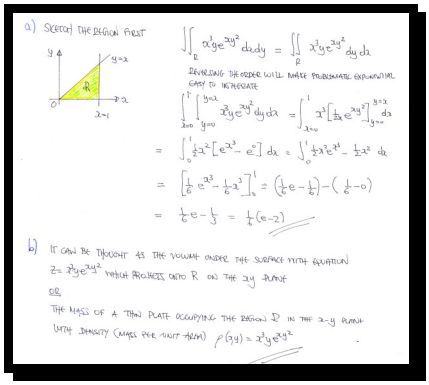
$$y = x\,, \qquad y = 0 \qquad \text{and} \qquad x = 1\,.$$

**a)** Determine the value of

$$I = \iint_R yx^3 \, \mathrm{e}^{xy^2} \, dx\, dy\,.$$

**b)** Give two distinct possible interpretations for the value of $I$.

$$\boxed{\frac{1}{6}(\mathrm{e}-2)}$$

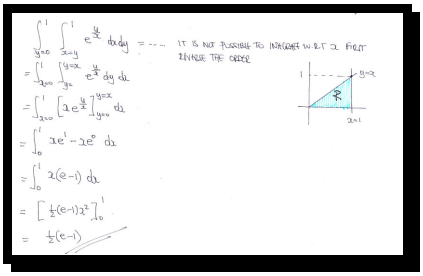



**Question 12**

Find the exact value of the following multiple integral

$$\int_0^1 \int_y^1 \mathrm{e}^{\frac{y}{x}} \, dx dy\,.$$

$$\boxed{\frac{1}{2}(\mathrm{e}-1)}$$



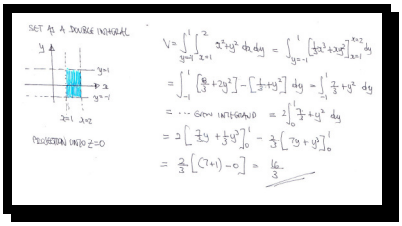
**Question 13**

The finite region $R$ is defined by the inequalities

$$1 \le x \le 2, \quad -1 \le y \le 1 \quad \text{and} \quad 0 \le z \le x^2 + y^2.$$

Determine the volume of $R$.

$$\frac{16}{3}$$

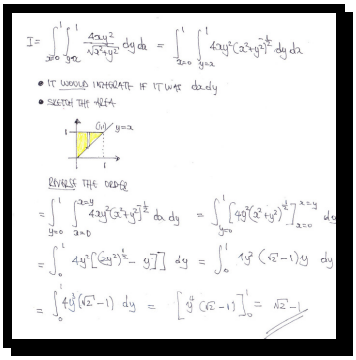


**Question 14**

Find the value of $I$ in exact surd form.

$$I = \int_0^1 \int_x^1 \frac{4xy^2}{\sqrt{x^2+y^2}} \, dy \, dx .$$

$$\sqrt{2}-1$$



**Question 15**

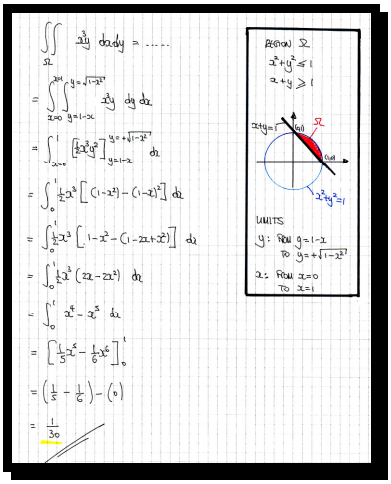
The finite region $\Omega$ in the $x$-$y$ plane, is defined by the inequalities

$$x^2 + y^2 \leq 1 \qquad \text{and} \qquad x + y \geq 1 .$$

Determine the value of

$$\int_{\Omega} x^3 y \ dx\,dy .$$

$\square$, $\boxed{\frac{1}{30}}$

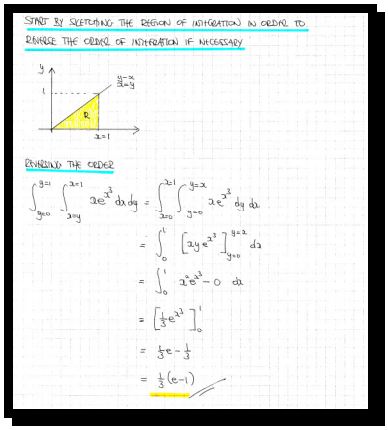


**Question 16**

Find the value of the following multiple integral

$$\int_0^1 \int_y^1 x\mathrm{e}^{x^3}\ dx\,dy\,.$$

$\square$, $\boxed{\frac{1}{3}(\mathrm{e}-1)}$



**Question 17**

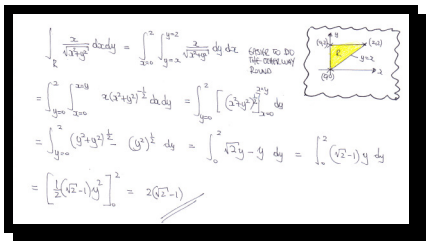
The finite region $R$ in the $x$-$y$ plane is defined as the rectilinear triangle with vertices at the points with coordinates

$$(0,0), \quad (0,2) \quad \text{and} \quad (2,2).$$

Determine an exact simplified value for

$$\int_R \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy .$$

$$\boxed{2\left(\sqrt{2}-1\right)}$$

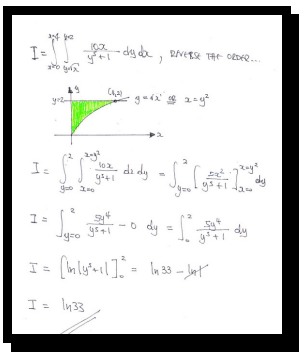


**Question 18**

Find the value of $I$ in exact logarithmic form.

