LINE INTEGRALS

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IN 2 DIMENSIONAL CARTESIAN COORDINATES

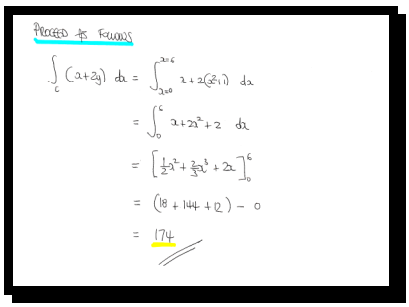
Question 1

Evaluate the integral

$$\int_C (x+2y)\,dx,$$

where C is the path along the curve with equation $y=x^2+1$, from $(0,1)$ to $(6,37)$.

$\boxed{174}$



Question 2

It is given that the vector function $\mathbf{F}$ satisfies

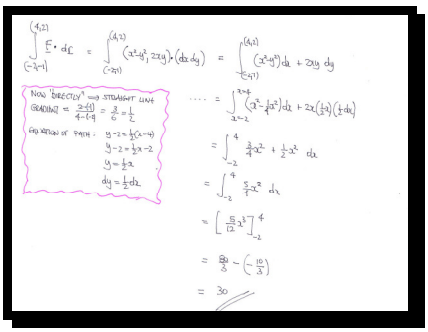
$$\mathbf{F}=\left(x^2-y^2\right)\mathbf{i}+\left(2xy\right)\mathbf{j}\,.$$

Evaluate the line integral

$$\int_{(-2,-1)}^{(4,2)}\mathbf{F}\cdot\mathbf{dr}\,,$$

along a path joining directly the points with Cartesian coordinates $(-2,-1)$ and $(4,2)$.

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Question 3

The path along the straight line with equation $y = x + 2$, from $A(0,2)$ to $B(3,5)$, is denoted by C.

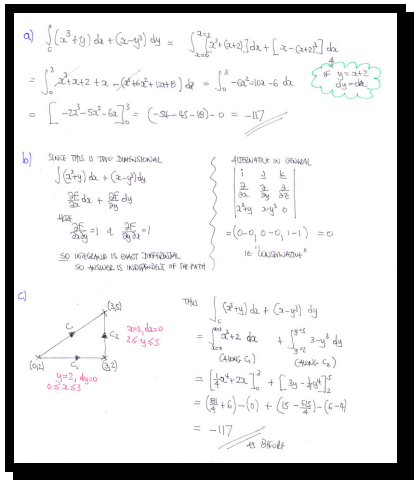
a) Evaluate the integral

$$\int_C \left(x^3 + y\right) dx + \left(x - y^3\right) dy .$$

b) Show that the integral is independent of the path chosen from A to B.

c) Verify the independence of the path by evaluating the integral of part **(a)** along a different path from A to B.

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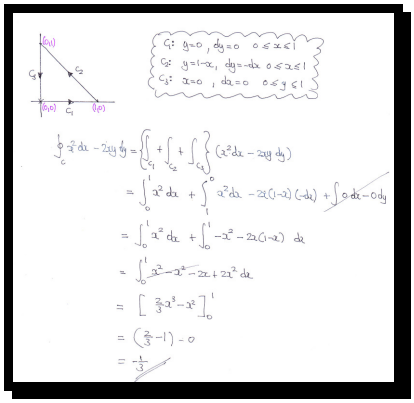
Question 4

The path along the perimeter of the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$, is denoted by C.

Evaluate the integral

$$\oint_C x^2\ dx - 2xy\ dy .$$

$$\boxed{-\frac{1}{3}}$$



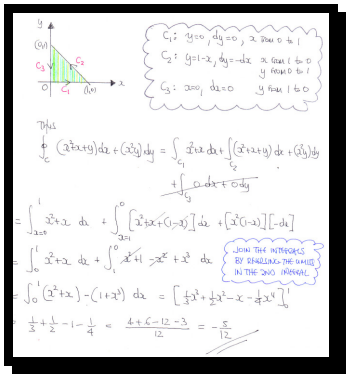
Question 5

The path along the perimeter of the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$, is denoted by C.

Evaluate the integral

$$\oint_C \left(x^2+x+y\right)\,dx+\left(x^2y\right)\,dy\,.$$

$$\boxed{-\frac{5}{12}}$$



Question 6

The functions F and G are defined as

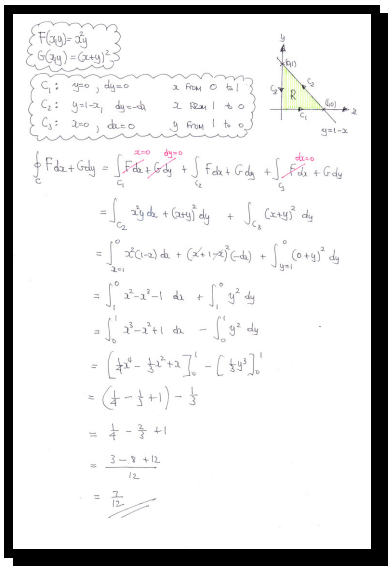
$$F(x,y)=x^2y \quad \text{and} \quad G(x,y)=(x+y)^2$$

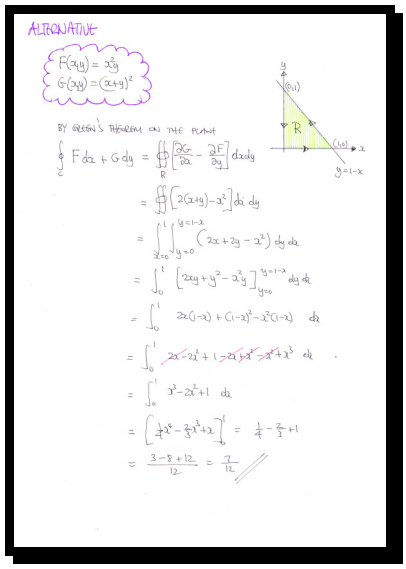
The anticlockwise path along the perimeter of the triangle whose vertices are located at $(0,0)$, $(1,0)$ and $(0,1)$, is denoted by C.

Evaluate the line integral

$$\int_C F\,dx+G\,dy\,.$$

$$\boxed{\frac{7}{12}}$$





Question 7

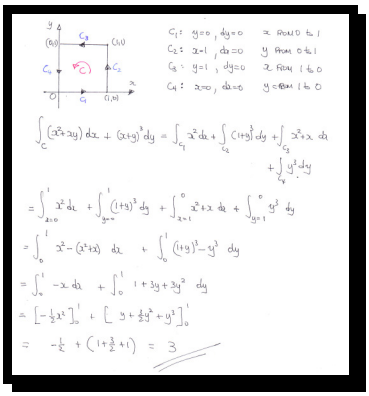
The anticlockwise path along the perimeter of the square whose vertices are located at the points $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, is denoted by C.

Evaluate the line integral

$$\int_C \left(x^2+xy\right)dx+\left(x+y\right)^3 dy\,.$$

You may not use Green's theorem in this question.

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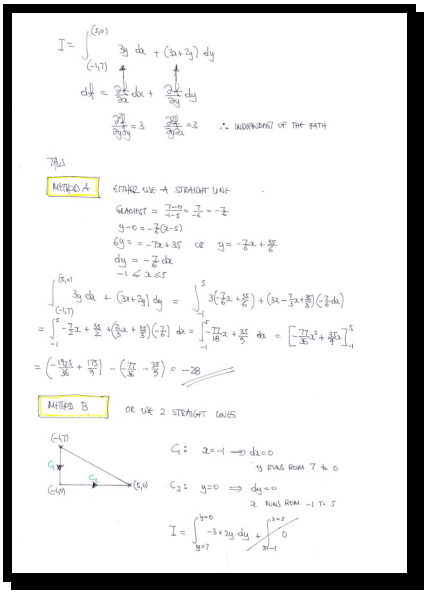
Question 8

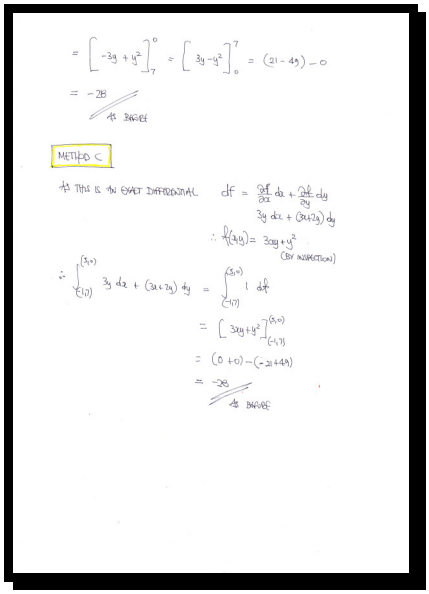
Evaluate the integral

$$\int_{(-1,7)}^{(5,0)} (3y)\,dx + (3x+2y)\,dy\,,$$

along a path joining the points with Cartesian coordinates $(-1,7)$ and $(5,0)$.

