LIMITS

LIMITS BY STANDARD EXPANSIONS

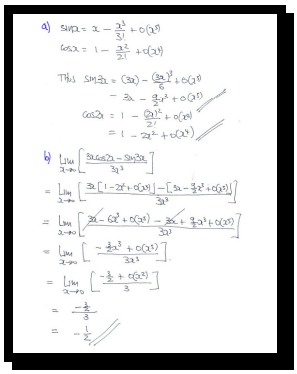
Question 1 (***)

a) Write down the first two non zero terms in the expansions of $\sin 3x$ and $\cos 2x$.

b) Hence find the exact value of

$$\lim_{x\to 0}\left[\frac{3x\cos 2x-\sin 3x}{3x^3}\right]$$

$$\boxed{\sin 3x\approx 3x-\frac{9}{2}x^3}\,,\quad \boxed{\cos 2x\approx 1-2x^2}\,,\quad \boxed{-\frac{1}{2}}$$



a) $\sin x = x - \frac{x^3}{3!} + o(x^3)$

$\cos x = 1 - \frac{x^2}{2!} + o(x^2)$

Thus $\sin 3x = (3x) - \frac{(3x)^3}{6} + o(x^3)$

$= 3x - \frac{9}{2}x^3 + o(x^3)$

$\cos 2x = 1 - \frac{(2x)^2}{2!} + o(x^2)$

$= 1 - 2x^2 + o(x^2)$

b) $\lim_{x\to 0}\left[\frac{3x\cos 2x - \sin 3x}{3x^3}\right]$

$= \lim_{x\to 0}\left[\frac{3x\left[1-2x^2+o(x^2)\right]-\left[3x-\frac{9}{2}x^3+o(x^3)\right]}{3x^3}\right]$

$= \lim_{x\to 0}\left[\frac{3x-6x^3+o(x^3)-3x+\frac{9}{2}x^3+o(x^3)}{3x^3}\right]$

$= \lim_{x\to 0}\left[\frac{-\frac{3}{2}x^3+o(x^3)}{3x^3}\right]$

$= \lim_{x\to 0}\left[\frac{-\frac{3}{2}+o(x^3)}{3}\right]$

$= \frac{-\frac{3}{2}}{3}$

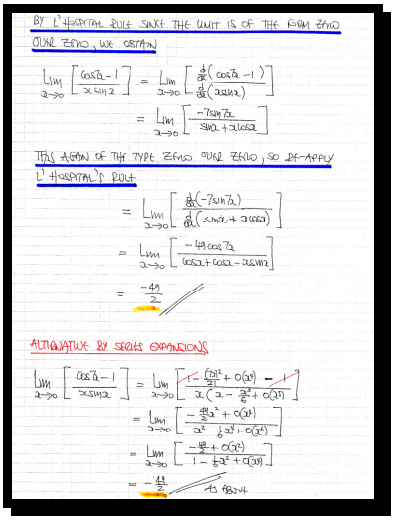
$= -\frac{1}{2}$

Question 2 (***)

Use standard expansions of functions to find the value of the following limit.

$$\lim_{x\to 0}\left[\frac{\cos 7x-1}{x\sin x}\right].$$

$\square$, $\boxed{-\frac{49}{2}}$

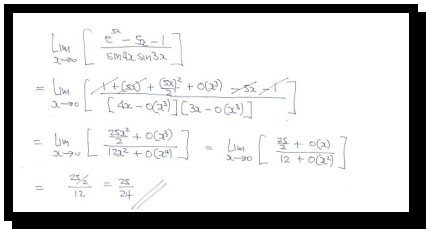


Question 3 (***)

Use standard expansions of functions to find the value of the following limit.

$$\lim_{x\to 0}\left[\frac{\mathrm{e}^{5x}-5x-1}{\sin 4x\sin 3x}\right].$$

$\boxed{\frac{25}{24}}$

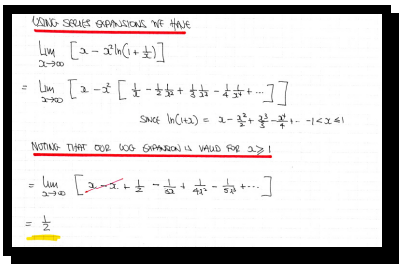


Question 4 (***)

Use standard series expansions to evaluate the following limit.

$$\lim_{x\to\infty}\left[x-x^2\ln\left[x+\frac{1}{x}\right]\right].$$

V, ☐, $\boxed{\frac{1}{2}}$

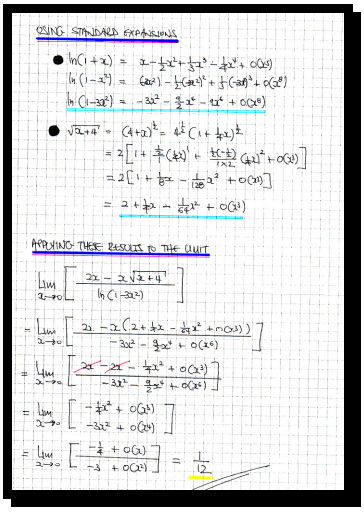


Question 5 (***)

By considering series expansion, determine the value of the following limit.

$$\lim_{x\to 0}\left[\frac{2x-x\sqrt{x+4}}{\ln\left(1-3x^2\right)}\right].$$

☐, $\boxed{\frac{1}{12}}$

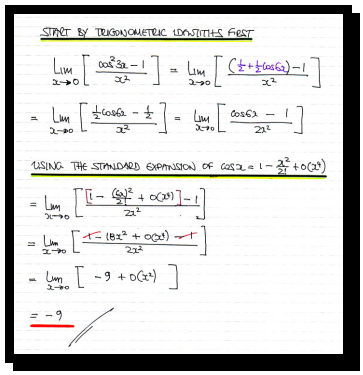


Question 6 (*+)**

Use standard expansions of functions to find the value of the following limit.

$$\lim_{x\to 0}\left[\frac{\cos^2 3x-1}{x^2}\right].$$

$\boxed{}$, $\boxed{-9}$

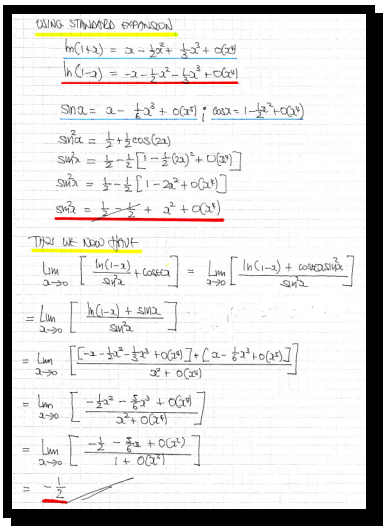


Question 7 (*+)**

Use standard expansions of functions to find the value of the following limit.

$$\lim_{x\to 0}\left[\frac{\ln(1-x)}{\sin^2 x}+\operatorname{cosec} x\right].$$

$\boxed{\phantom{-\frac{1}{2}}}$, $\boxed{-\frac{1}{2}}$



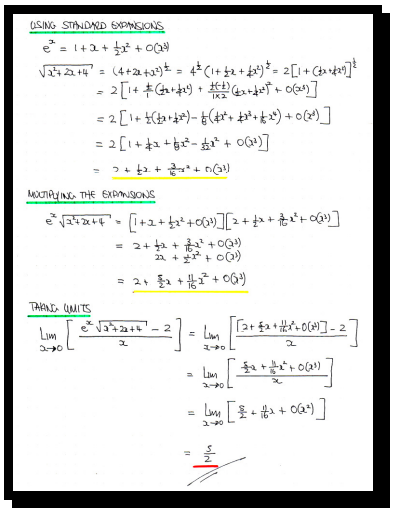
Question 8 (**+)**

Use standard expansions of functions to find the value of the following limit.

$$\lim_{x\to 0}\left[\frac{e^x\sqrt{x^2+2x+4}\ -2}{x}\right].$$

No credit will be given for using alternative methods such as L' Hospital's rule.