$$I = \int_0^4 \int_{\sqrt{x}}^2 \frac{10x}{y^5+1}\, dy\, dx .$$

$$\boxed{\ln 33}$$

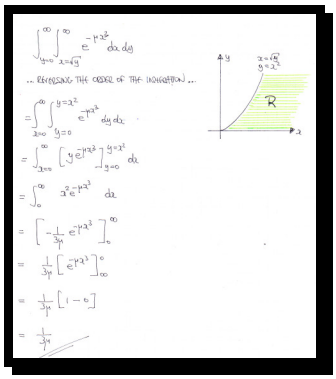


**Question 19**

Given that $\mu$ is a positive constant, find the value of the following integral.

$$\int_0^{\infty}\int_{\sqrt{y}}^{\infty} \mathrm{e}^{-\mu x^3}\, dx\, dy\,.$$

$$\boxed{\frac{1}{3\mu}}$$

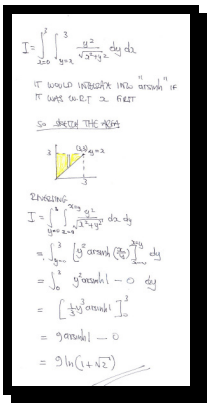


**Question 20**

Find the value of $I$ in exact logarithmic form.

$$I=\int_0^3\int_x^3 \frac{y^2}{\sqrt{x^2+y^2}}\, dy\, dx\,.$$

$$\boxed{9\ln\left(1+\sqrt{2}\right)}$$

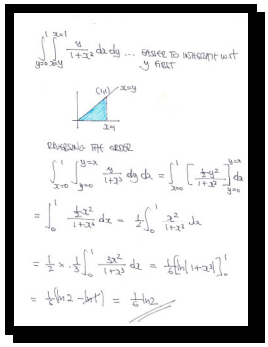


**Question 21**

Find the value of $I$ in exact logarithmic form.

$$I=\int_0^1\int_y^1 \frac{y}{1+x^3}\,dx\,dy\,.$$

$$\boxed{\frac{1}{6}\ln 2}$$

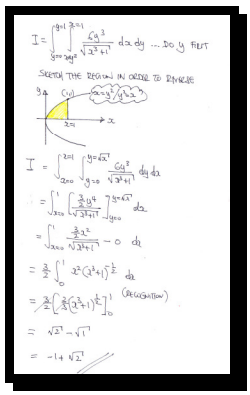


**Question 22**

Find the value of $I$ in exact surd form.

$$I=\int_0^1\int_{y^2}^1 \frac{6y^3}{\sqrt{x^3+1}}\,dx\,dy\,.$$

$$\boxed{\sqrt{2}-1}$$

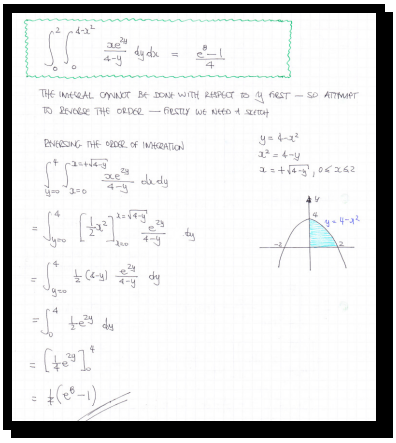


**Question 23**

Find the value of the following multiple integral

$$\int_0^2 \int_0^{4-x^2} \frac{x\mathrm{e}^{2y}}{4-y}\, dy\, dx\,.$$

$$\boxed{\frac{1}{4}\left(\mathrm{e}^8-1\right)}$$

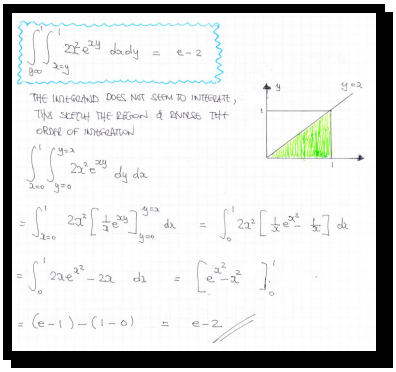


**Question 24**

Find the value of the following multiple integral

$$\int_0^1 \int_y^1 2x^2\,\mathrm{e}^{xy}\, dx\, dy\,.$$

$$\boxed{\mathrm{e}-2}$$

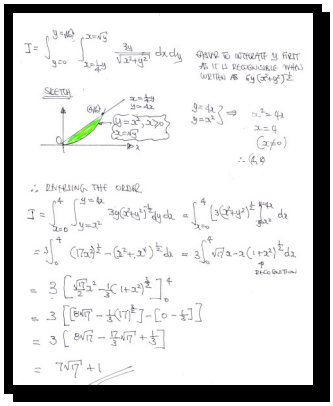


**Question 25**

Find the value of $I$ in exact surd form.

$$I=\int_{0}^{16}\int_{\frac{1}{4}y}^{\sqrt{y}}\frac{3y}{\sqrt{x^2+y^2}}\,dx\,dy\,.$$

$$\boxed{1+7\sqrt{17}}$$

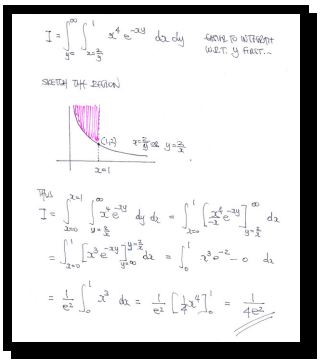


**Question 26**

Find the value of $I$ in exact simplified form.

$$I=\int_{2}^{\infty}\int_{\frac{2}{y}}^{1}x^4\,\mathrm{e}^{-xy}\,dx\,dy\,.$$

$$\boxed{\frac{1}{4\mathrm{e}^2}}$$



**Question 27**

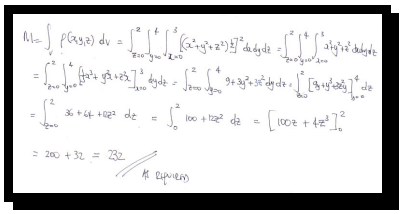
A rectangular block $B$ is bounded in the first octant by the planes with equations

$$x = 3, \quad y = 4, \text{ and } z = 2.$$

The density $\rho$ at any point $P$ within $B$, is equal to the square of the distance of $P$ from the origin $O$.

