LEIBNIZ' S RULE OF DIFFERENTIATION

Leibniz Theorem

If $y = u(x)v(x)$ then

$$y_n = \sum_{r=1}^{n} \binom{n}{r} u_r v_{n-r} = u_n + nu_{n-1}v_1 + \frac{n(n-1)}{2!}u_{n-2}v_2 + \frac{n(n-1)(n-2)}{3!}u_{n-3}v_3 + \ldots \ ,$$

where $u_m = \dfrac{d^m u}{dx^m}$ and $v_m = \dfrac{d^m v}{dx^m}$.

nth order differential coefficients

$$\frac{d^n}{dx^n}\left(x^a\right) = y_n = \frac{a!}{(a-n)!}a^{a-n}$$

$$\frac{d^n}{dx^n}\left(\mathrm{e}^{ax}\right) = y_n = a^n \mathrm{e}^{ax}$$

$$\frac{d^n}{dx^n}(\sin ax) = y_n = a^n \sin\left[ax + \frac{n\pi}{2}\right]$$

$$\frac{d^n}{dx^n}(\cos ax) = y_n = a^n \cos\left[ax + \frac{n\pi}{2}\right]$$

$$\frac{d^n}{dx^n}(\sinh ax) = y_n = \frac{1}{2}a^n\left[\left[1-(-1)^n\right]\sinh ax + \left[1+(-1)^n\right]\cosh ax\right]$$

$$\frac{d^n}{dx^n}(\cosh ax) = y_n = \frac{1}{2}a^n\left[\left[1+(-1)^n\right]\sinh ax + \left[1-(-1)^n\right]\cosh ax\right]$$

Question 1 **(***)**

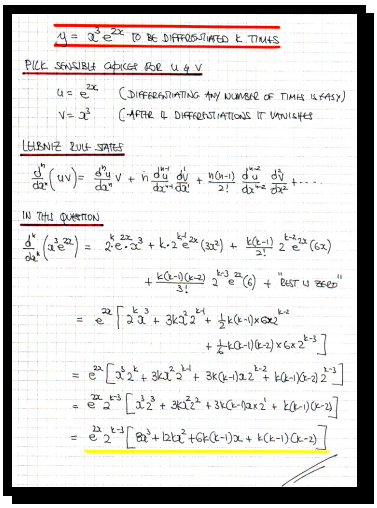
$$y = x^3 \mathrm{e}^{2x}, \ x \in \mathbb{R}.$$

Use the Leibniz rule to show that

$$\frac{d^k y}{dx^k} = \mathrm{e}^{2x} 2^{k-3} f(x,k), \ k \in \mathbb{N},$$

where $f(x,k)$ is a function to be found.

$\square$, $\boxed{\dfrac{d^k y}{dx^k} = \mathrm{e}^{2x} 2^{k-3}\left[8x^3 + 12kx^2 + 6k(k-1)x + k(k-1)(k-2)\right]}$



Question 2 **(***)**

$$y = x^3 \mathrm{e}^{2x}, \ x \in \mathbb{R}.$$

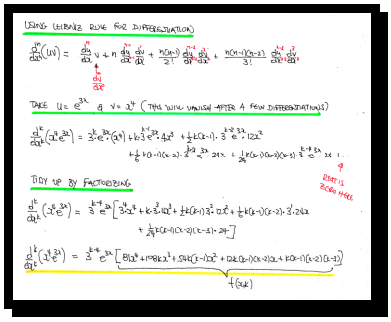
Use the Leibniz rule to show that

$$\frac{d^k y}{dx^k} = \mathrm{e}^{3x} 3^{k-4} f(x,k), \ k \in \mathbb{N},$$

where $f(x,k)$ is a function to be found.

V, ,

$$f(x,k) = \left[81x^4 + 108kx^3 + 54k(k-1)x^2 + 12k(k-1)(k-2)x + k(k-1)(k-2)(k-3) \right]$$

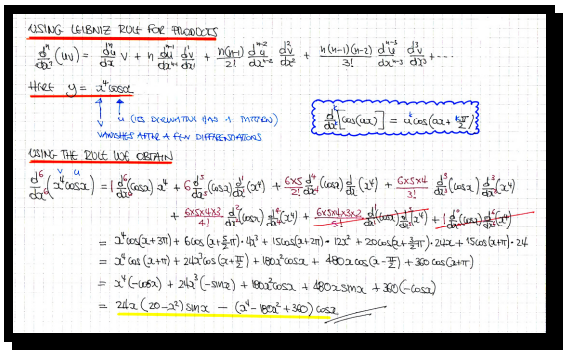


Question 3 **(***+)**

$$y = x^4 \cos x\,,\ \ x \in \mathbb{R}\,.$$

Use the Leibniz rule to find a simplified expression for $\dfrac{d^6 y}{dx^6}$.

$\square$, $\boxed{\dfrac{d^6 y}{dx^6} = 24x\left(20 - x^2\right)\sin x - \left(x^4 - 180x^2 + 360\right)\cos x}$

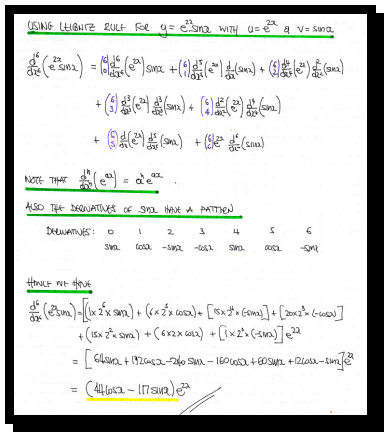


Question 4 **(***+)**

$$y = \mathrm{e}^{2x} \sin x \,, \quad x \in \mathbb{R} \,.$$

Use the Leibniz rule to find a simplified expression for $\dfrac{d^6 y}{dx^6}$.

$\boxed{}$, $\boxed{\dfrac{d^6 y}{dx^6} = \mathrm{e}^{2x}\left(44\cos x - 117\sin x\right)}$



Question 5 **(***+)**

The function with equation $y=f\left(x\right)$ is differentiable n times, $n\in\mathbb{N}$, and satisfies the following relationship.

$$\left(1+x^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0\,.$$

Use the Leibniz rule to show that at $x=0$

$$\frac{d^{n+2}y}{dx^{n+2}}=\left(4-n^2\right)\frac{d^ny}{dx^n}\,.$$

$\boxed{}$, $\boxed{\text{proof}}$