$\boxed{}$, $\boxed{\frac{5}{2}}$



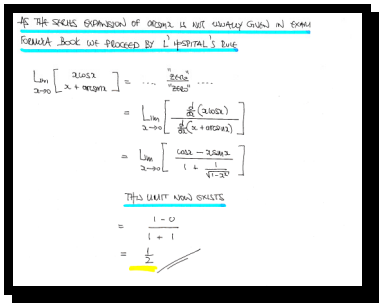
LIMITS BY L'HOSPITAL RULE

Question 1 (**)

Find the value of the following limit

$$\lim_{x\to 0}\left[\frac{x\cos x}{x+\arcsin x}\right].$$

$\boxed{}$, $\boxed{\frac{1}{2}}$

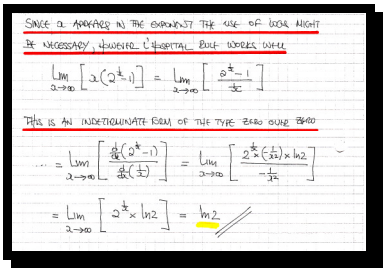


Question 2 (+)**

Find the value of the following limit

$$\lim_{x\to\infty}\left[x\left(2^{\frac{1}{x}}-1\right)\right].$$

$\boxed{}$, $\boxed{\ln 2}$

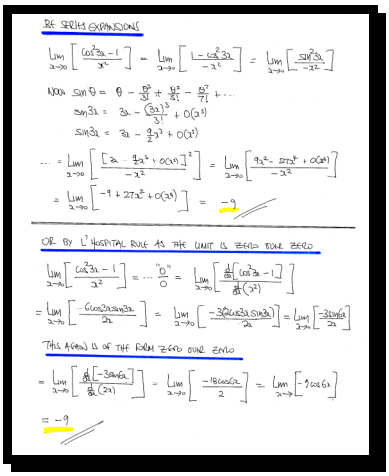


Question 3 **(***)**

Find the value of the following limit

$$\lim_{x\to 0}\left[\frac{\cos^2 3x-1}{x^2}\right].$$

$\boxed{}$, $\boxed{-9}$

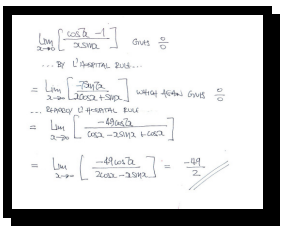


Question 4 **(***)**

Find the value of the following limit

$$\lim_{x\to 0}\left[\frac{\cos 7x-1}{x\sin x}\right].$$

$\boxed{\phantom{-\frac{49}{2}}}$, $\boxed{-\frac{49}{2}}$

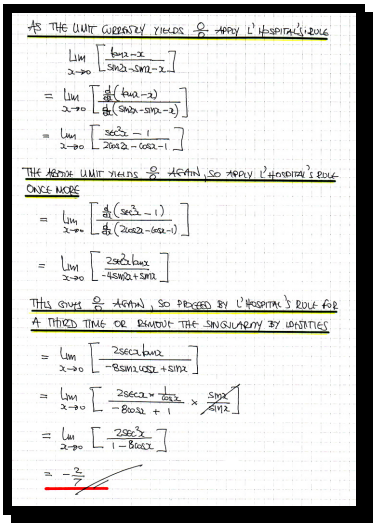


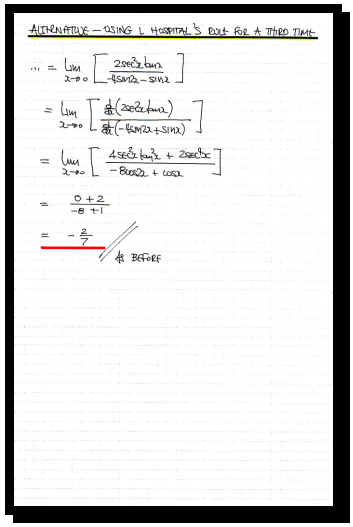
Question 5 (***)

Use L'Hospital's rule to find the value of the following limit

$$\lim_{x\to 0}\left[\frac{\tan x - x}{\sin 2x - \sin x - x}\right].$$

$\square$, $\boxed{-\frac{2}{7}}$





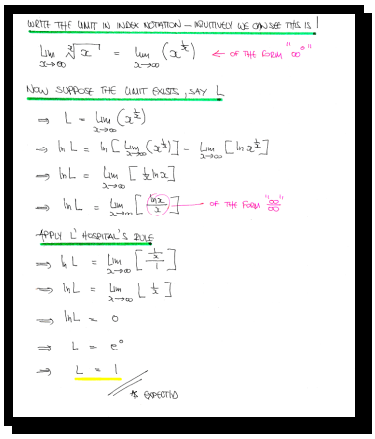
Question 6 **(***+)**

Show clearly that the following limit converges to 1.

$$\lim_{x\to\infty}\left[\sqrt[x]{x}\right]$$

You must justify the evaluation.

, proof

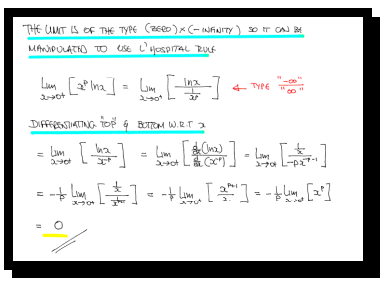


Question 7 **(****)**

If $p\in(0,\infty)$, show that

$$\lim_{x\to 0^+}\left[x^p\ln x\right]=0,\quad x\in(0,\infty).$$

, proof

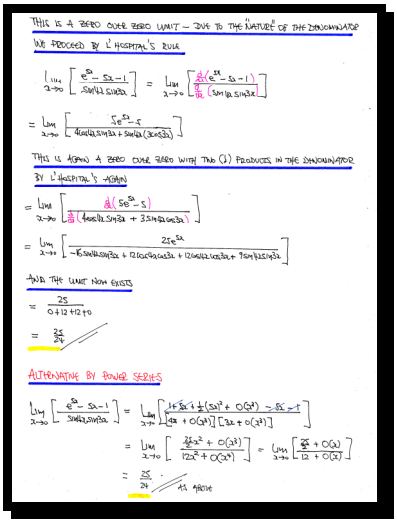


Question 8 **(****)**

Find the value of the following limit

$$\lim_{x\to 0}\left[\frac{\mathrm{e}^{5x}-5x-1}{\sin 4x\sin 3x}\right].$$

V, [ ], $\frac{25}{24}$

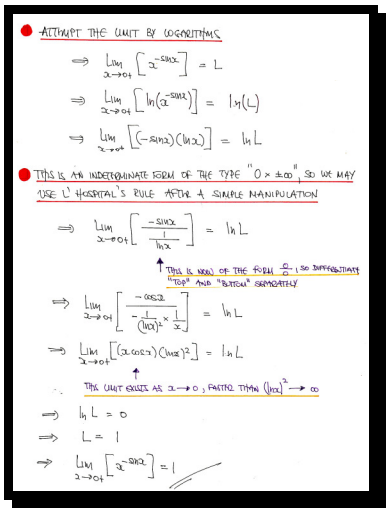


Question 9 **(****)**

Find the value of the following limit

$$\lim_{x\to 0+}\left[x^{-\sin x}\right].$$

$\boxed{}$, $\boxed{1}$

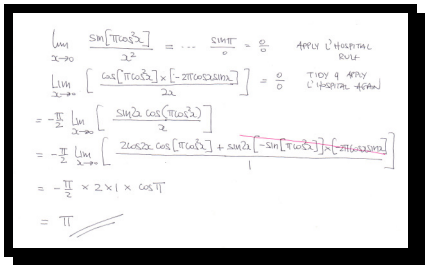


Question 10 **(****+)**

Find the value of the following limit

$$\lim_{x\to 0}\left[\frac{\sin\left(\pi\cos^2 x\right)}{x^2}\right].$$

$\boxed{\pi}$

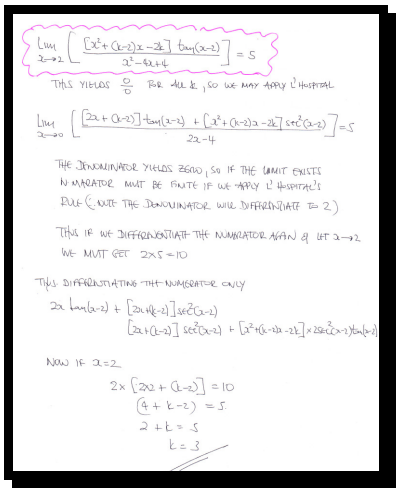


Question 11 **(****+)**

Find the value of the constant k, given that

$$\lim_{x\to 2}\left\{\frac{\left[x^2+(k-2)x-2k\right]\tan(x-2)}{x^2-4x+4}\right\}=5.$$