Show that the mass of $B$ is $232$.

proof



**Question 28**

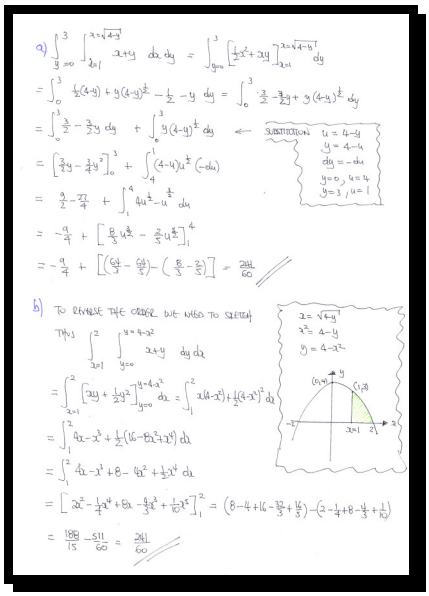
$$I=\int_0^3\int_1^{\sqrt{4-y}}\left(x+y\right)\ dxdy$$

**a)** Show clearly that $I=\dfrac{241}{60}$.

You may not reverse the order of integration in this part.

**b)** Carry out the integration in reverse order and hence verify the answer of $I$, obtained in part **(a)**.

proof

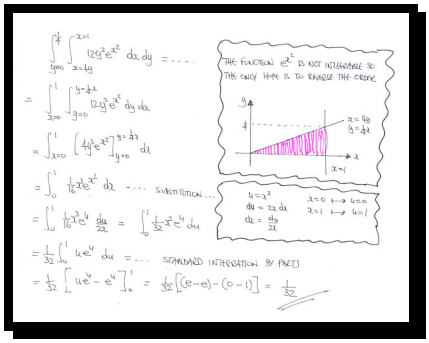


**Question 29**

Find the value of the following multiple integral

$$\int_0^{\frac{1}{4}} \int_{4y}^{1} 12y^2 \mathrm{e}^{x^2} \, dxdy \, .$$

$$\boxed{\frac{1}{32}}$$

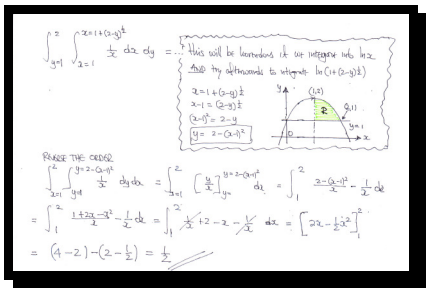


**Question 30**

Find the value of the following multiple integral

$$\int_1^2 \int_1^{1+\sqrt{2-y}} \frac{1}{x} \, dxdy \, .$$

$$\boxed{\frac{1}{2}}$$



**Question 31**

The finite region $R$ is in the first octant and is bounded by the plane with equation

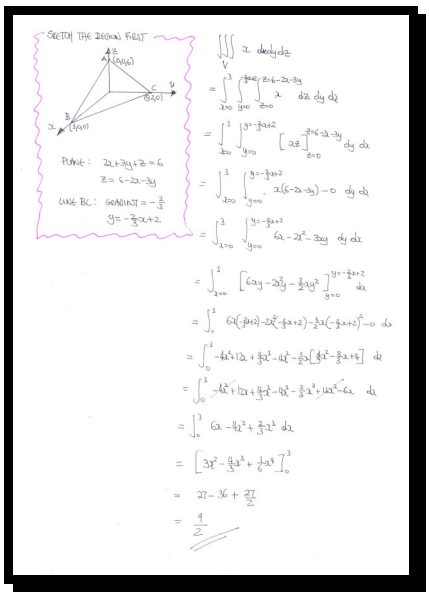
$$2x+3y+z=6.$$

Find the value of the integral

$$\iiint_V x \ dV,$$

where $V$ is the volume of $R$.

$$\frac{9}{2}$$



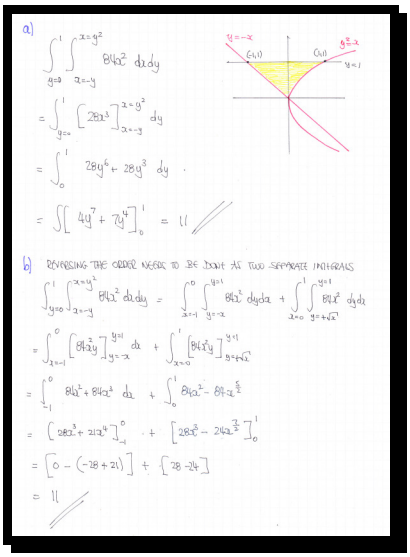
**Question 32**

$$I = \int_0^1 \int_{-y}^{y^2} 84x^2 \, dx\, dy \, .$$

**a)** Evaluate the above double integral.

**b)** Verify the answer of part **(a)** by reversing the order of integration.

11



**Question 33**

A surface has equation

$$z \equiv y^3 - xy \, .$$

A cuboid of infinite length is generated. This cuboid intersects the $x$-$y$ plane at the points with coordinates

$$(1,0,0) \, , \, (3,0,0) \, , \, (3,2,0) \text{ and } (1,2,0) \, .$$

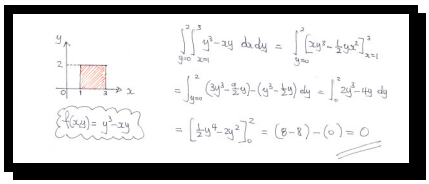
Show clearly that the volume of the finite region inside the cuboid for which