-28





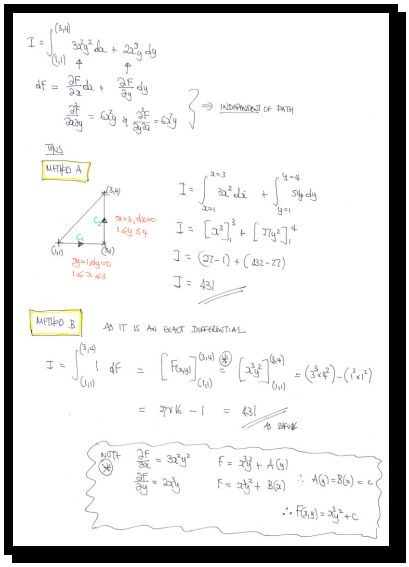
Question 9

Evaluate the integral

$$\int_{(1,1)}^{(3,4)} \left(3x^2y^2\right)\,dx \;+\; \left(2x^3y\right)\,dy\,,$$

along a path joining the points with Cartesian coordinates $(1,1)$ and $(3,4)$.

$$\boxed{431}$$



Question 10

$$\mathbf{F}(x,y)\equiv\left(2xy^2+\cos x\right)\mathbf{i}+\left(2x^2y-\sin y\right)\mathbf{j}.$$

Show that the vector field $\mathbf{F}$ is conservative, and hence evaluate the integral

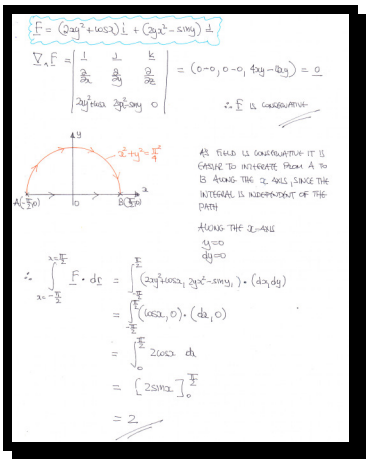
$$\int_C \mathbf{F}\cdot\mathbf{dr},$$

where C is the arc of the circle with equation

$$x^2+y^2=\frac{\pi^2}{4},\ y\geq 0,$$

from $A\left(-\frac{\pi}{2},0\right)$ to $B\left(\frac{\pi}{2},0\right)$.

[2]



Question 11

In this question α, β *and* γ *are positive constants.*

$$\mathbf{F} = -\alpha y^2 \mathbf{e_x} + \beta x^2 \mathbf{e_y}.$$

A particle of mass m is moving on the x-y plane, under the action of $\mathbf{F}$.

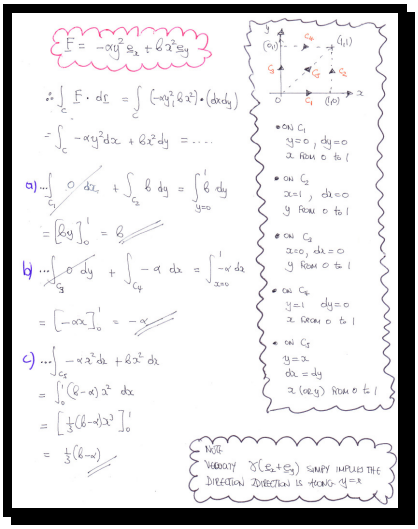
Find the work done by $\mathbf{F}$ on the particle in moving it from the Cartesian origin O to the point $(1,1)$, in each of the following cases.

a) Directly from O to $(1,0)$, then directly from $(1,0)$ to $(1,1)$.

b) Directly from O to $(0,1)$, then directly from $(0,1)$ to $(1,1)$.

c) Moving the particle with velocity $\mathbf{v} = \gamma\left(\mathbf{e_x} + \mathbf{e_y}\right)$.

$$\boxed{W_1 = \beta}, \quad \boxed{W_2 = -\alpha}, \quad \boxed{W_3 = \tfrac{1}{3}(\beta - \alpha)}$$



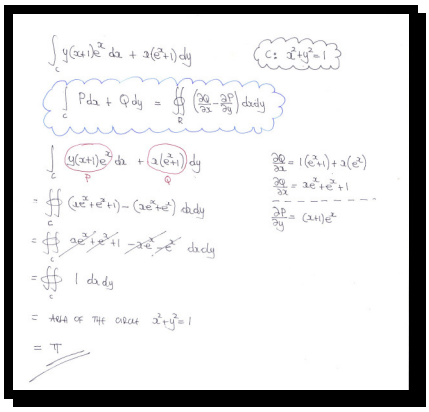
Question 12

Evaluate the line integral

$$\oint_C \left[y(x+1)\mathrm{e}^x \, dx \; + \; x\left(\mathrm{e}^x+1\right) \; dy \right],$$

where C is a circle of radius 1, centre at the origin O, traced anticlockwise.

$\boxed{\pi}$



Question 13

It is given that the vector function $\mathbf{F}$ satisfies

$$\mathbf{F}=\left(\sin x^3-xy\right)\mathbf{i}+\left(x+y^3\sin y\right)\mathbf{j}.$$

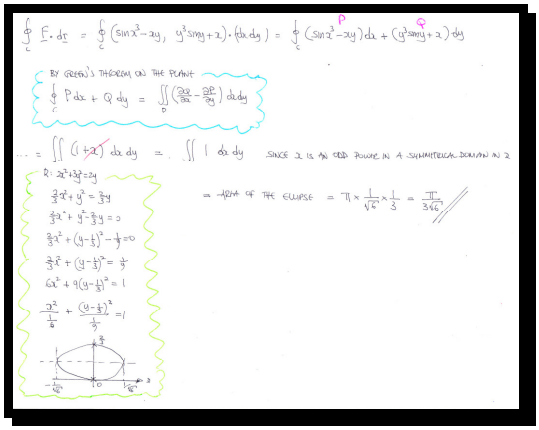
Evaluate the line integral

$$\oint_C \mathbf{F}\cdot\mathbf{dr}\,,$$

where C is the ellipse with Cartesian equation

$$2x^2+3y^2=2y\,.$$

$$\boxed{\frac{\pi}{3\sqrt{6}}}$$



Question 14

It is given that the vector function $\mathbf{F}$ satisfies

$$\mathbf{F}=\left[x\cos x\right]\mathbf{i}+\left[15xy+\ln\left(1+y^3\right)\right]\mathbf{j}.$$

Evaluate the line integral

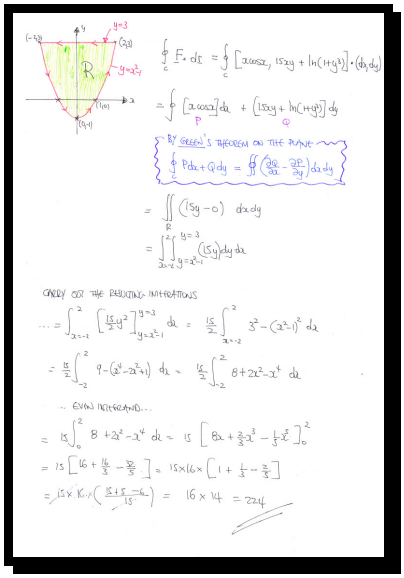
$$\oint_C \mathbf{F}\cdot\mathbf{dr},$$

where C is the curve

$$\left\{(x,y):y=3,\ -2\leq x\leq 2\right\}\cup\left\{(x,y):y=x^2-1,\ -2\leq x\leq 2\right\},$$

traced in an anticlockwise direction.

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LINE INTEGRALS

2 DIMENSIONAL PARAMETERIZATIONS

Question 1

The path along the semicircle with equation

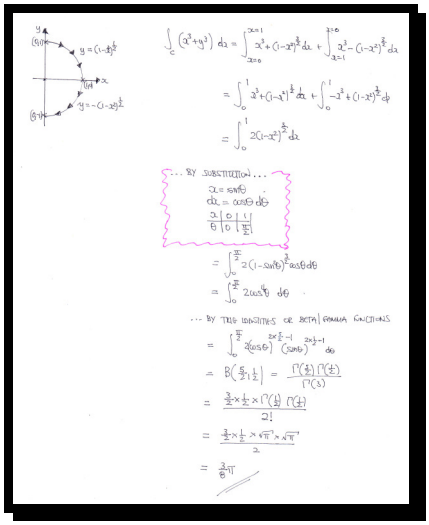
$$x^2+y^2=1,\ x\geq 0$$

from $A(0,1)$ to $B(0,-1)$, is denoted by C.

Evaluate the integral

$$\int_C \left(x^3+y^3\right)dx.$$

$$\frac{3}{8}\pi$$



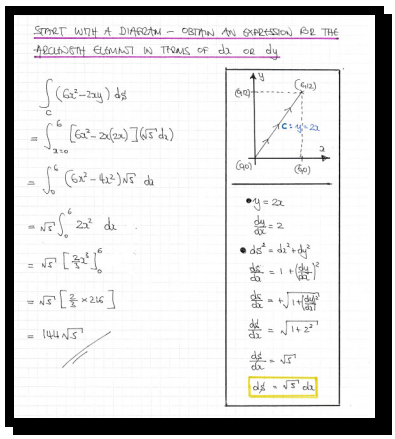
Question 2

Evaluate the integral

$$\int_{(0,0)}^{(6,12)} \left(6x^2 - 2xy\right) ds \,,$$

where s is the arclength along the straight line segment from $(0,0)$ to $(6,12)$.