$$\boxed{k=3}$$

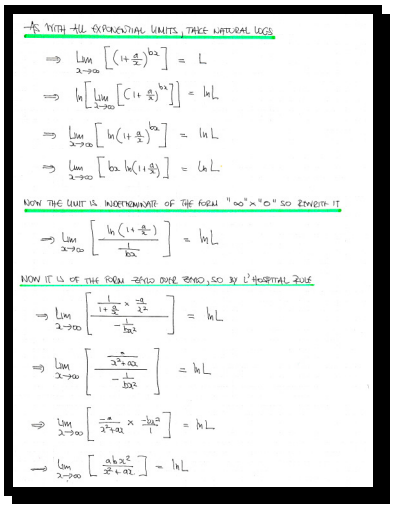


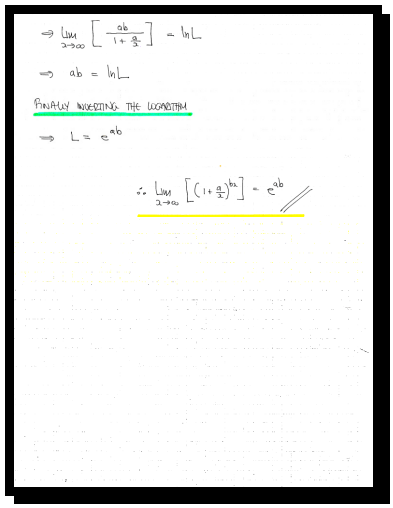
Question 12 (**+)**

Show with detailed workings that

$$\lim_{x\to\infty}\left[\left(1+\frac{a}{x}\right)^{bx}\right] = \mathrm{e}^{ab}.$$

[ ], [proof]



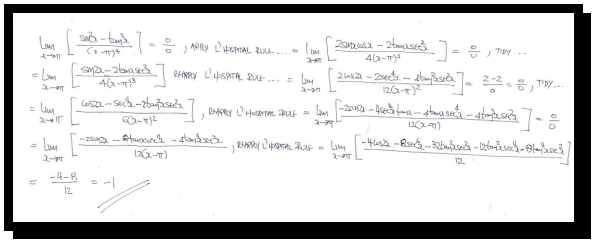


Question 13 (*****)

Find the value of the following limit

$$\lim_{x\to\pi}\left[\frac{\sin^2 x-\tan^2 x}{(x-\pi)^4}\right].$$

$\boxed{-1}$

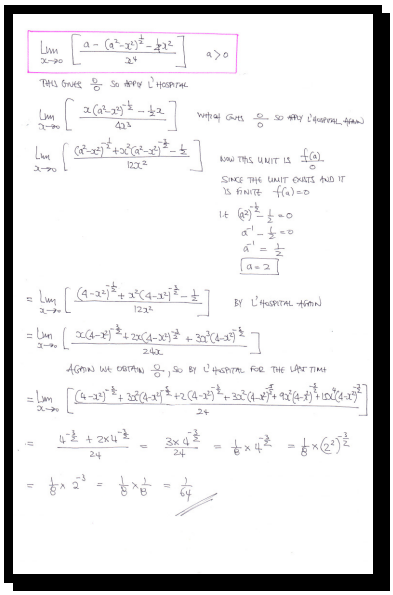


Question 14 (*****)

$$L=\lim_{x\to 0}\left[\frac{a-\sqrt{a^2-x^2}-\frac{1}{4}x^2}{x^4}\right],\ a>0.$$

Given that L is finite, determine its value.

$\boxed{\frac{1}{64}}$

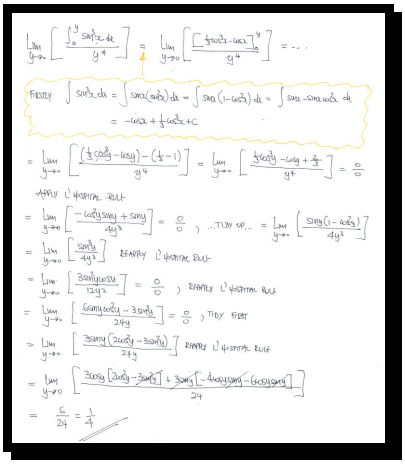


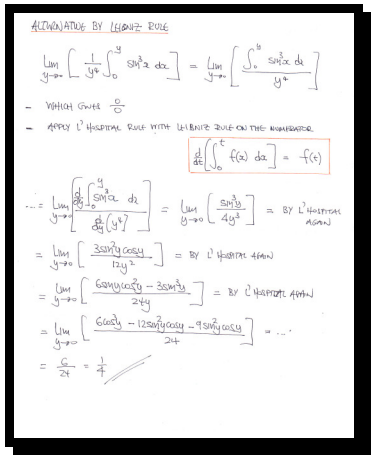
Question 15 (*****)

Find the value of the following limit

$$\lim_{y\to 0}\left[\frac{1}{y^4}\int_0^y \sin^3 x \ dx\right].$$

$$\boxed{\frac{1}{4}}$$





VARIOUS LIMITS

Question 1 (**)

Find the value of the following limit

$$\lim_{x\to\infty}\left[\frac{3x^2+7x-1}{x^2+5}\right].$$

$\boxed{3}$

$$\lim_{x\to\infty}\left[\frac{3x^2+7x-1}{x^2+5}\right]=\lim_{x\to\infty}\left[\frac{3+\frac{7}{x}-\frac{1}{x^2}}{1+\frac{5}{x^2}}\right]$$
$$=\frac{3}{1}=3$$

Question 2 (**)

Find the value of the following limit

$$\lim_{x\to 2}\left[\frac{x^3-x^2-x-2}{x-2}\right].$$

$\boxed{7}$

$$\lim_{x\to 2}\left[\frac{x^3-x^2-x-2}{x-2}\right]=\lim_{x\to 2}\left[\frac{(x-2)(x^2+x+1)}{x-2}\right]$$
$$=2^2+2+1=7$$

OR BY L'HOSPITAL SINCE $\frac{0}{0}$

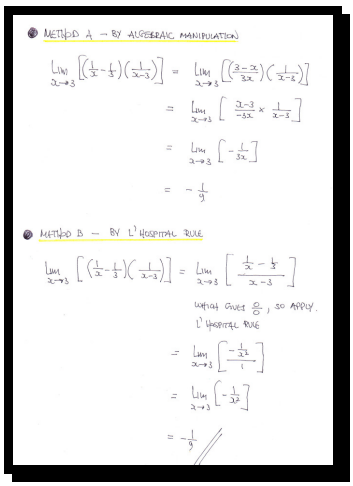
$$\lim_{x\to 2}\left[\frac{3x^2-2x-1}{1}\right]=\frac{3\times 2^2-2\times 2-1}{1}=7$$

Question 3 (+)**

Find the value of the following limit

$$\lim_{x\to 3}\left[\left(\frac{1}{x}-\frac{1}{3}\right)\left(\frac{1}{x-3}\right)\right].$$

$$\boxed{-\frac{1}{9}}$$

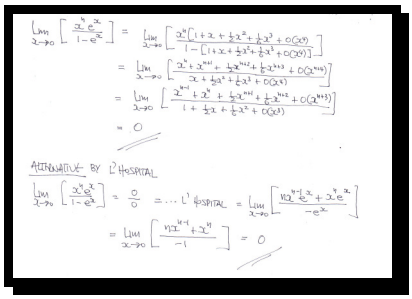


Question 4 (+)**

Given that n is a positive integer determine

$$\lim_{x\to 0}\left[\frac{x^n\,\mathrm{e}^x}{1-\mathrm{e}^x}\right].$$

$$\boxed{0}$$



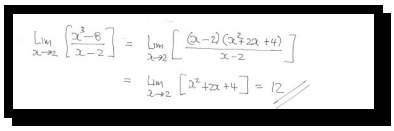
Question 5 (***)

Find the value of the following limit

$$\lim_{x\to 2}\left[\frac{x^3-8}{x-2}\right].$$

You may not use the L' Hospital's rule in this question.