$$z \geq 0 \text{ and } z \leq y^3 - xy \, ,$$

is equal to the volume of the finite region inside the cuboid for which

$$z \leq 0 \text{ and } z \geq y^3 - xy \, .$$

$\boxed{0}$

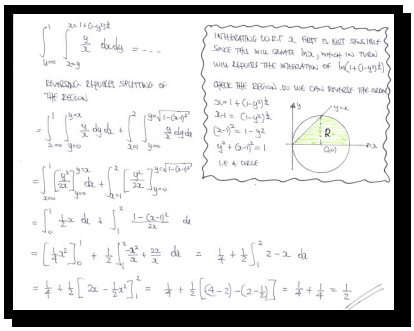


**Question 34**

Find the value of the following multiple integral

$$\int_0^1 \int_y^{1+\sqrt{1-y^2}} \frac{y}{x}\, dx\, dy\,.$$

$$\boxed{\frac{1}{2}}$$

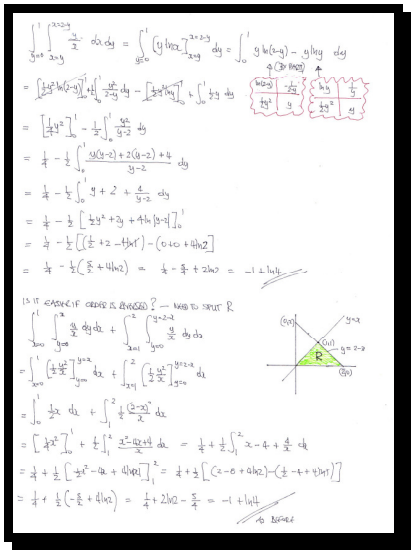


**Question 35**

Find the value of the following multiple integral

$$\int_0^1 \int_y^{2-y} \frac{y}{x}\, dx\, dy\,.$$

$$\boxed{-1+\ln 4}$$



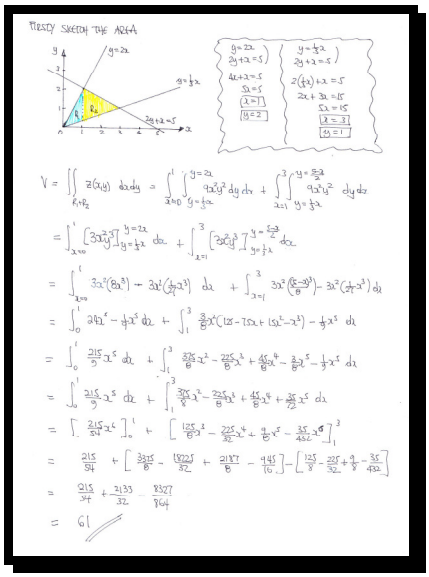
**Question 36**

The cross section of a right prism is the finite region in the $x$-$y$ plane, defined by the following inequalities.

$$y \leq 2x \ , \quad y \geq \frac{1}{3}x \quad \text{and} \quad 2y + x \leq 5 \, .$$

Show that the volume of the prism, for which $0 \leq z \leq 9x^2 y^2$, is $61$ cubic units.

proof



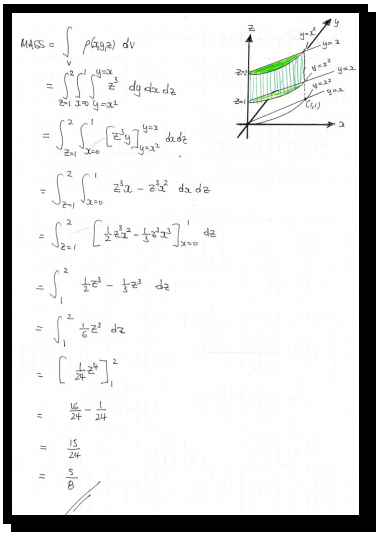
**Question 37**

A solid is bounded by the surfaces

$$y = x^2, \quad y = x, \quad z = 1 \quad \text{and} \quad z = 2.$$

Find the mass of the solid if its density is given by $\rho(z) = z^3$.

$\frac{5}{8}$

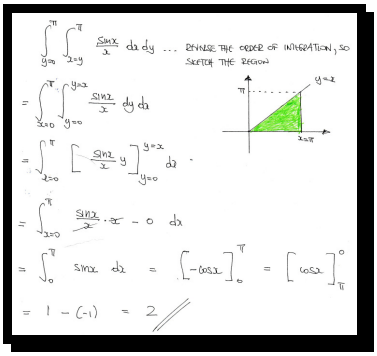


**Question 38**

Evaluate the integral

$$\int_0^{\pi}\int_y^{\pi} \frac{\sin x}{x}\, dxdy\,.$$

2

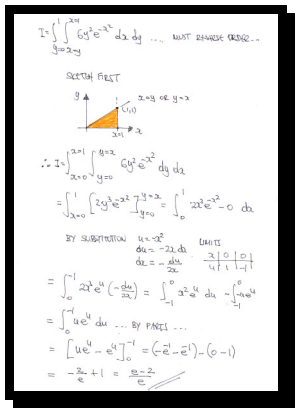


**Question 39**

Find the value of $I$ in exact simplified form.

$$I=\int_0^1\int_y^1 6y^2\,\mathrm{e}^{-x^2}\,dx\,dy\,.$$

$$\boxed{\frac{\mathrm{e}-2}{\mathrm{e}}}$$

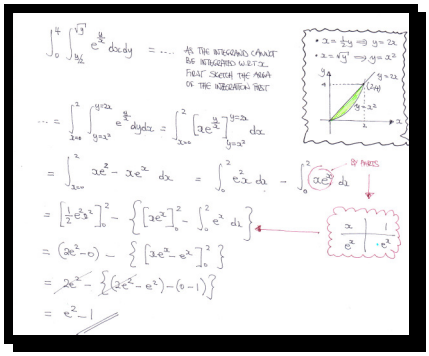


**Question 40**

Find the value of the following integral.