$\boxed{\phantom{144\sqrt{5}}}$, $\boxed{144\sqrt{5}}$



Question 3

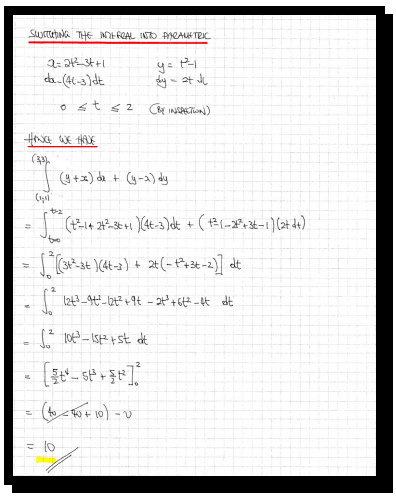
Evaluate the integral

$$\int_{(1,-1)}^{(3,3)} \left(y+x\right) dx + \left(y-x\right) dy ,$$

along the curve with parametric equations

$$x = 2t^2 - 3t + 1 \quad \text{and} \quad y = t^2 - 1 .$$

[ ], [10]



Question 4

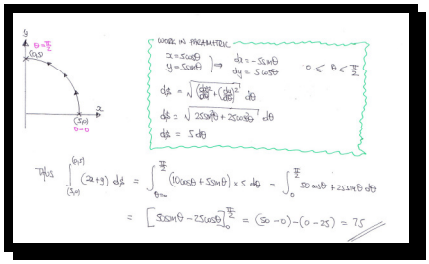
Evaluate the line integral

$$\int_{(5,0)}^{(0,5)} (2x+y)\, ds\,,$$

where s is the arclength along the quarter circle with equation

$$x^2 + y^2 = 25\,.$$

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Question 5

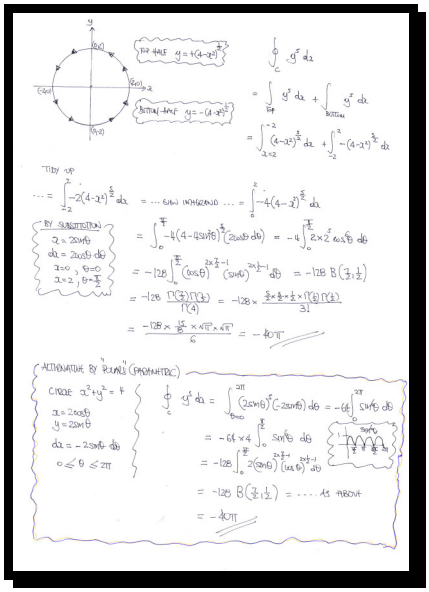
Evaluate the line integral

$$\oint_C y^5 \, dx,$$

where C is a circle of radius 2, centre at the origin O, traced anticlockwise.

You may not use Green's theorem in this question.

$\boxed{-40\pi}$



Question 6

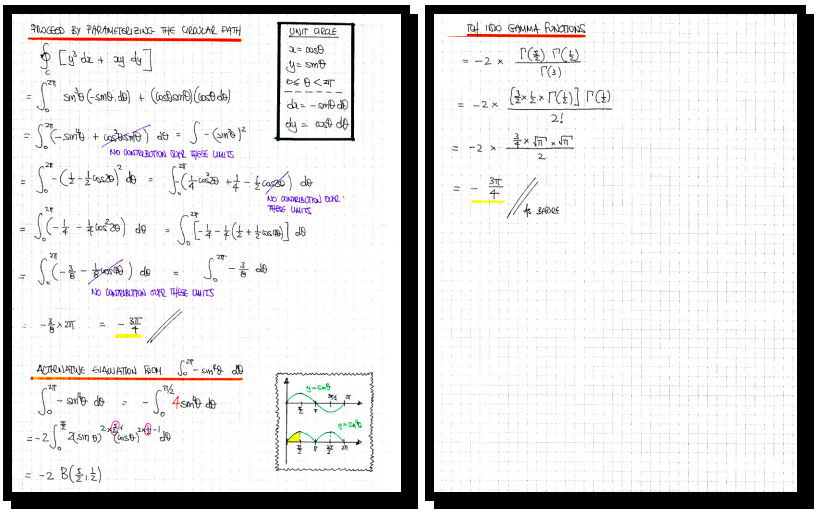
Evaluate the line integral

$$\oint_C \left[y^3 dx + (xy) dy \right] ,$$

where C is a circle of radius 1, centre at the origin O, traced anticlockwise.

You may not use Green's theorem in this question.

$\square$, $\boxed{-\frac{3\pi}{4}}$



Question 7

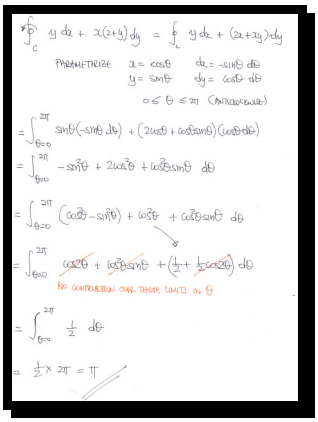
Evaluate the line integral

$$\oint_C \left[y\,dx + x(2+y)\,dy \right] ,$$

where C is a circle of radius 1, centre at the origin O, traced anticlockwise.

You may not use Green's theorem in this question.

$\boxed{\pi}$



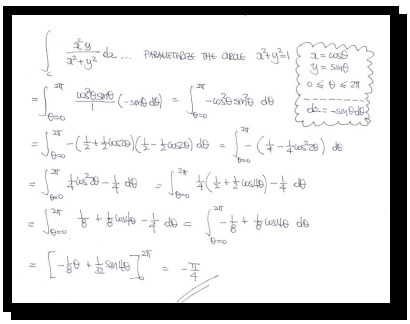
Question 8

Evaluate the line integral

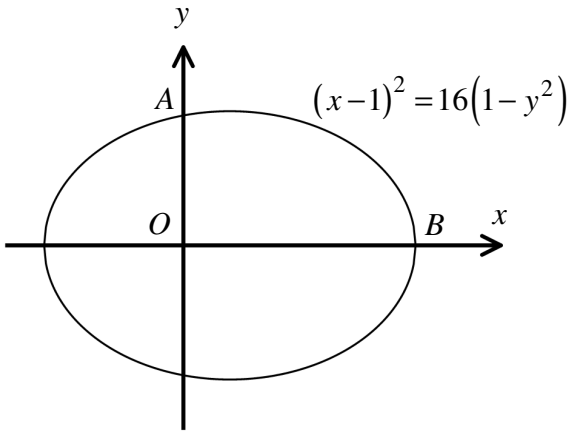
$$\oint_C \frac{x^2 y}{x^2+y^2}\, dx,$$

where C is a circle of radius 1, centre at the origin O, traced anticlockwise.

$$-\frac{\pi}{4}$$



Question 9



The figure above shows the ellipse with equation

$$(x-1)^2=16(1-y^2).$$

The ellipse meets the positive y and x axes at the points A and B, respectively, as shown in the figure.

The elliptic path C is the clockwise section from A to B.

Determine the value of each of the following line integrals.

a) $\displaystyle\int_C \left[\left(x^2+xy\right)\,dx+\left(y^2+\frac{1}{2}x^2\right)\,dy\right]$.

b) $\displaystyle\int_C \left[y^3\,dx+\frac{1}{16}(x-1)^3\,dy\right]$.

$\boxed{}$, $\boxed{\frac{125}{3}-\frac{5}{64}\sqrt{15}}$, $\boxed{\frac{1}{4}\sqrt{15}}$

[solution overleaf]



a) FIRSTLY DETERMINE THE CO-ORDINATES OF A & B

- $x=0$: $1=16(1-y^2)$, $\frac{1}{16}=1-y^2$, $y^2=\frac{15}{16}$, $y=\pm\frac{1}{4}\sqrt{15}$, $A\left(0,\frac{1}{4}\sqrt{15}\right)$
- $y=0$: $(x-1)^2=16$, $x-1=\pm 4$, $x=5$, $B(5,0)$

NEXT WE TEST WHETHER THE INTEGRAL IS INDEPENDENT OF THE PATH

$$\frac{\partial \phi}{\partial x}dx+\frac{\partial \phi}{\partial y}dy=d\phi$$

x^2+xy , $y^2+\frac{1}{2}x^2$

$\frac{\partial}{\partial y}\left(x^2+xy\right)=x$, $\frac{\partial}{\partial x}\left(y^2+\frac{1}{2}x^2\right)=x$ IS EXACT

- $\frac{\partial \phi}{\partial x}=x^2+xy \Rightarrow \phi(x,y)=\frac{1}{3}x^3+\frac{1}{2}x^2y+f(y)$
- $\frac{\partial \phi}{\partial y}=y^2+\frac{1}{2}x^2 \Rightarrow \phi(x,y)=\frac{1}{3}y^3+\frac{1}{2}x^2y+g(x)$