$\boxed{12}$

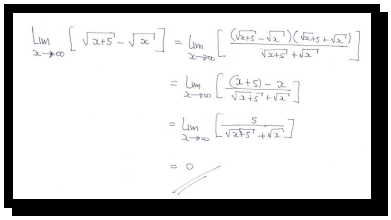


Question 6 (***)

Find the value of the following limit.

$$\lim_{x\to\infty}\left[\sqrt{x+5}-\sqrt{x}\right].$$

$\boxed{0}$

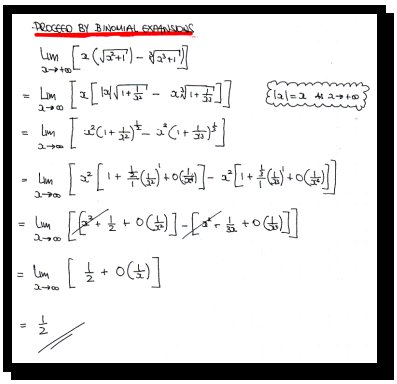


Question 7 (***)

Find the value of the following limit.

$$\lim_{x\to\infty}\left[x\sqrt{x^2+1}-\sqrt[3]{x^3+1}\right].$$

$\square$, $\boxed{\frac{1}{2}}$



Question 8 **(***)**

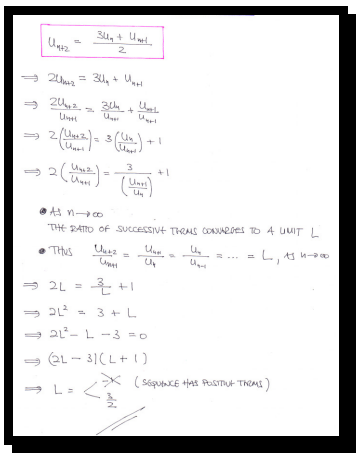
The Fibonacci sequence is given by the recurrence formula

$$u_{n+2} = u_{n+1} + u_n, \quad u_1 = 1,\ u_2 = 1.$$

It is further given that in this sequence **the ratio of consecutive terms** converges to a limit ϕ, known as the *Golden Ratio*.

Show, by using the above recurrence formula, that $\phi = \frac{1}{2}\left(1+\sqrt{5}\right)$.

[ ] , [proof]



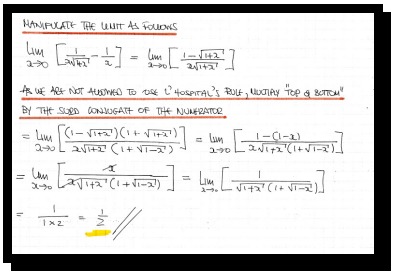
Question 9 (*+) Limits**

Evaluate the following limit.

$$\lim_{x\to 0}\left[\frac{1}{x\sqrt{1+x}}-\frac{1}{x}\right].$$

You may NOT use L'Hospital's rule in this question

$\boxed{}$, $\boxed{x=\frac{1}{2}}$

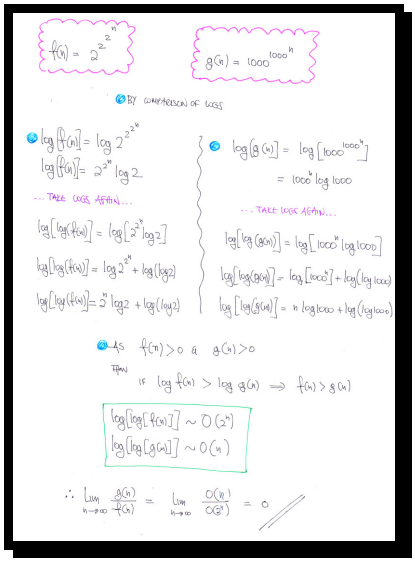


Question 10 (*+)**

$$f(n)=2^{2^{2^n}},\ n\in\mathbb{R} \qquad \text{and} \qquad f(n)=1000^{1000^n},\ n\in\mathbb{R}\ .$$

Determine whether or not $\lim_{n\to\infty}\left[\frac{g(n)}{f(n)}\right]$ exists.

$$\boxed{\lim_{n\to\infty}\left[\frac{g(n)}{f(n)}\right]=0}$$



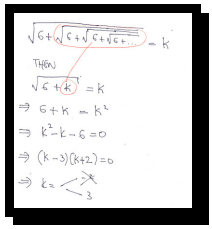
Question 11 (*+)**

Show clearly without the use of any calculating aid that

$$\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+...}}}}=k\,,$$

where k is an integer to be found.

$\boxed{k=3}$



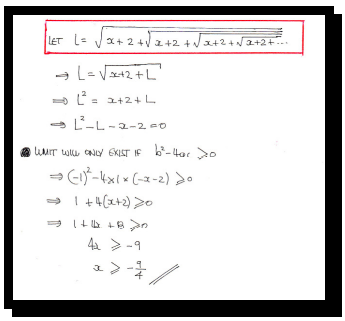
Question 12 (*+)**

$$\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+...}}}}}\,,$$

It is given that the above nested radical converges to a limit L, $L\in\mathbb{R}$.

Determine the range of possible values of x.

$\boxed{}$, $\boxed{x\geq -\frac{9}{4}}$

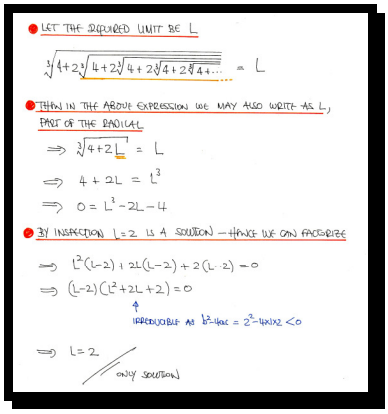


Question 13 **(***+)**

$$\sqrt[3]{4+2\sqrt[3]{4+2\sqrt[3]{4+2\sqrt[3]{4+2\sqrt[3]{4+...}}}}}$$

Given that the above nested radical converges, determine its limit.

$\boxed{}$, $\boxed{L=2}$



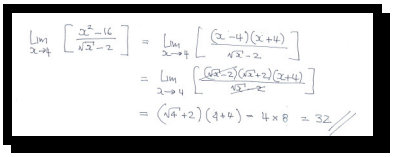
Question 14 **(****)**

Find the value of the following limit

$$\lim_{x\to 4}\left[\frac{x^2-16}{\sqrt{x}-2}\right].$$

You may not use the L' Hospital's rule in this question.

$\boxed{32}$



Question 15 (****)

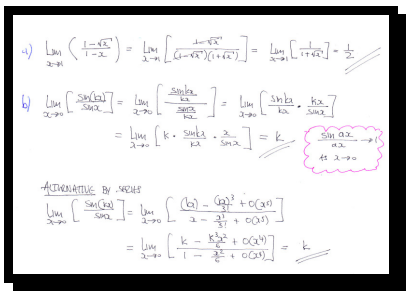
Find the value of each of the following limits.

a) $\lim\limits_{x\to 1}\left[\dfrac{1-\sqrt{x}}{1-x}\right]$.

b) $\lim\limits_{x\to 0}\left[\dfrac{\sin(kx)}{\sin x}\right]$.

You may not use the L' Hospital's rule in this question.

$\boxed{32}$



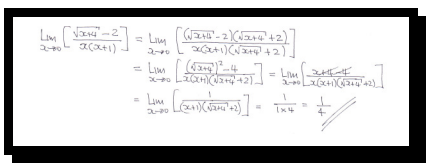
Question 16 (****)

Find the value of the following limit

$$\lim_{x\to 0}\left[\frac{\sqrt{x+4}-2}{x(x+1)}\right].$$

You may not use the L' Hospital's rule in this question.

$\boxed{\dfrac{1}{4}}$



Question 17 (****)

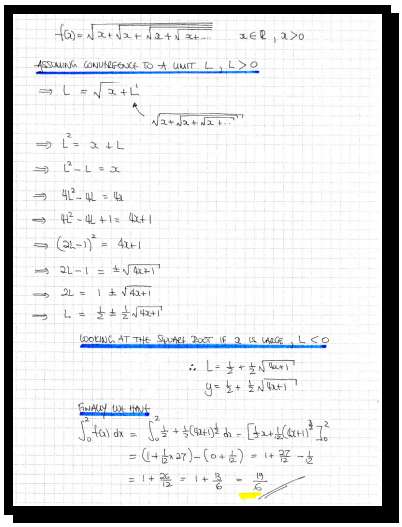
The function f is defined as

$$f(x)\equiv\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+...}}}}}\ ,\quad x\in(0,\infty).$$