$$\int_0^4\int_{\frac{1}{2}y}^{\sqrt{y}} \mathrm{e}^{\frac{y}{x}}\,dx\,dy\,.$$

$$\boxed{\mathrm{e}^2-1}$$



**Question 41**

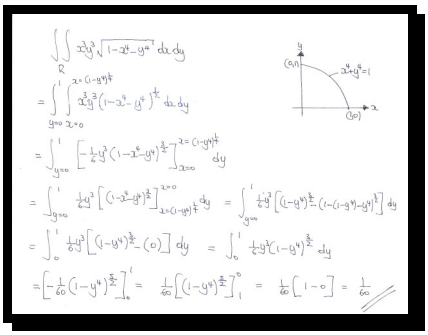
The finite region $R$ is bounded by the curve with Cartesian equation

$$x^4 + y^4 = 1, \quad x \geq 0, \quad y \geq 0.$$

Find the value of

$$\iint_R x^3 y^3 \sqrt{1 - x^4 - y^4} \, dx \, dy .$$

$$\boxed{\frac{1}{60}}$$



**Question 42**

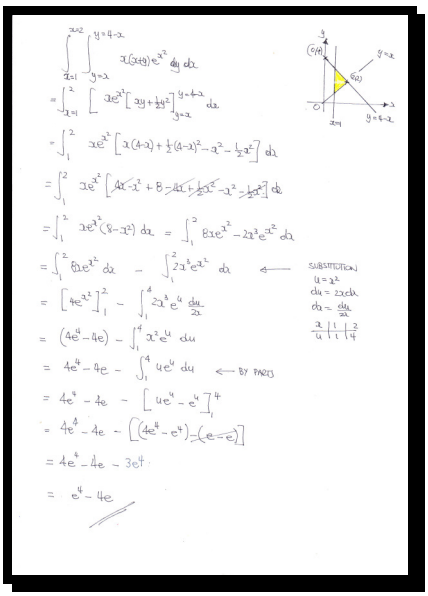
The finite region $R$ in the $x$-$y$ plane is bounded by the straight lines with equations

$$y = x \ , \quad x = 1 \quad \text{and} \quad y = 4 - x \, .$$

Determine an exact simplified value for

$$\int_R x(x+y)\mathrm{e}^{x^2} \, dx \, dy \, .$$

$$\boxed{\mathrm{e}^4 - 4\mathrm{e}}$$



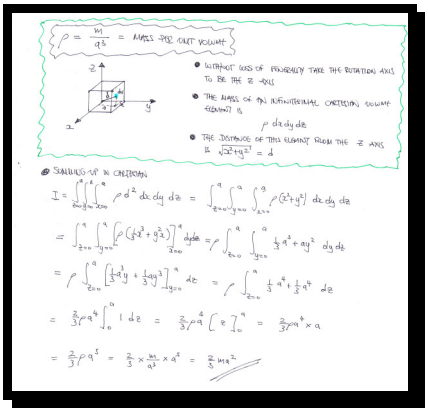
**Question 43**

A uniform solid cube, of mass $m$ and side length $a$, is free to rotate about one of its edges, $L$.

Use multiple integration in Cartesian coordinates, to find the moment of inertia of this cube about $L$, giving the answer in terms of $m$ and $a$.

*You may **not** use any standard rules or standard results about moments of inertia in this question apart from the definition of moment of inertia.*

$$\boxed{\frac{2}{3}ma^2}$$



**Question 44**

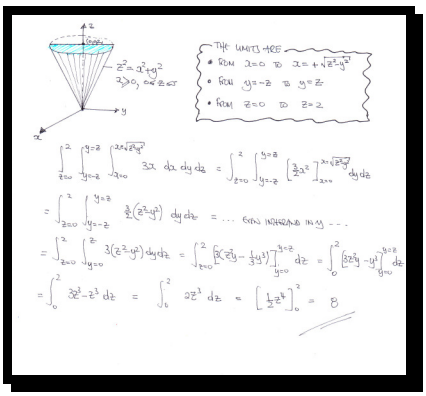
The finite region $V$ is defined as

$$z^2 = x^2 + y^2 \ , \quad x \geq 0 \, , \quad 0 \leq z \leq 2 \, .$$

Use Cartesian integration to determine the value of

$$\int_V 3x \ dV \, .$$

8



**Question 45**

The finite region $R$ in the $x$-$y$ plane is defined as

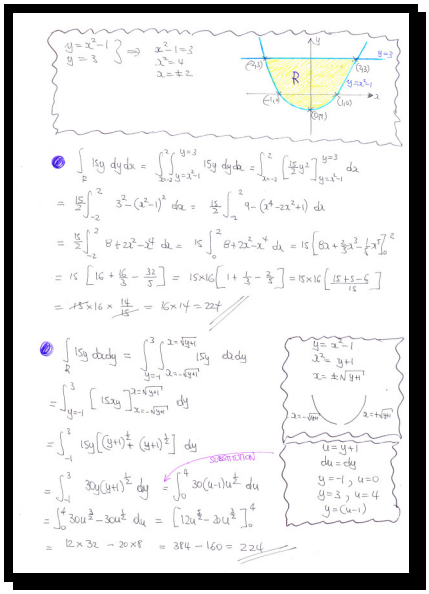
$$R=\left\{(x,y):y\le 3\cap y\ge x^2-1\right\}.$$

**a)** Evaluate, without reversing the order of integration

$$\int_R 15y\ dy\,dx\,.$$

**b)** Verify the answer of part **(a)**, by carrying out the integration in reverse order.

both yield 224



**Question 46**

A thin plate occupies the region in the $x$-$y$ plane with equation

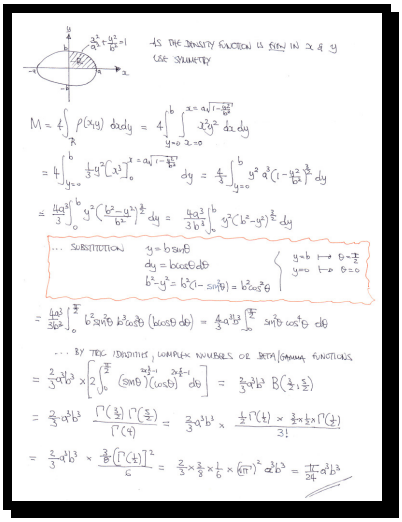
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The mass per unit area of the plate $\rho$, is given by

$$\rho(x,y)=x^2y^2.$$

Find a simplified expression for the mass of the plate.