$\therefore \phi(x,y)=\frac{1}{3}x^3+\frac{1}{3}y^3+\frac{1}{2}x^2y+C$

HENCE WE OBTAIN

$$\int_A^B\left[\left(x^2+xy\right)dx+\left(y^2+\frac{1}{2}x^2\right)dy\right]=\int_A^B d\phi$$

$$=\left[\frac{1}{3}x^3+\frac{1}{3}y^3+\frac{1}{2}x^2y\right]_{(0,\frac{1}{4}\sqrt{15})}^{(5,0)}$$

$$=\frac{1}{3}\times 5^3-\frac{1}{3}\left(\frac{1}{4}\sqrt{15}\right)^3$$

$$=\frac{125}{3}-\frac{1}{3}\frac{15}{64}\sqrt{15} \Rightarrow \frac{125}{3}-\frac{5}{64}\sqrt{15}$$

b) CHECKING FOR PATH INDEPENDENCE FOR (b)

$\frac{\partial}{\partial y}\left(y^3\right)=3y^2$ $\frac{\partial}{\partial x}\left(\frac{1}{16}(x-1)^3\right)=\frac{3}{16}(x-1)^2$

DIFFERENTIAL IS NOT EXACT SO IT DEPENDS ON THE PATH

PARAMETRIZE THE ELLIPSE

$\Rightarrow (x-1)^2=16(1-y^2)$

$\Rightarrow (x-1)^2=16-16y^2$

$\Rightarrow (x-1)^2+16y^2=16$

$\Rightarrow \frac{(x-1)^2}{16}+y^2=1$

$\left[\cos^2\theta+\sin^2\theta=1\right]$

$\therefore \cos\theta=\frac{x-1}{4}$ & $\sin\theta=y$

$\boxed{x=1+4\cos\theta}$ & $\boxed{y=\sin\theta}$

$dx=-4\sin\theta$ & $dy=\cos\theta\,d\theta$

TRANSFORMING THE LIMITS

$A\left(0,\frac{1}{4}\sqrt{15}\right) \longmapsto \theta=\arcsin\frac{\sqrt{15}}{4}=\arccos\left(\frac{1}{4}\right)$

$B(5,0) \longmapsto \theta=0$

TRANSFORMING THE INTEGRAL INTO PARAMETRIC

$$\int_C y^3\,dx+\frac{1}{16}(x-1)^3\,dy$$

$$=\int_{\arccos\frac{1}{4}}^{0}\left[\sin^3\theta(-4\sin\theta)+\frac{1}{16}(1+4\cos\theta-1)^3\cos\theta\right]d\theta$$

$$=\int_{\arcsin\frac{\sqrt{15}}{4}}^{0}-4\sin^4\theta+4\cos^4\theta\,d\theta$$

$$=\int_{\arcsin\frac{\sqrt{15}}{4}}^{0}4\left(\cos^4\theta-\sin^4\theta\right)d\theta$$

$$=\int_{\arcsin\frac{\sqrt{15}}{4}}^{0}4\left(\cos^2\theta-\sin^2\theta\right)\left(\cos^2\theta+\sin^2\theta\right)d\theta$$

$$=\int_{\arcsin\frac{\sqrt{15}}{4}}^{0}4\cos 2\theta\,d\theta$$

$$=\left[2\sin 2\theta\right]_{\theta=\arcsin\frac{\sqrt{15}}{4}}^{0}$$

$$=\left[4\sin\theta\cos\theta\right]_{\theta=\arcsin\frac{\sqrt{15}}{4}}^{0}$$

$$=0-4\times\frac{\sqrt{15}}{4}\times\frac{1}{4}$$

$$=\frac{1}{4}\sqrt{15}$$

Question 10

The closed curve C bounds the finite region R in the x-y plane defined as

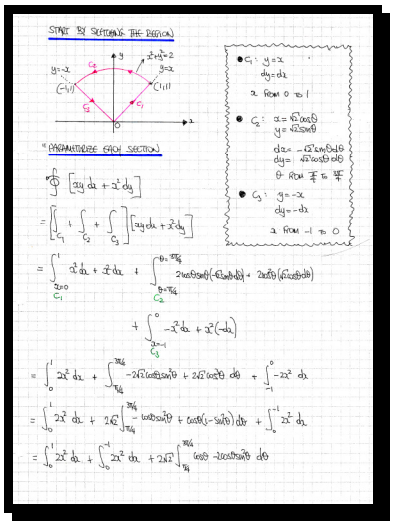
$$R(x,y)=\{x+y\geq 0\ \cap\ x-y\leq 0\ \cap\ x^2+y^2\leq 2\}.$$

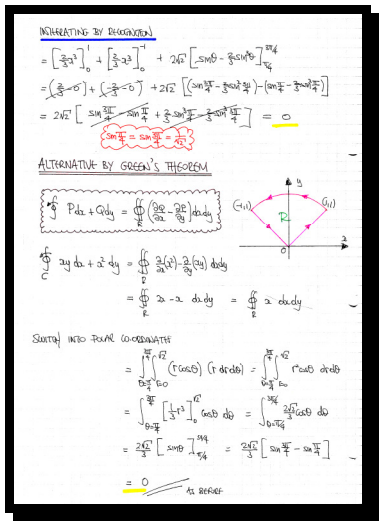
Evaluate the line integral

$$\oint_C \left(xy\,dx\ +\ x^2\,dy\right),$$

where C is traced anticlockwise.

$\boxed{}$, $\boxed{0}$





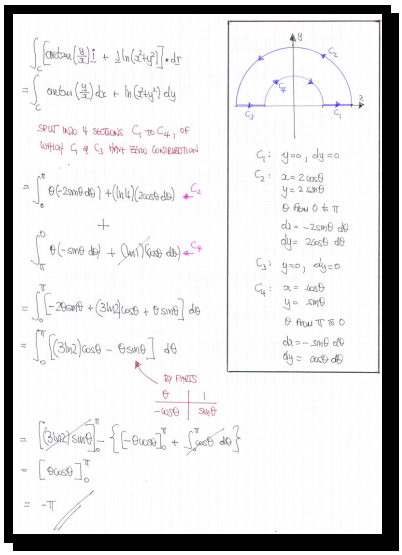
Question 11

Evaluate the line integral

$$\oint_C \left[\arctan\left(\frac{y}{x}\right) dx \ + \ \ln\left(x^2+y^2\right) \ dy \right],$$

where C is the polar rectangle such that $1 \le r \le 2$, $0 \le \theta \le \pi$, traced anticlockwise.

$\boxed{-\pi}$



Question 12

Evaluate the line integral

$$\oint_C \left[(2x-y)\,dx \; + \; (2y-x) \;\; dy \right],$$

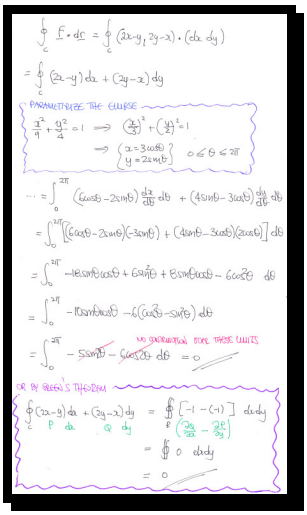
where C is an ellipse with Cartesian equation

$$\frac{x^2}{9}+\frac{y^2}{4}=1,$$

traced anticlockwise.

You may not use Green's theorem in this question.

$\boxed{0}$



Question 13

It is given that the vector function $\mathbf{F}$ satisfies

$$\mathbf{F}=(x-3y)\mathbf{i}+(y-2x)\mathbf{j}.$$

Evaluate the line integral

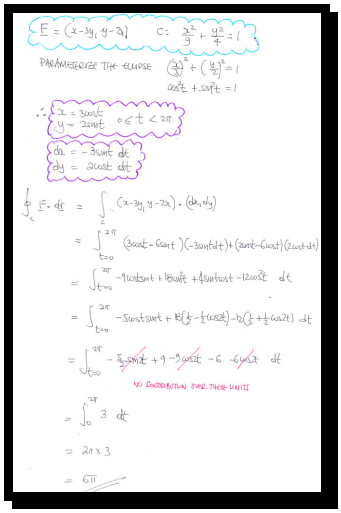
$$\oint_C \mathbf{F}\cdot\mathbf{dr},$$

where C is the ellipse with cartesian equation

$$\frac{x^2}{9}+\frac{y^2}{4}=1.$$

You may not use Green's theorem in this question.