Determine the value of

$$\int_0^2 f(x)\ dx\ .$$

$\square$, $\boxed{\frac{19}{6}}$

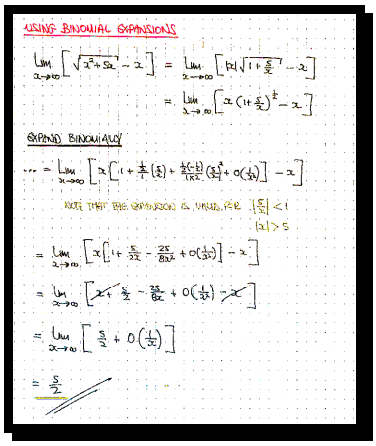


Question 18 **(****+)**

Find the value of the following limit

$$\lim_{x\to\infty}\left[\sqrt{x^2+5x}-x\right].$$

$\boxed{}$, $\boxed{\frac{5}{2}}$

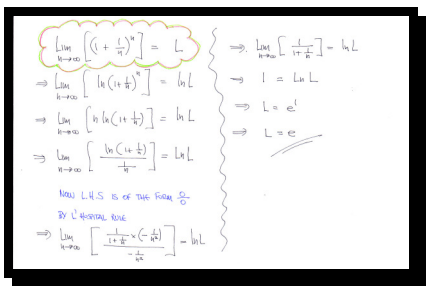


Question 19 **(****+)**

Find the value of the following limit

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right].$$

$\boxed{\text{e}}$

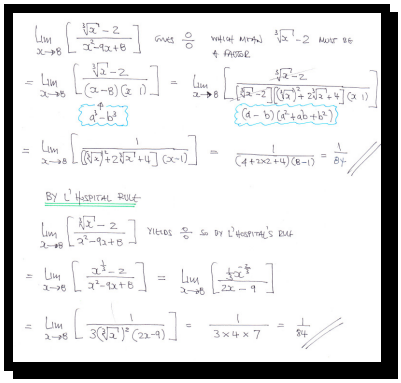


Question 20 (**+)**

Use two distinct methods to evaluate the following limit

$$\lim_{x\to 8}\left[\frac{\sqrt[3]{x}-2}{x^2-9x+8}\right].$$

$$\boxed{\frac{1}{84}}$$



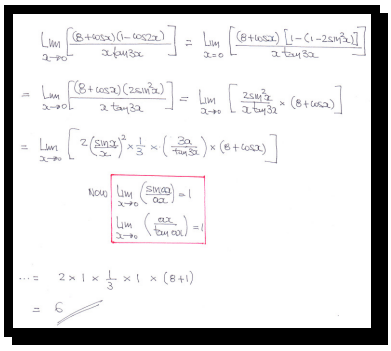
Question 21 (**+)**

Find the value of the following limit

$$\lim_{x\to 0}\left[\frac{(8+\cos x)(1-\cos 2x)}{x\tan 3x}\right].$$

You may not use the L' Hospital's rule in this question.

$$\boxed{6}$$

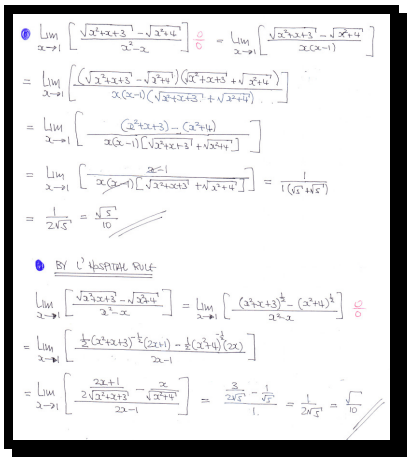


Question 22 (**+)**

Use two distinct methods to evaluate the following limit

$$\lim_{x\to 1}\left[\frac{\sqrt{x^2+x+3}-\sqrt{x^2+4}}{x^2-x}\right].$$

$$\boxed{\frac{\sqrt{10}}{5}}$$



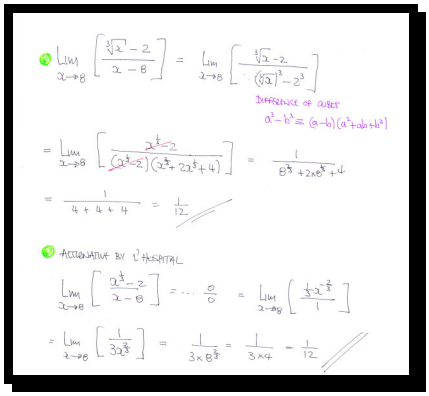
Question 23 **(****+)**

Find the value of the following limit

$$\lim_{x\to 8}\left[\frac{\sqrt[3]{x}-2}{x-8}\right].$$

You may not use the L' Hospital's rule in this question.

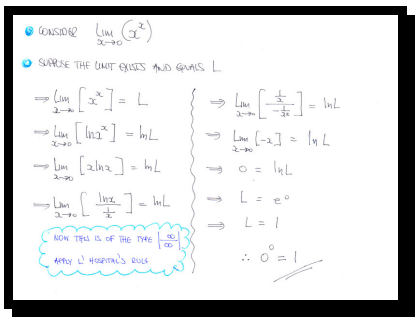
$$\frac{1}{12}$$



Question 24 **(****+)**

By considering the limit of an appropriate function show that $0^0=1$.

proof



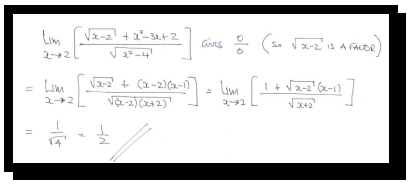
Question 25 (**+)**

Find the value of the following limit

$$\lim_{x\to 2}\left[\frac{\sqrt{x-2}+x^2-3x+2}{\sqrt{x^2-4}}\right].$$

You may not use the L' Hospital's rule in this question.

$$\boxed{\frac{1}{2}}$$



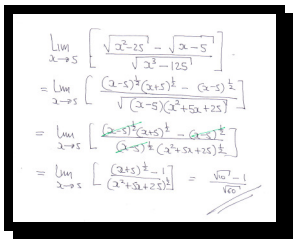
Question 26 (**+)**

Find the value of the following limit

$$\lim_{x\to 5}\left[\frac{\sqrt{x^2-25}-\sqrt{x-5}}{\sqrt{x^3-125}}\right].$$

You may not use the L' Hospital's rule in this question.

$$\boxed{\frac{\sqrt{10}-1}{\sqrt{60}}}$$

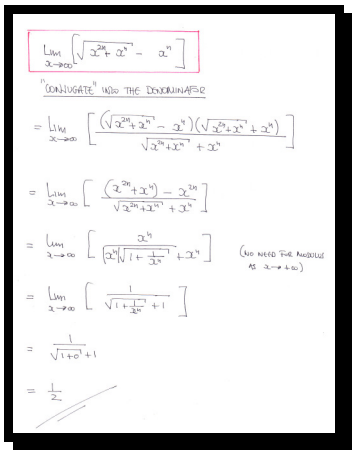


Question 27 (**+)**

Find the value of the following limit

$$\lim_{x\to\infty}\left[\sqrt{x^{2n}-x^{n}}\ -\ x^{n}\right],\ n\in\mathbb{N}.$$

$\boxed{\frac{1}{2}}$

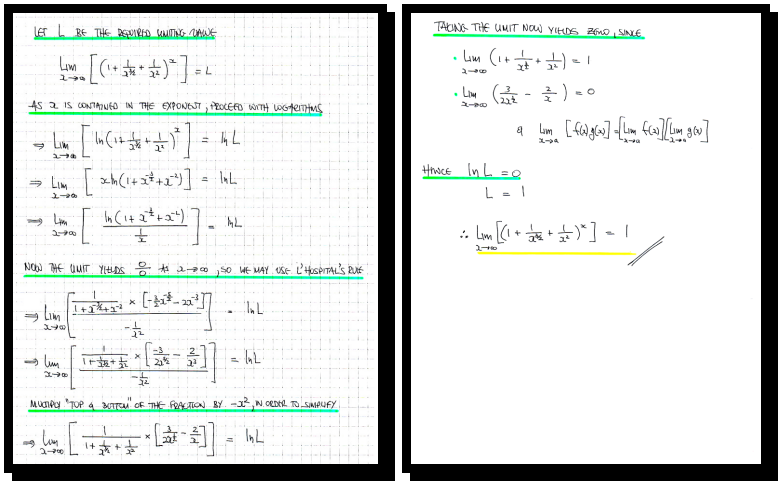


Question 28 (**+)**

Find the value of the following limit

$$\lim_{x\to\infty}\left[\left(1+\frac{1}{x^{\frac{3}{2}}}+\frac{1}{x^2}\right)^x\right].$$

$\boxed{}$, $\boxed{1}$

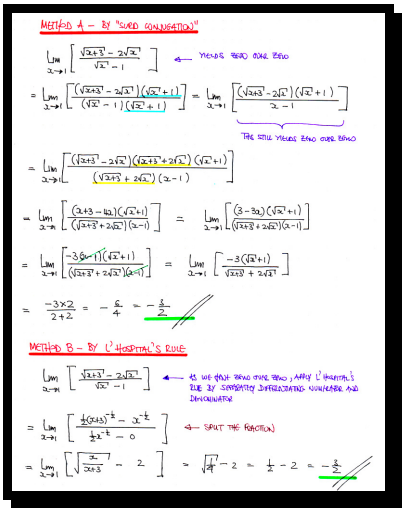


Question 29 (**+)**

Use two distinct methods to evaluate the following limit.