$$\frac{\pi a^3 b^3}{24}$$

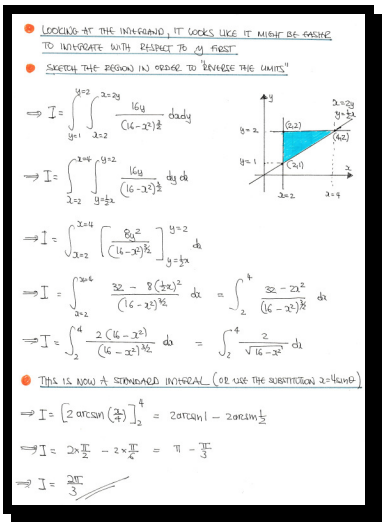


**Question 47**

Find the value of $I$ in exact simplified form.

$$I = \int_{1}^{2} \int_{2}^{2y} \frac{16y}{\left(16-x^2\right)^{\frac{3}{2}}} \, dx \, dy \, .$$

$\square$, $\boxed{\dfrac{2\pi}{3}}$



**Question 48**

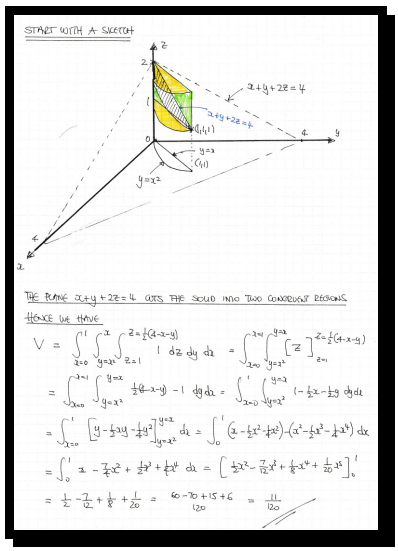
A solid $S$ is bounded by the surfaces

$$y = x^2, \quad y = x, \quad z = 1 \quad \text{and} \quad z = 2.$$

Find the volume of the finite region bounded by $S$ and the plane with equation

$$x + y + 2z = 4.$$

$$\frac{11}{120}$$



**Question 49**

A thin plate occupies the region in the $x$-$y$ plane defined by the inequalities

$$0 \le x \le 2 \quad \text{and} \quad 0 \le y \le 2x .$$

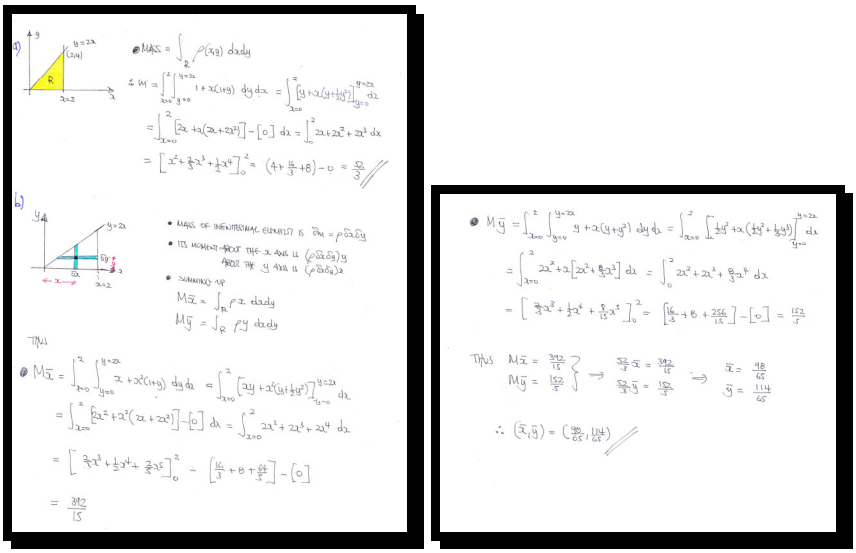
The mass per unit area of the plate $\rho$, is given by

$$\rho(x, y) = 1 + x(1 + y) .$$

**a)** Find the mass of the plate.

**b)** Determine the coordinates of the centre of mass of the plate.

$$\boxed{m = \frac{52}{3}}, \quad \boxed{(\bar{x}, \bar{y}) = \left(\frac{98}{65}, \frac{114}{65}\right)}$$



**Question 50**

The finite region $R$ in the $x$-$y$ plane is defined as the rectilinear triangle with vertices at the points with coordinates

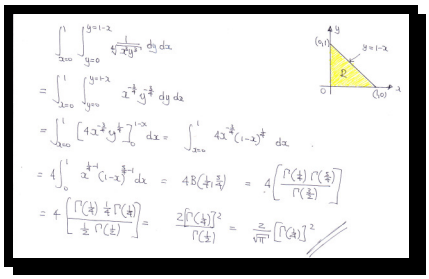
$$(0,0),\ (0,1) \text{ and } (1,0).$$

Determine an exact simplified value for

$$\int_R \frac{1}{\sqrt[4]{x^3y^3}}\,dx\,dy.$$

[You may leave the final answer in terms of $\pi$ and Gamma functions]