6π



Question 14

$$\mathbf{F}(x,y) \equiv \left(-\frac{y}{x^2+y^2}\right)\mathbf{i} + \left(\frac{x}{x^2+y^2}\right)\mathbf{j}.$$

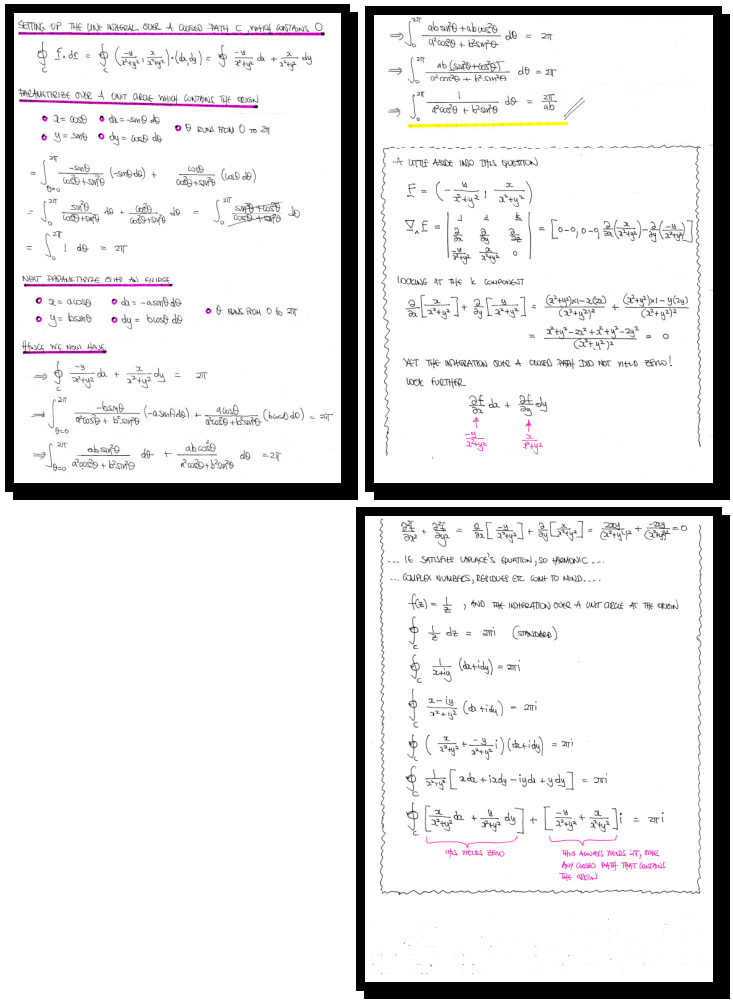
By considering the line integral of $\mathbf{F}$ over two different suitably parameterized closed paths, show that

$$\int_0^{2\pi} \frac{1}{a^2\cos^2\theta + b^2\sin^2\theta}\, d\theta = \frac{2\pi}{ab},$$

where a and b are real constants.

You may assume without proof that the line integral of $\mathbf{F}$ yields the same value over any simple closed curve which contains the origin.

[ ], [proof]



Created by T. Madas

LINE INTEGRALS

IN 3 DIMENSIONS

Created by T. Madas

Question 1

It is given that the vector function $\mathbf{F}$ satisfies

$$\mathbf{F}=\left(x^2y\right)\mathbf{i}+\left(4xy^2\right)\mathbf{j}+\left(-6xz\right)\mathbf{k}\,.$$

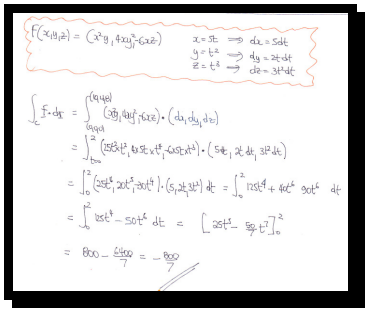
Evaluate the line integral

$$\int_{(0,0,0)}^{(10,4,8)}\mathbf{F}\cdot\mathbf{dr}\,,\qquad \text{where}\quad \mathbf{dr}=\left(dx,dy,dz\right)^{\mathrm{T}}\,,$$

along a path given by the parametric equations

$$x=5t\,,\qquad y=t^2\,,\qquad z=t^3\,.$$

$$\boxed{-\frac{800}{7}}$$



Question 2

It is given that the vector function $\mathbf{F}$ satisfies

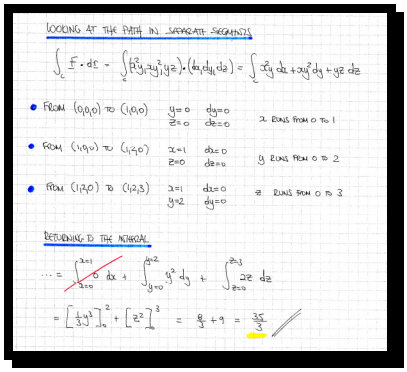
$$\mathbf{F}=\left(x^2y\right)\mathbf{i}+\left(xy^2\right)\mathbf{j}+\left(yz\right)\mathbf{k}\,.$$

Evaluate the line integral

$$\int_{(0,0,0)}^{(1,2,3)}\mathbf{F}\cdot\mathbf{dr}\,,\qquad \text{where}\quad \mathbf{dr}=\left(dx,dy,dz\right)^{\mathrm{T}}\,,$$

along a path of three straight line segments joining $(0,0,0)$ to $(1,0,0)$, $(1,0,0)$ to $(1,2,0)$ and $(1,2,0)$ to $(1,2,3)$.

$\square$, $\boxed{\frac{35}{3}}$



Question 3

It is given that

$$\mathbf{F}(x,y,z)\equiv \mathbf{j}\wedge\mathbf{r}\,,$$

where $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

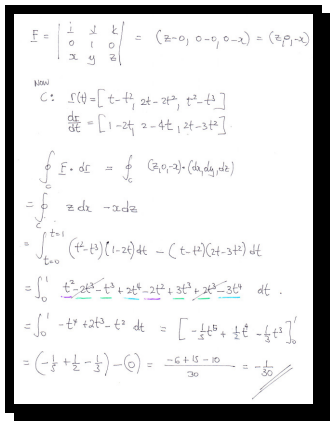
Evaluate the line integral

$$\int_C \mathbf{F}\cdot\mathbf{dr}\,,$$

where C is the closed curve given parametrically by

$$\mathbf{R}(t)=\left(t-t^2\right)\mathbf{i}+\left(2t-2t^2\right)\mathbf{j}+\left(t^2-t^3\right)\mathbf{k}\,,\ 0\le t\le 1\,.$$

$$\boxed{-\frac{1}{30}}$$



Question 4

The simple closed curve C has Cartesian equation

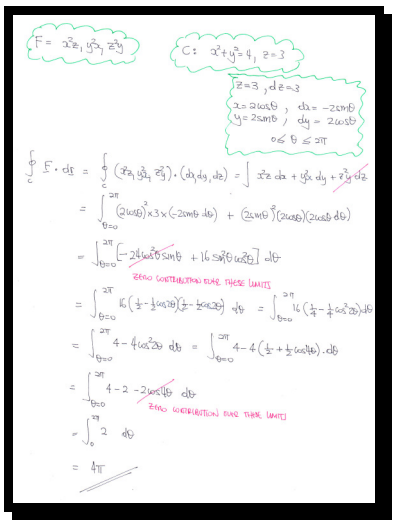
$$x^2 + y^2 = 4\,,\ z = 3\,.$$

Given that $\mathbf{F} = x^2 z\,\mathbf{i} + y^2 x\,\mathbf{j} + z^2 y\,\mathbf{k}$, evaluate the integral

$$\oint_C \mathbf{F} \cdot \mathbf{dr}\,.$$

You may not use Green's theorem in this question.

$\boxed{4\pi}$



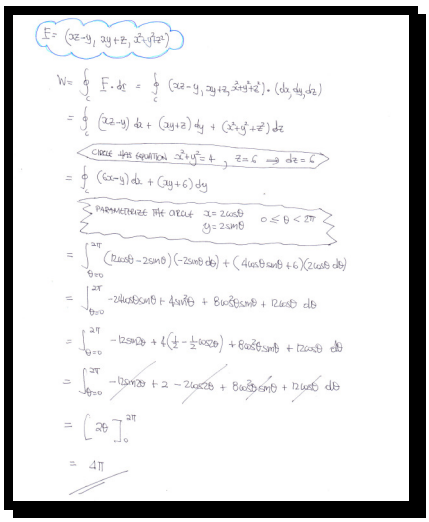
Question 5

$$\mathbf{F}=\left(xz-y\right)\mathbf{i}+\left(xy+z\right)\mathbf{j}+\left(x^2+y^2+z^2\right)\mathbf{k}\,.$$

Determine the work done by $\mathbf{F}$, when it moves in a complete revolution in a circular path of radius 2 around the z axis, at the level of the plane with equation $z=6$.

You may not use Green's theorem in this question.

$\boxed{4\pi}$



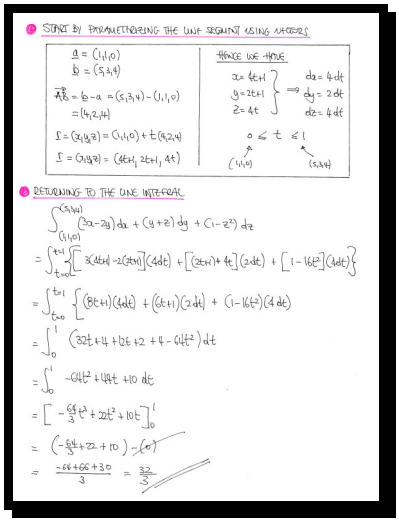
Question 6

Evaluate the integral

$$\int_{(1,1,0)}^{(5,3,4)} (3x-2y)\,dx+(y+z)\,dy+\left(1-z^2\right)dz\,,$$

along the straight line segment joining the points with Cartesian coordinates $(1,1,0)$ and $(5,3,4)$.