$$\lim_{x\to 1}\left[\frac{\sqrt{x+3}-2\sqrt{x}}{\sqrt{x}-1}\right].$$

$\square$, $\boxed{-\frac{3}{2}}$



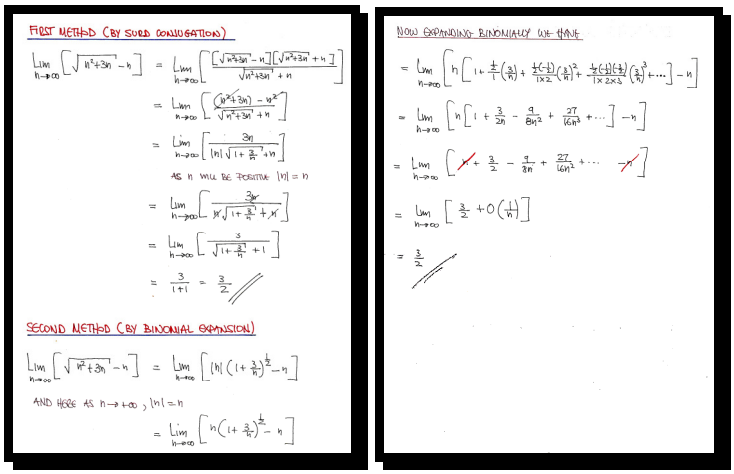
Question 30 (**+)**

Use two distinct methods to evaluate the following limit

$$\lim_{n\to\infty}\left[\sqrt{n^2+3n}-n\right].$$

You may not use the L' Hospital's rule in this question.

$\square$, $\boxed{\frac{3}{2}}$



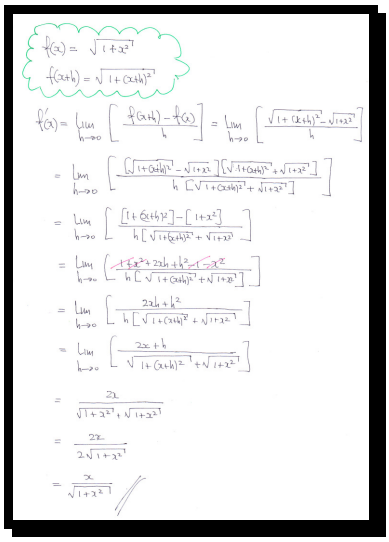
Question 31 (**+)**

$$f(x)=\sqrt{1+x^2}\ ,\ x\in\mathbb{R}\ .$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x)=\frac{x}{\sqrt{1+x^2}}\ .$$

proof



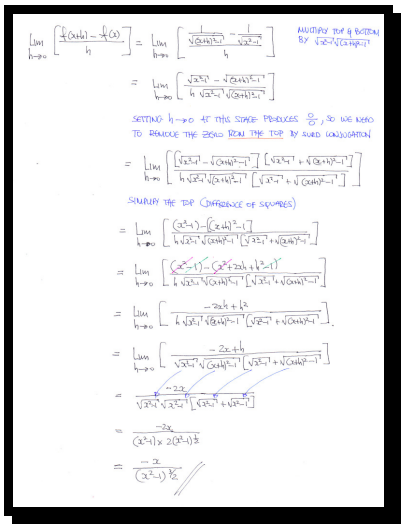
Question 32 (**+)**

$$f(x)=\frac{1}{\sqrt{x^2-1}},\ x\in\mathbb{R},\ |x|>1.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x)=-\frac{x}{\left(x^2-1\right)^{\frac{3}{2}}}.$$

proof



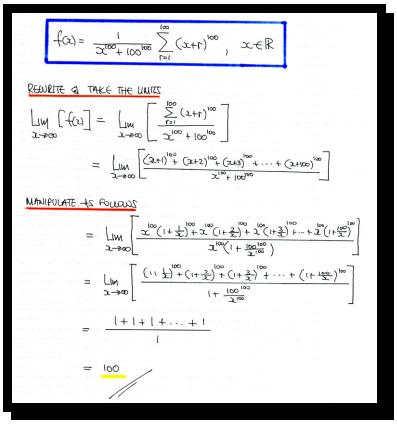
Question 33 (**+)**

$$f(x) \equiv \frac{1}{x^{100}+100^{100}} \sum_{r=1}^{100} (x+r)^{100}, \quad x \in \mathbb{R}.$$

Use a formal method to find

$$\lim_{x \to \infty} f(x).$$

$\square$, $\boxed{100}$

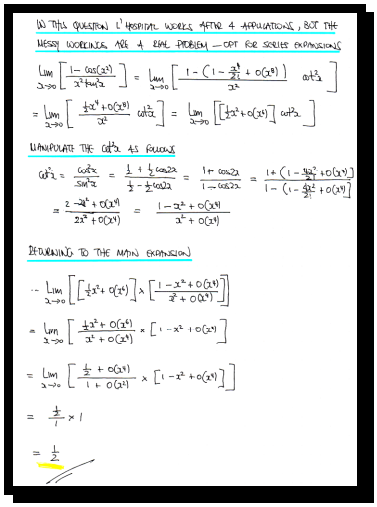


Question 34 **(****+)**

Find the value of the following limit

$$\lim_{x\to 0}\left[\frac{1-\cos\left(x^2\right)}{x^2\tan^2 x}\right].$$

$\square$, $\boxed{\frac{1}{2}}$



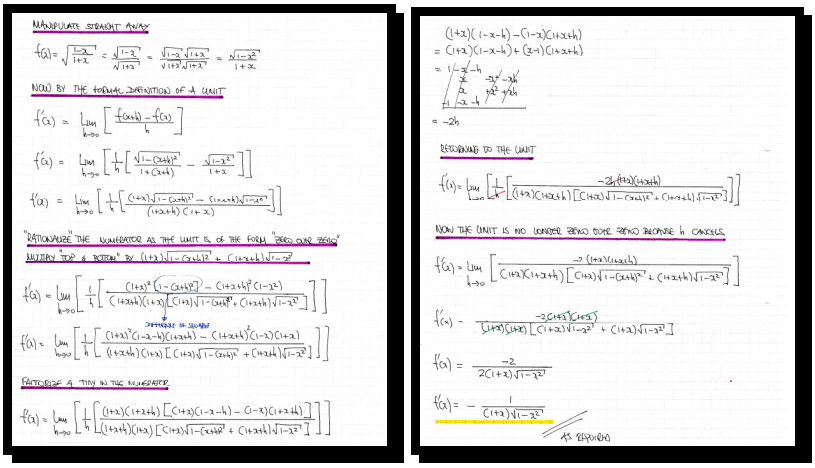
Question 35 (*****)

$$f(x)=\sqrt{\frac{1-x}{1+x}},\ x\in\mathbb{R},\ |x|<1.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x)=-\frac{1}{(1+x)\sqrt{1-x^2}}.$$

[ ], [proof]



Question 36 (*****)

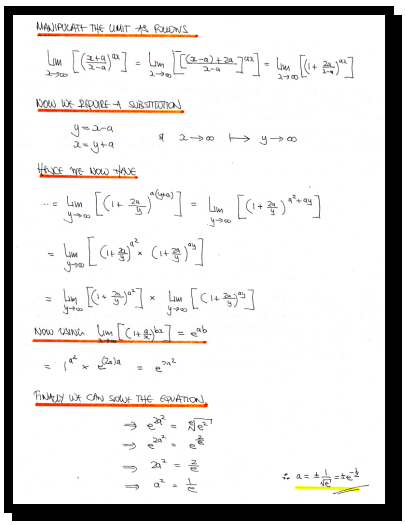
Solve the following equation over the set of real numbers.

$$\lim_{x\to\infty}\left[\left(\frac{x+a}{x-a}\right)^{ax}\right] = \sqrt[e]{e^2}\,.$$

You may assume that the limit in the left hand side of the equation exists.

You must clearly state any results used in the solution.