$$\frac{2}{\sqrt{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

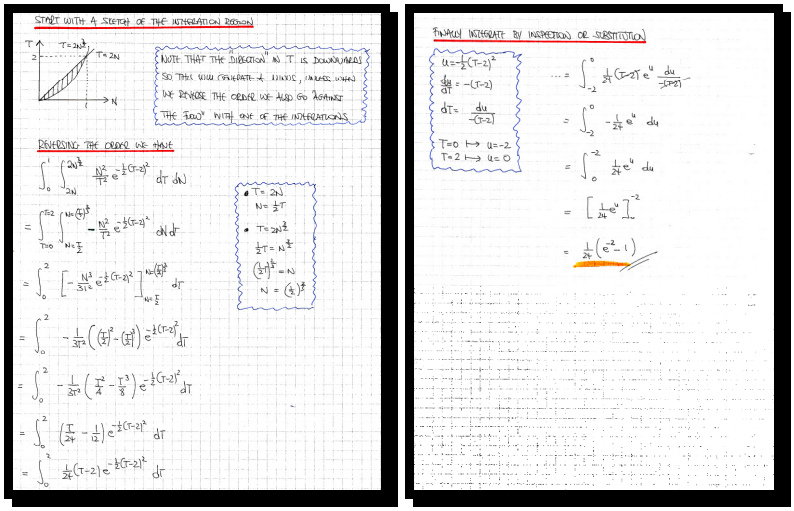


**Question 51**

$$I=\int_{0}^{1}\int_{2N}^{2N^{\frac{3}{2}}}\frac{N^2\exp\left[-\frac{1}{2}(T-2)^2\right]}{T^2}\,dT\,dN\,.$$

Determine, in exact simplified form, the value of $I$.

$\boxed{\phantom{00}}$, $\boxed{I=\frac{1}{24}\left(\mathrm{e}^{-2}-1\right)}$



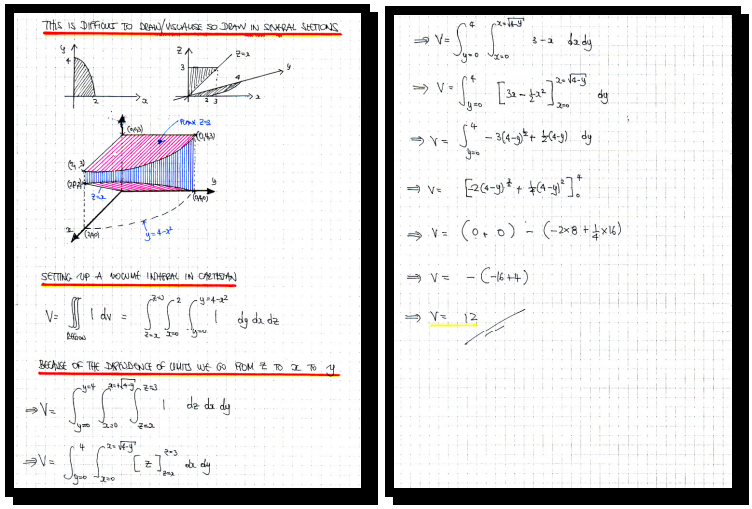
**Question 52**

The finite region $V$ in the first octant, is bounded by the surfaces with equations

$$y = 4 - x^2 \ , \quad z = x \quad \text{and} \quad z = 3 \, .$$

Find the volume of the solid.

$\boxed{\phantom{00}}$, $\boxed{12}$



**Question 53**

The finite region $R$ is defined as

$$R=\left\{ (x,y,z):0\le z\le 1-|x|-|y| \right\}.$$

Evaluate the triple integral

$$\iiint_R xy+z^2\ dxdydz\ .$$

$$\frac{1}{15}$$



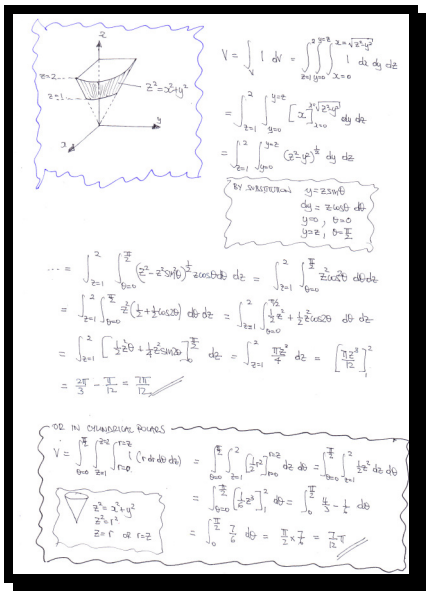
**Question 54**

A finite region is defined as

$$z^2 = x^2 + y^2 \,,\ x \geq 0 \,,\ y \geq 0 \,,\ 1 \leq z \leq 2$$

Show by integration in Cartesian that the volume of this region is $\dfrac{7\pi}{12}$.

proof

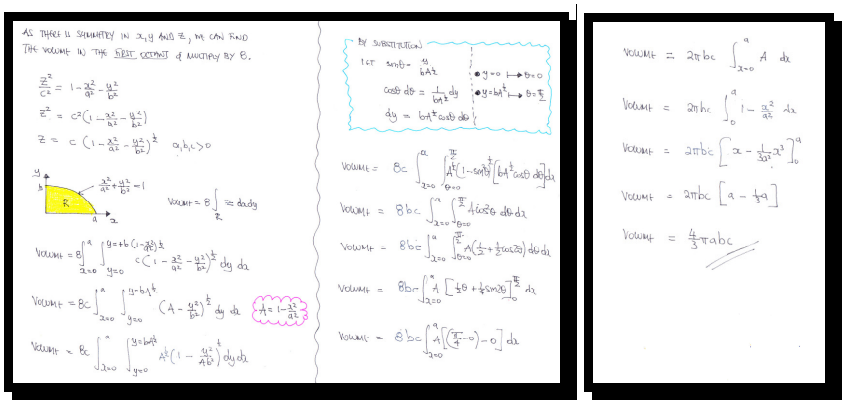


**Question 55**

Use integration in Cartesian coordinates to find, in exact simplified form, the volume of the ellipsoid with equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

$$\boxed{V=\frac{4}{3}\pi abc}$$



**Question 56**

The finite region $V$ is defined as

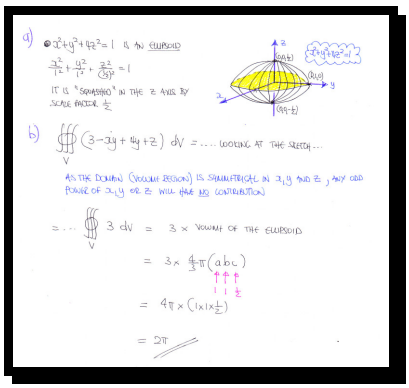
$$V = \left\{ (x, y, z) : x^2 + y^2 + 4z^2 \leq 1 \right\}.$$