$\square$, $\boxed{\frac{32}{3}}$



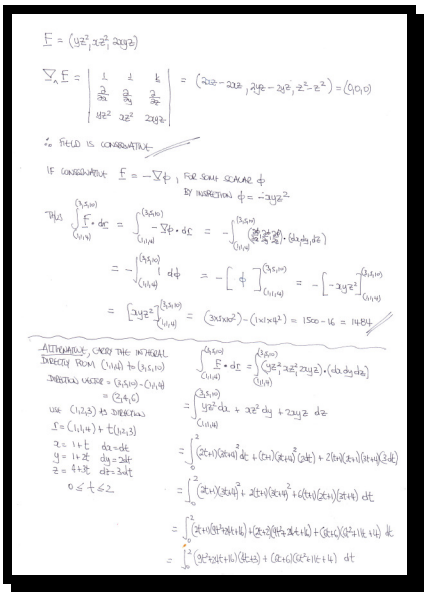
Question 7

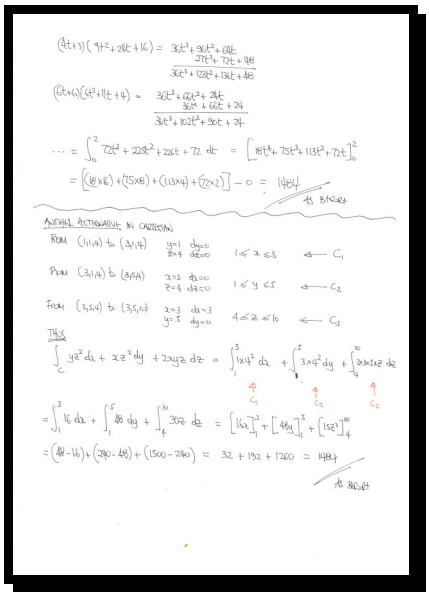
$$\mathbf{F}(x,y,z) \equiv yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k}\,.$$

Show that the vector field $\mathbf{F}$ is conservative, and hence evaluate the integral

$$\int_{(1,1,4)}^{(3,5,10)} \mathbf{F}\cdot\mathbf{dr}\,.$$

1484





Question 8

A vector field $\mathbf{F}$ is defined as

$$\mathbf{F}(x,y,z)\equiv[x+yz]\mathbf{i}+[y+xz]\mathbf{j}+\left[x(y+1)+z^2\right]\mathbf{k}\,.$$

The closed path C joins $(0,0,0)$ to $(1,1,1)$, $(1,1,1)$ to $(1,1,0)$, $(1,1,0)$ to $(0,0,0)$, in that order.

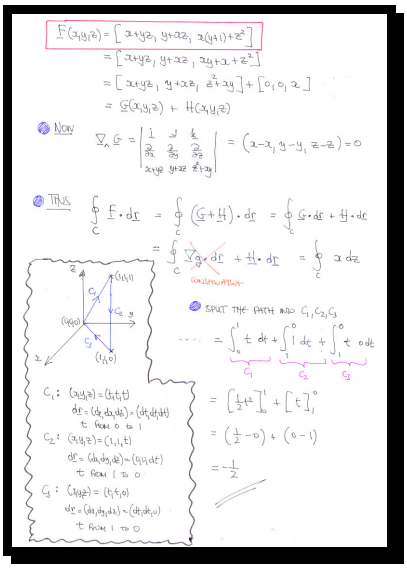
By writing

$$\mathbf{F}(x,y,z)=\mathbf{G}(x,y,z)+\mathbf{H}(x,y,z),$$

for some vector functions $\mathbf{G}$ and $\mathbf{H}$, where $\nabla g(x,y,z)=\mathbf{G}(x,y,z)$ for some smooth scalar function $g(x,y,z)$, evaluate the line integral

$$\oint_C \mathbf{F}\cdot\mathbf{dr}\,.$$

$$\boxed{-\frac{1}{2}}$$



Question 9

A vector field $\mathbf{F}$ is defined as

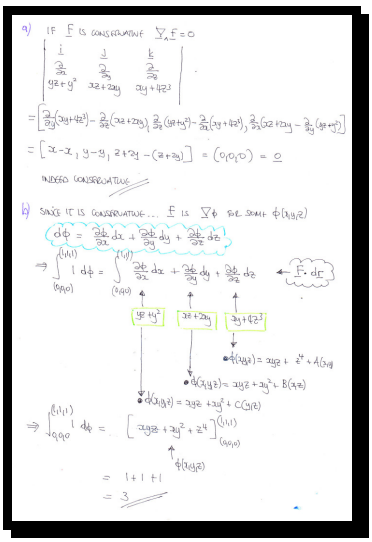
$$\mathbf{F}(x,y,z)\equiv\left(yz+y^2\right)\mathbf{i}+\left(xz+2xy\right)\mathbf{j}+\left(xy+4z^3\right)\mathbf{k}\,.$$

a) Show that $\mathbf{F}$ is conservative.

b) Hence evaluate the integral

$$\int_{(0,0,0)}^{(1,1,1)}\mathbf{F}\cdot\mathbf{dr}\,.$$

3



Question 10

A curve C is defined as

$$(x, y, z) = (\cos 3t, \sin 3t, t), \ 0 \le t \le 2\pi .$$

a) Sketch the graph of C.

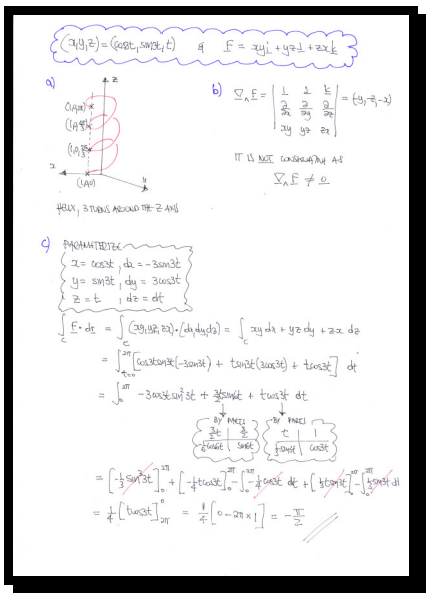
$$\mathbf{F}(x, y, z) \equiv xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k} .$$

b) Determine whether the vector field $\mathbf{F}$ is conservative.

c) Evaluate the integral

$$\int_C \mathbf{F} \cdot \mathbf{dr} .$$

$$\boxed{-\frac{\pi}{2}}$$



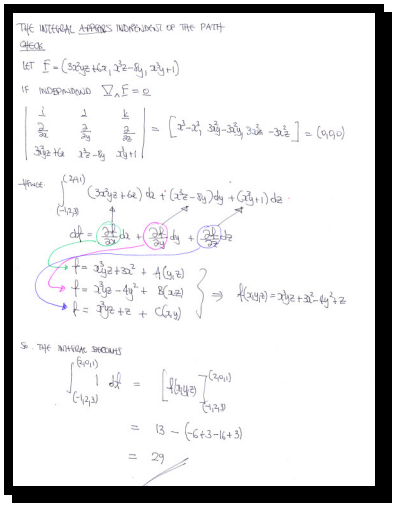
Question 11

Evaluate the integral

$$\int_{(-1,2,3)}^{(2,0,1)} \left(3x^2yz+6x\right)dx+\left(x^3z-8y\right)dy+\left(x^3y+1\right)dz\,,$$

along a path joining the points with Cartesian coordinates $(-1,2,3)$ and $(2,0,1)$.

29



Question 12

A curve C is defined by $\mathbf{r}=\mathbf{r}(t)$, $0\le t\le 2\pi$ as

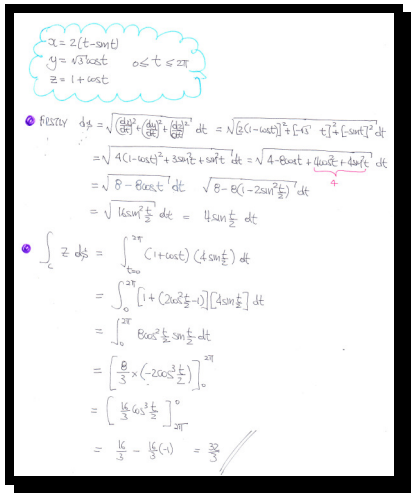
$$\mathbf{r}(t)=(x,y,z)=\left[2(t-\sin t)\,,\sqrt{3}\cos t\,,1+\cos t\right].$$

Evaluate the integral

$$\int_C z\,ds\,,$$

where s is the arclength along C.

$$\frac{32}{3}$$



Question 13

A vector field $\mathbf{F}$ and a scalar field ψ are given.

$$\mathbf{F}=\left(3x^3y\right)\mathbf{i}+\left(15\sqrt{z}\right)\mathbf{j}-\left(\frac{13}{96}xz\right)\mathbf{k} \quad \text{and} \quad \psi(x,y,z)=x\mathrm{e}^{\frac{2y}{\sqrt{z}}}.$$