$\square$, $\boxed{a=\pm e^{-\frac{1}{2}}}$



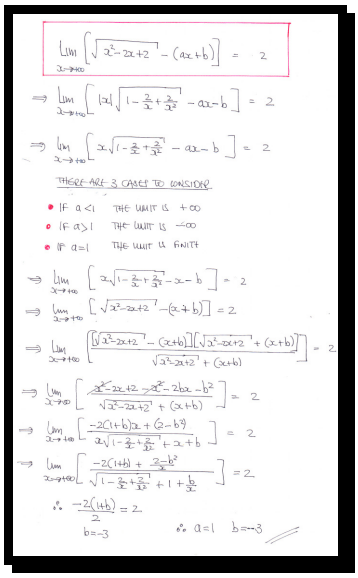
Question 37 (*****)

It is given that for some real constants a and b,

$$\lim_{x\to+\infty}\left[\sqrt{x^2-2x+2}-(ax+b)\right]=2,\ x\in\mathbb{R},\ x>0.$$

Determine the value of a and the value of b.

$\boxed{a=1}$, $\boxed{b=-3}$



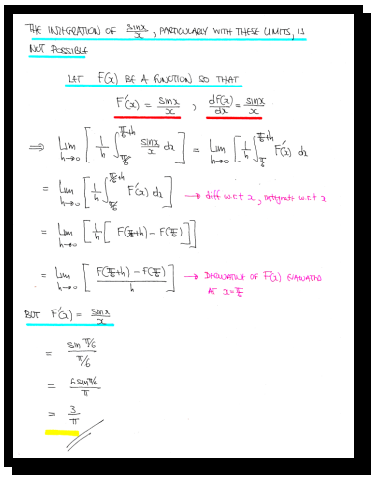
Question 38 (*****)

Determine the exact value of the following limit.

$$\lim_{h\to 0}\left[\frac{1}{h}\left[\int_{\frac{1}{6}\pi}^{\frac{1}{6}\pi+h}\frac{\sin x}{x}\,dx\right]\right]$$

You must justify the evaluation.

$\square$, $\boxed{\dfrac{3}{\pi}}$

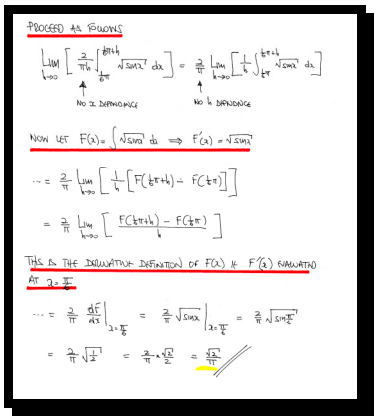


Question 39 (*****)

Evaluate the following limit.

$$\lim_{h\to 0}\left[\int_{\frac{1}{6}\pi}^{\frac{1}{6}\pi+h}\frac{2\sqrt{\sin x}}{\pi h}\,dx\right].$$

$\square$, $\boxed{\dfrac{\sqrt{2}}{\pi}}$



Question 40 **(*****)**

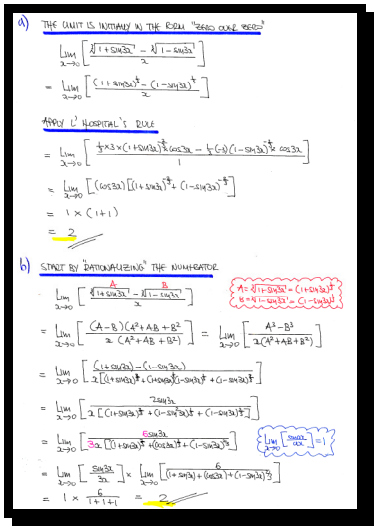
a) Use L'Hospital's rule to evaluate

$$\lim_{x\to 0}\left[\frac{\sqrt[3]{1+\sin 3x}-\sqrt[3]{1-\sin 3x}}{x}\right].$$

b) Verify the answer to part **(a)** by an alternative method.

You must state clearly any additional results used.

$\boxed{}$, $\boxed{2}$



Question 41 (*****)

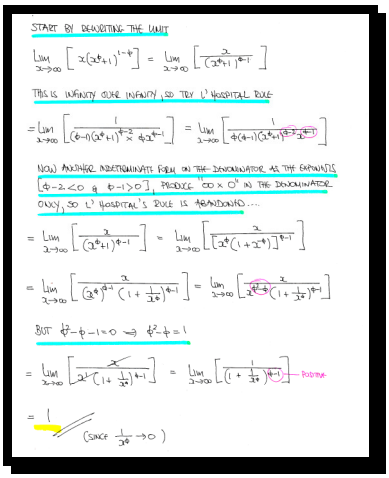
The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

This implies that $\phi^2 - \phi - 1 = 0$, $\phi = \frac{1}{2}\left(1+\sqrt{5}\right) \approx 1.62$.

Show, with full justification, that

$$\lim_{x\to\infty}\left[x\left(x^{\phi}+1\right)^{1-\phi} \right] = 1.$$

$\square$, proof



Question 42 **(*****)**

A curve has equation $y = f(x)$.

The finite region R is bounded by the curve, the x axis and the straight lines with equations $x = a$ and $x = b$, and hence the area of R is given by

$$I(a,b) = \int_a^b f(x)\ dx .$$

The area of R is also given by the limiting value of the sum of the areas of rectangles of width δx and height $f(x_i)$, known as a "right (upper) Riemann sum"

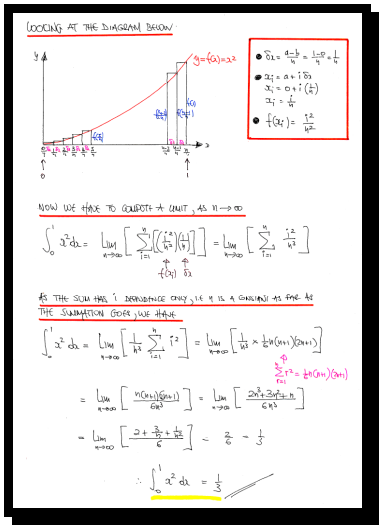
$$I(a,b) = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[f(x_i)\,\delta x\right]\right],$$

where $\delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\delta x$.

Using the "right (upper) Riemann sum" definition, and with the aid of a diagram where appropriate, show clearly that

$$\int_0^1 x^2\ dx = \frac{1}{3}.$$

, proof



Question 43 **(*****)**

The Lambert W function, also called the omega function or product logarithm, is a multivalued function which has the property

$$W\left(x\mathrm{e}^{x}\right)\equiv x\,,$$

and hence if $x\mathrm{e}^{x}=y$ then $x=W\left(y\right)$.

For example

$$-x\mathrm{e}^{-x}=2\ \ \Rightarrow\ \ -x=W\left(2\right),\ \ \left(x+\pi\right)\mathrm{e}^{x+\pi}=\frac{1}{2}\ \ \Rightarrow\ \ x+\pi=W\left(\frac{1}{2}\right)\text{ and so on.}$$

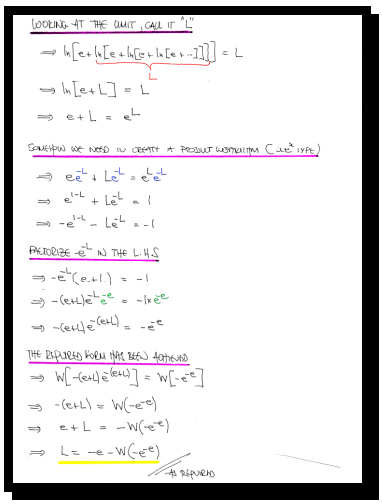
Use this result to show that the limit of

$$\ln\left(\mathrm{e}+\ln\left(\mathrm{e}+\ln\left(\mathrm{e}+\ln\left(\mathrm{e}+...\right)\right)\right)\right)$$

is given by

$$-\mathrm{e}-W\left[-\mathrm{e}^{-\mathrm{e}}\right].$$

[ ], [proof]



Question 44 **(*****)**

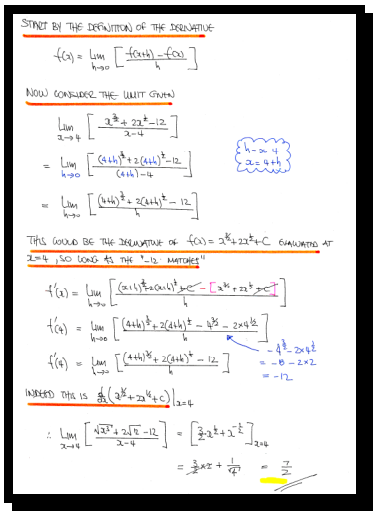
No credit will be given for using L'Hospital's rule in this question.