**a)** Sketch and name the region $V$.

**b)** Evaluate

$$\iiint_V (3 - xy + 4y + z)\, dx\, dy\, dz.$$

$2\pi$



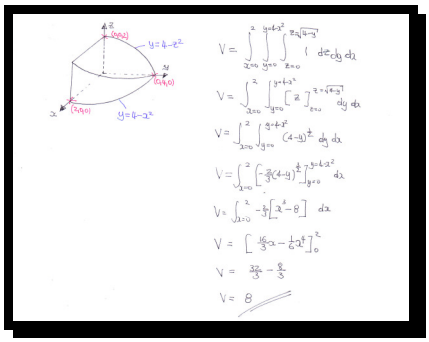
**Question 57**

The finite region $V$ in the first octant, is bounded by the surfaces with equations

$$y = 4 - x^2 \quad \text{and} \quad y = 4 - z^2 .$$

Find the volume of the solid defined as the interior of $V$.

8



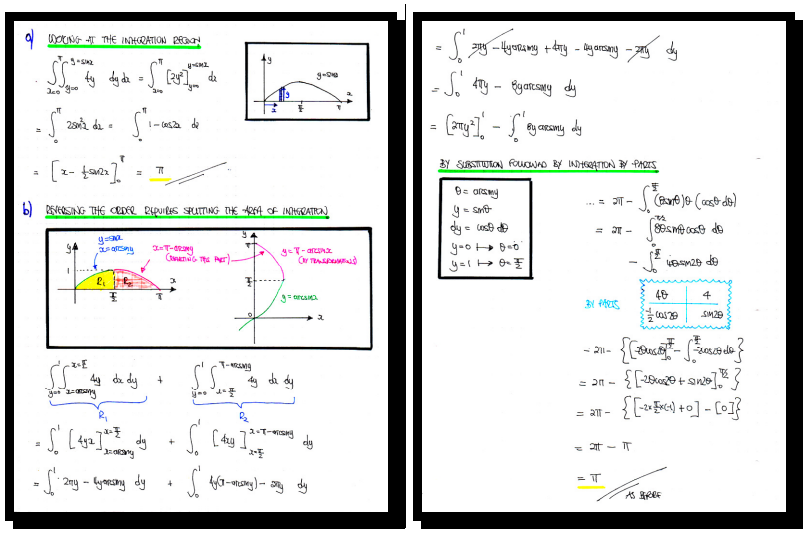
## Question 58

$$I = \int_0^{\pi} \int_0^{\sin x} 4y \; dy \, dx \, .$$

**a)** Determine the exact value of $I$ .

**b)** Verify the answer of part **(a)** by reversing the order of integration.

$\boxed{\phantom{00}}$ , $\boxed{I = \pi}$

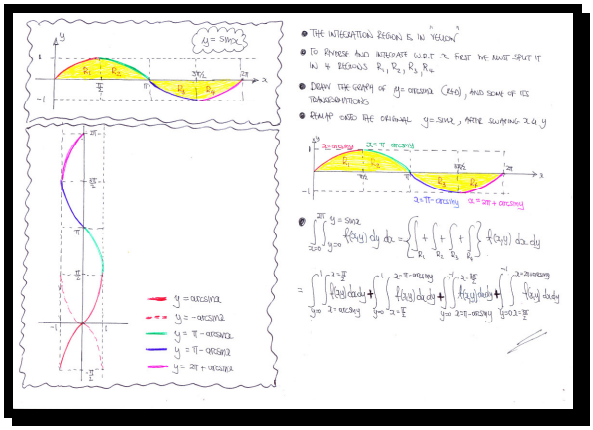


**Question 59**

$$I=\int_{0}^{2\pi}\int_{0}^{\sin x} f\left(x,y\right)\ dy\,dx\,.$$

Write an expression for $I$, by reversing the order of integration.

$$I=\left\{\int_{0}^{1}\int_{\arcsin y}^{\frac{\pi}{2}}+\int_{0}^{1}\int_{\frac{\pi}{2}}^{\pi-\arcsin y}+\int_{0}^{-1}\int_{\pi-\arcsin y}^{\frac{3\pi}{2}}+\int_{0}^{-1}\int_{\frac{3\pi}{2}}^{2\pi-\arcsin y}\right\} f\left(x,y\right)\ dx\ dy$$

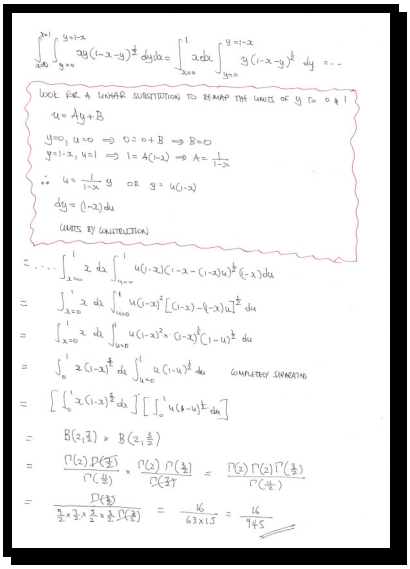


**Question 60**

$$I=\int_0^1\int_0^{1-x} xy\left(1-x-y\right)^{\frac{1}{2}}\,dydx\,.$$

Determine the exact value of $I$ by transforming it into an expression involving Beta functions.

$$\frac{16}{945}$$



**Question 61**

A finite region $R$ defined by the inequalities

$$x^3+y^3+z^3\le 1,\quad x\ge 0,\quad y\ge 0,\quad z\ge 0.$$

Show that the volume of $R$ is

$$\left[\frac{1}{k}\Gamma\left(\frac{1}{k}\right)\right]^k,$$

where $k$ is a positive integer to be found.

$$\boxed{k=3}$$