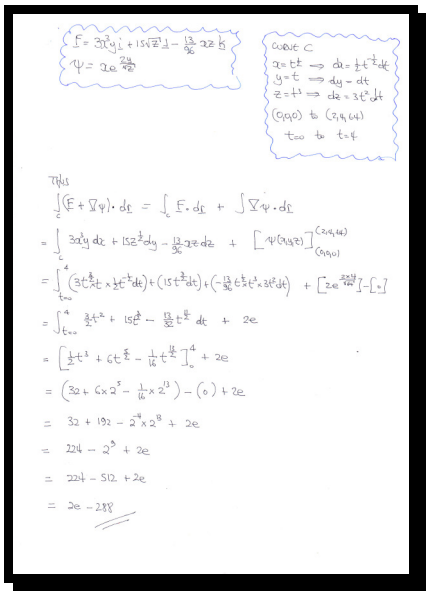
Evaluate the integral

$$\int_{(0,0,0)}^{(2,4,64)}\left[\mathbf{F}+\nabla\psi\right]\cdot\mathbf{dr},$$

along the curve with parametric equations

$$x=\sqrt{t}, \quad y=t \quad \text{and} \quad z=t^3.$$

$$\boxed{2\mathrm{e}-288}$$



Question 14

It is given that the vector function $\mathbf{F}$ satisfies

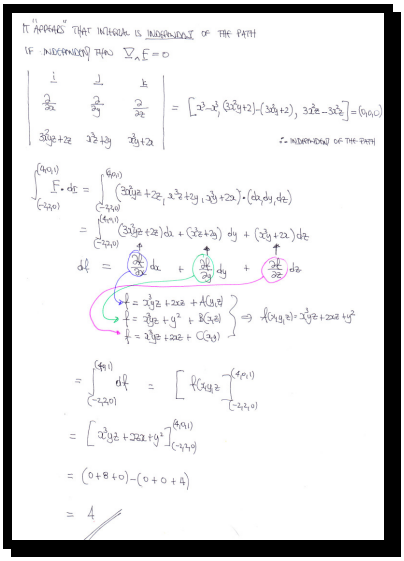
$$\mathbf{F}=\left(3x^2yz+2z\right)\mathbf{i}+\left(x^3z+2y\right)\mathbf{j}+\left(x^3y+2x\right)\mathbf{k}\,.$$

Evaluate the line integral

$$\int_{(-2,2,0)}^{(4,0,1)}\mathbf{F}\cdot\mathbf{dr}\,,$$

along a path joining the points with Cartesian coordinates $(4,0,1)$ and $(-2,2,0)$.

4



Question 15

It is given that the vector function $\mathbf{F}$ satisfies

$$\mathbf{F}=\left(1+xy^2\right)\mathbf{i}+\left(x+xyz\right)\mathbf{j}+\left(y\sin z\right)\mathbf{k}\,.$$

Evaluate the line integral

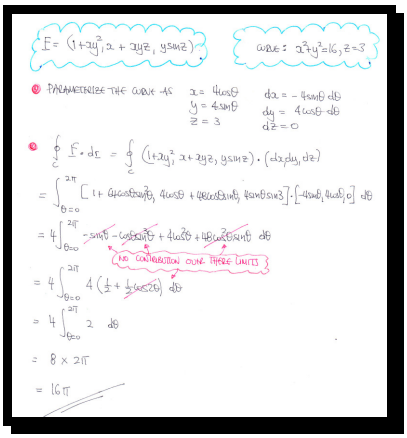
$$\oint_C \mathbf{F}\cdot\mathbf{dr}\,,$$

where C is the anticlockwise cartesian path

$$x^2+y^2=16\,,\quad z=3\,.$$

You may not use Green's theorem in this question.

16π



Question 16

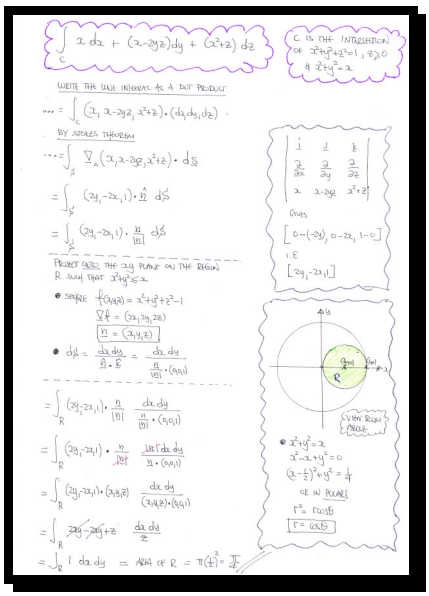
Evaluate the line integral

$$\oint_C \left[x\,dx + (x - 2yz)\,dy + \left(x^2 + z\right)dz \right],$$

where C is the intersection of the surfaces with respective Cartesian equations

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0 \qquad \text{and} \qquad x^2 + y^2 = x, \quad z \geq 0.$$

$$\frac{\pi}{4}$$



Question 17

It is given that the vector field $\mathbf{F}$ satisfies

$$\mathbf{F} = 8z\,\mathbf{i} + 4x\,\mathbf{j} + y\,\mathbf{k}\,.$$

Evaluate the line integral

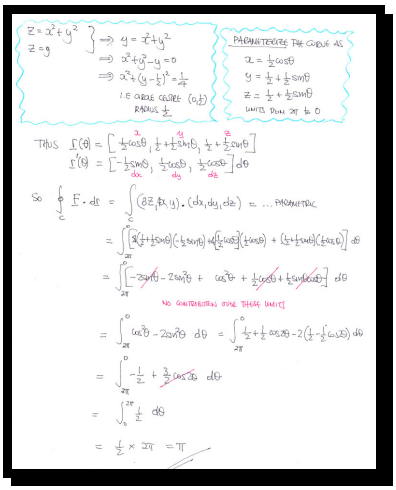
$$\oint_C \mathbf{F}\cdot\mathbf{dr}\,,$$

where C is the intersection of the surfaces with respective Cartesian equations

$$z = x^2 + y^2 \qquad \text{and} \qquad z = y\,.$$

*You may **not** use Stokes' Theorem in this question.*

$\boxed{\pi}$



Question 18

It is given that the vector field $\mathbf{F}$ satisfies

$$\mathbf{F} = y^2\,\mathbf{i} + z^2\,\mathbf{j} + x^2\,\mathbf{k}\,.$$

Evaluate the line integral

$$\oint_C \mathbf{F}\cdot\mathbf{dr}\,,$$

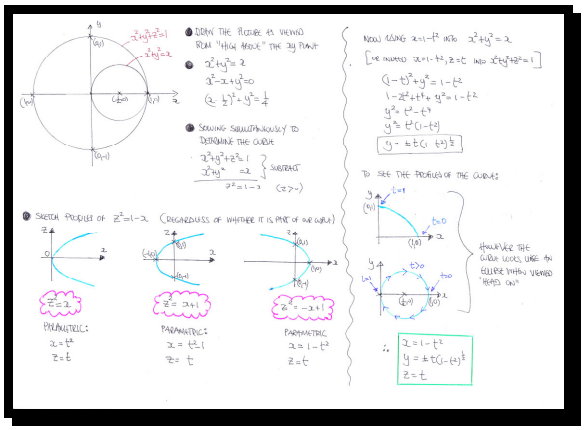
where C is the intersection of the surfaces with respective Cartesian equations

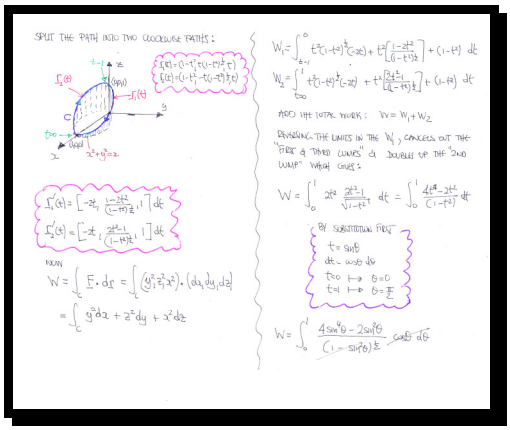
$$x^2 + y^2 + z^2 = 1, \quad z \geq 0 \qquad \text{and} \qquad x^2 + y^2 = x, \quad z \geq 0\,.$$

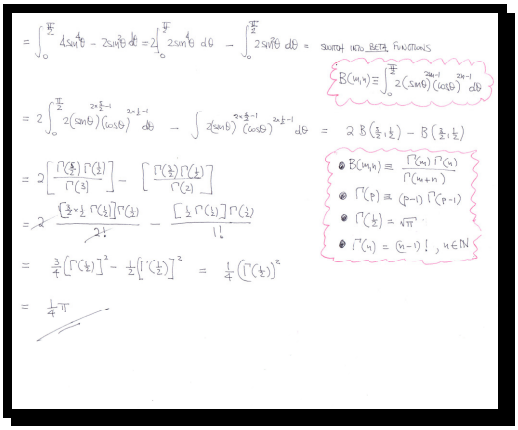
*You may **not** use Stokes' Theorem in this question.*