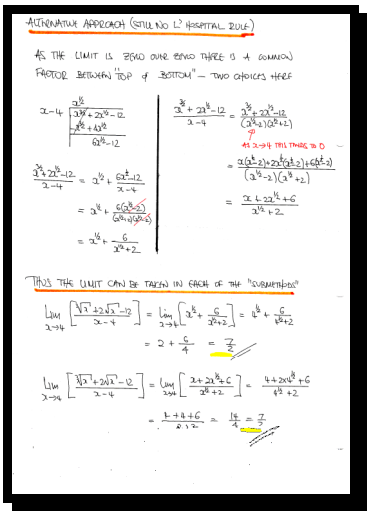
a) Use the formal definition of the derivative of a suitable expression, to find the value for the following limit

$$\lim_{x\to 4}\left[\frac{\sqrt{x^3}+2\sqrt{x}-12}{x-4}\right].$$

b) Verify the answer to part **(a)** by an alternative method.

$\square$, $\boxed{\frac{7}{2}}$





Question 45 **(*****)**

Use the formal definition of the derivative to prove that if

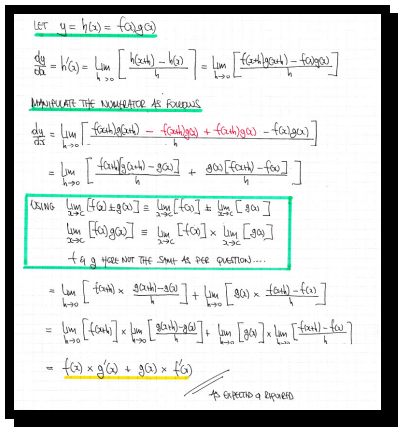
$$y = f(x)\, g(x),$$

then $\dfrac{dy}{dx} = f'(x)\, g(x) + f(x)\, g'(x)$

You may assume that

- $\lim\limits_{x \to c}\left[f(x) + g(x) \right] = \lim\limits_{x \to c}\left[f(x) \right] + \lim\limits_{x \to c}\left[g(x) \right]$
- $\lim\limits_{x \to c}\left[f(x) \times g(x) \right] = \lim\limits_{x \to c}\left[f(x) \right] \times \lim\limits_{x \to c}\left[g(x) \right]$

V, ☐, proof



Question 46 (*****)

A curve has equation $y = f(x)$.

The finite region R is bounded by the curve, the x axis and the straight lines with equations $x = a$ and $x = b$, and hence the area of R is given by

$$I(a,b) = \int_a^b f(x)\ dx .$$

The area of R is also given by the limiting value of the sum of the areas of rectangles of width δx and height $f(x_i)$, known as a "right (upper) Riemann sum"

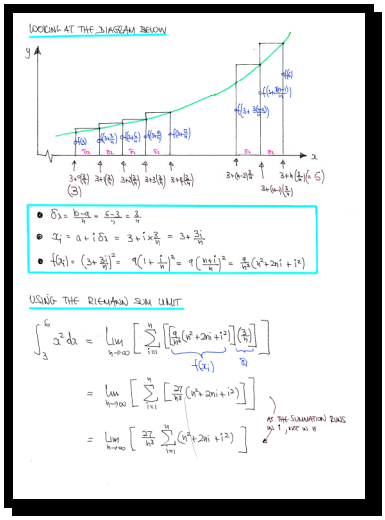
$$I(a,b) = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[f(x_i)\,\delta x\right]\right],$$

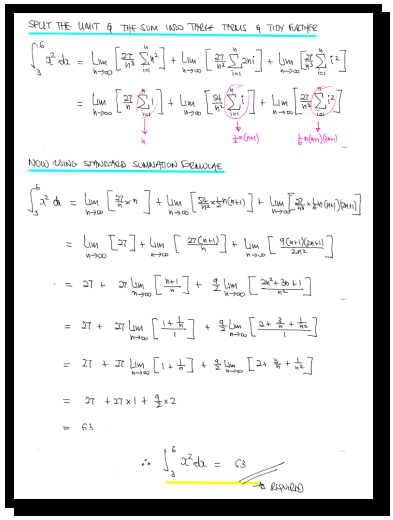
where $\delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\delta x$.

Using the "right (upper) Riemann sum" definition, and with the aid of a diagram where appropriate, show clearly that

$$\int_3^6 x^2\ dx = 63 .$$

, proof





Question 47 (*****)

A curve has equation $y = f(x)$.

The finite region R is bounded by the curve, the x axis and the straight lines with equations $x = a$ and $x = b$, and hence the area of R is given by

$$I(a,b) = \int_a^b f(x)\ dx .$$

The area of R is also given by the limiting value of the sum of the areas of rectangles of width δx and height $f(x_i)$, known as a "right (upper) Riemann sum"

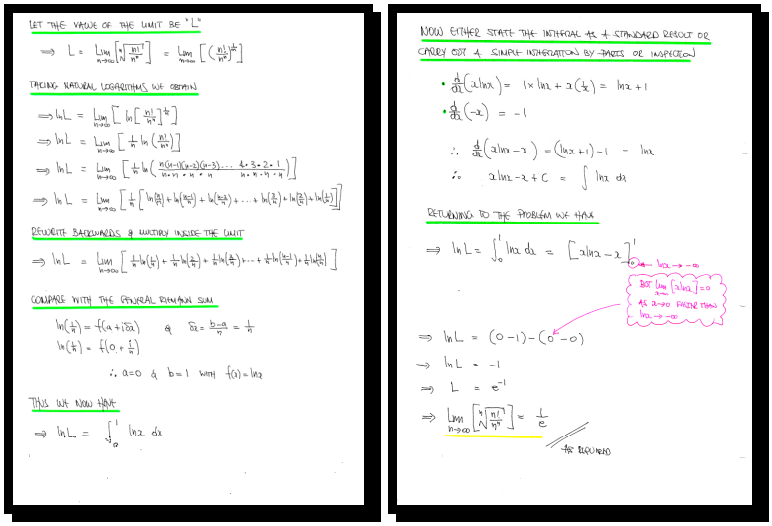
$$I(a,b) = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[f(x_i)\,\delta x\right]\right],$$

where $\delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\delta x$.

Using the "right (upper) Riemann sum" definition, and with the aid of a diagram where appropriate, show clearly that

$$\lim_{n\to\infty}\left[\sqrt[n]{\frac{n!}{n^n}}\right] = \frac{1}{\mathrm{e}}.$$

, proof

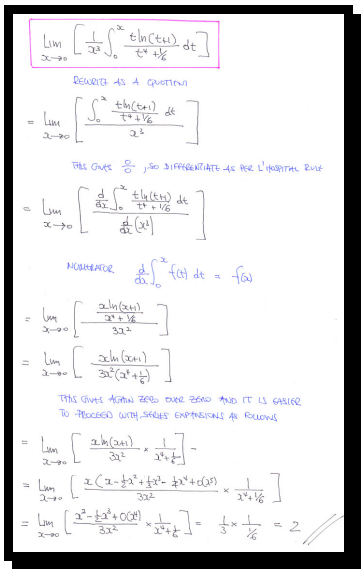


Question 48 **(*****)**

Use Leibniz rule and standard series expansions to evaluate the following limit

$$\lim_{x\to 0}\left[\frac{1}{x^3}\int_0^x \frac{t\ln(t+1)}{t^4+\frac{1}{6}}\,dt\right].$$

$\boxed{}$, $\boxed{2}$

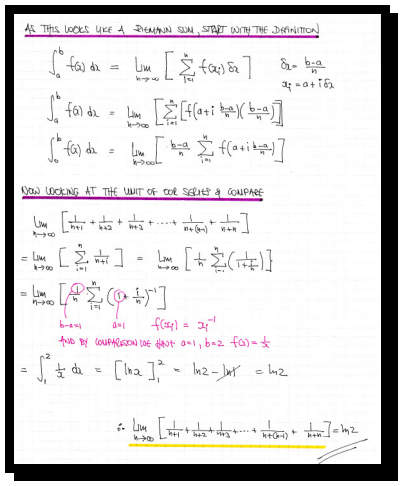


Question 49 (*****)

Determine the limit of the following series.

$$\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\frac{1}{n+4}+\ldots+\frac{1}{n+n-2}+\frac{1}{n+n-1}+\frac{1}{n+n}+\right].$$

V, [ ], ln2



Question 50 **(*****)**

a) Show with detailed workings that

$$\lim_{x\to\infty}\left[\sqrt{x^2+2x-1}-\sqrt{x^2-1}\right]=1.$$

b) Hence determine in exact simplified form the value of

$$\lim_{x\to\infty}\left[\left(\sqrt{x^2+2x-1}-\sqrt{x^2-1}\right)^x\right].$$

$\square$, $\boxed{e^{-\frac{1}{2}}=\frac{1}{\sqrt{e}}}$