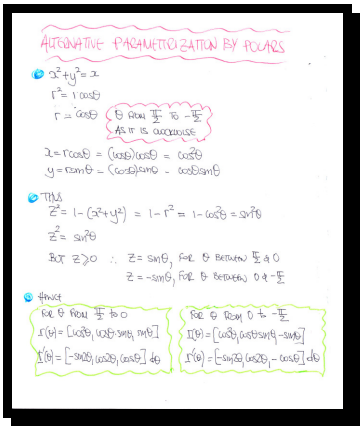
$$\boxed{\frac{\pi}{4}}$$

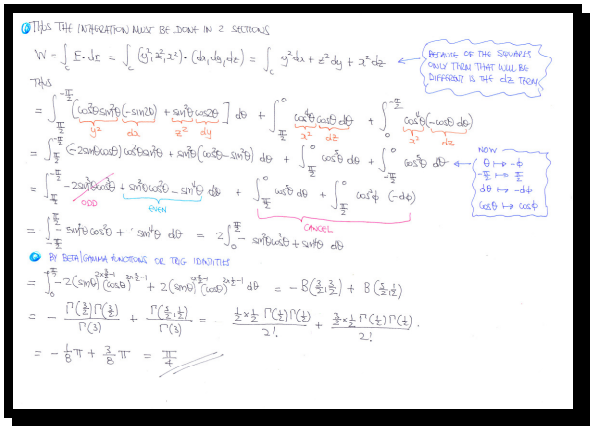
[solution overleaf]











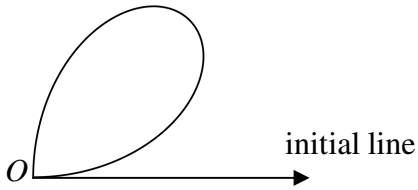
Created by T. Madas

LINE INTEGRALS

IN POLAR COORDINATES

Created by T. Madas

Question 1



The figure above shows the closed curve C with polar equation

$$r = \sin 2\theta\,,\ 0 \le \theta \le \frac{\pi}{2}.$$

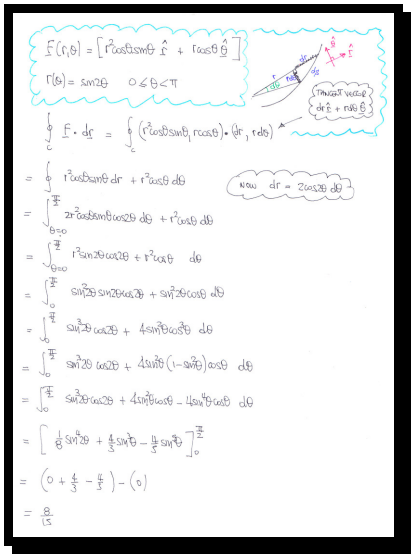
The vector field $\mathbf{F}$ is given in plane polar coordinates (r,θ) by

$$\mathbf{F}(r,\theta) = \left(r^2 \cos\theta \sin\theta\right)\hat{\mathbf{r}} + \left(r\cos\theta\right)\hat{\boldsymbol{\theta}}.$$

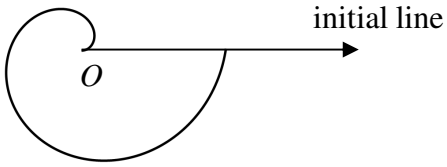
Evaluate the line integral

$$\oint_C \mathbf{F}\cdot\mathbf{dr}\,.$$

$$\frac{8}{15}$$



Question 2



The figure above shows the curve C with polar equation

$$r=\theta,\ 0\le\theta\le 2\pi .$$

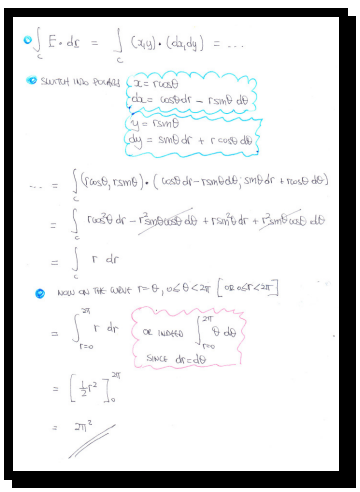
The vector field $\mathbf{F}$ is given in Cartesian coordinates by

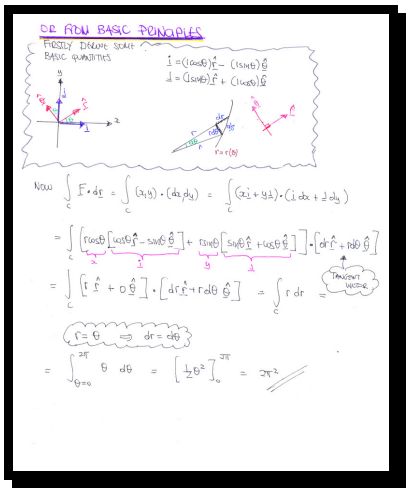
$$\mathbf{F}(x,y)=x\mathbf{i}+y\mathbf{j}.$$

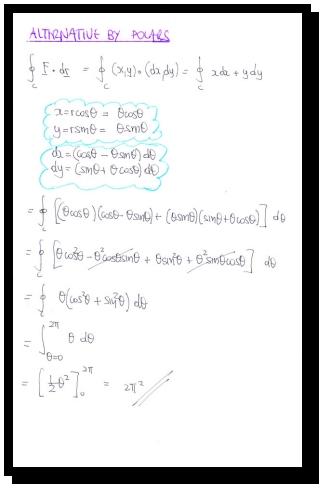
Evaluate the line integral

$$\int_C \mathbf{F}\cdot\mathbf{dr}\ .$$

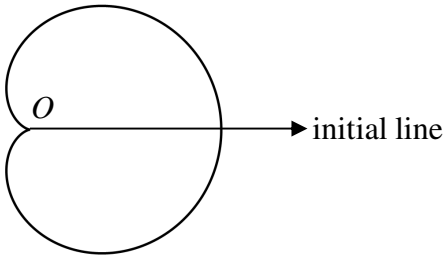
$$\boxed{2\pi^2}$$







Question 3



The figure above shows the closed curve C with polar equation

$$r = 1 + \cos\theta\,,\ 0 \le \theta \le 2\pi\,.$$

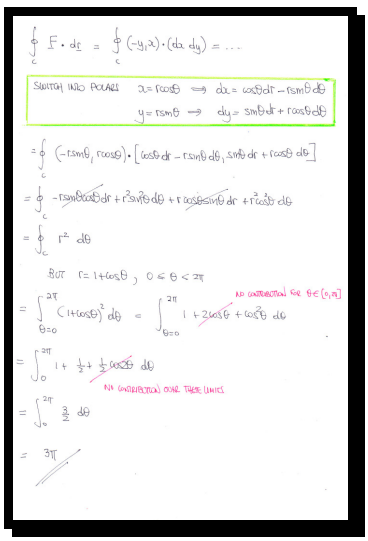
The vector field $\mathbf{F}$ is given in Cartesian coordinates by

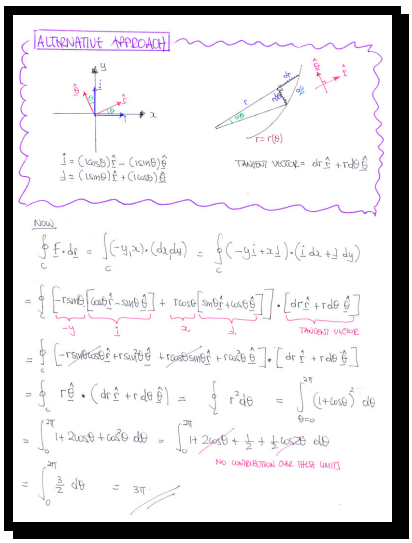
$$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}\,.$$

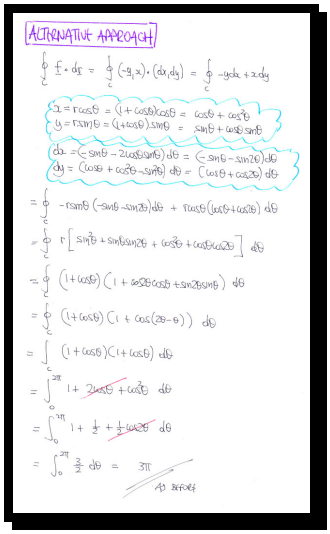
Evaluate the line integral

$$\oint_C \mathbf{F}\cdot\mathbf{dr}\,.$$

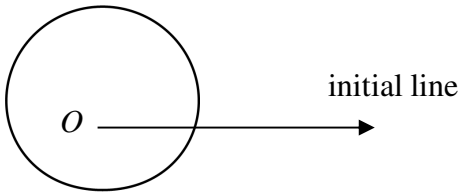
3π







Question 4



The figure above shows the closed curve C with polar equation

$$r = 3 + \sin\theta, \ 0 \le \theta \le 2\pi .$$

The vector field $\mathbf{F}$ is given in Cartesian coordinates by

$$\mathbf{F}(x, y) = (x + y)\mathbf{i} + (-x + y)\mathbf{j}.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{dr} .$$

[ ], 19π

