Created by T. Madas

LEGENDRE'S EQUATION

including Legendre's functions and Legendre's polynomials

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Summary on Legendre Functions/Polynomials

Legendre's Differential Equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n-1\right)y=0,\ \ n\in\mathbb{R}.$$

General Solution of Legendre's Differential Equation

$$y=A\left[1-\frac{(n+1)n}{2!}x^2+\frac{(n+3)(n+1)n(n-2)}{4!}x^4-\frac{(n+5)(n+3)(n+1)n(n-2)(n-4)}{6!}x^6+\ldots\right]$$

$$+$$

$$B\left[x-\frac{(n+2)(n-1)}{3!}x^3+\frac{(n+4)(n+2)(n-1)(n-3)}{3!}x^5-\frac{(n+6)(n+4)(n+2)(n-1)(n-3)(n-5)}{7!}x^7+\ldots\right]$$

- If n is an even integer, the first solution terminates after a finite number of terms, while the second one produces an infinite series.

- If n is an odd integer, the second solution terminates after a finite number of terms, while the first solution produces an infinite series.

- The finite solutions are the Legendre Polynomials, also known as solutions of the first kind, denoted by $P_n\left(x\right)$.

- The infinite series solutions are known as solutions of the second kind, denoted by $Q_n\left(x\right)$.

The second solution $Q_n\left(x\right)$ can be written in terms of $P_n\left(x\right)$ by

$$Q_n\left(x\right)=P_n\left(x\right)\int\frac{1}{\left(1-x^2\right)\left(P_n\left(x\right)\right)^2}\,dx$$

The infinite series form for the Legendre's polynomial $P_n(x)$ is given by

$$P_n(x)=\sum_{k=0}^{N}\left[\frac{(2n-2k)!}{2^n\ k!\,(n-k)!(n-2k)!}(-1)^k\ x^{n-2k}\right],$$

where N is the floor function

$$N=\begin{cases}\frac{1}{2}n & \text{if } n \text{ is even}\\ \frac{1}{2}(n-1) & \text{if } n \text{ is odd}\end{cases}$$

The generating function for the Legendre's polynomial $P_n(x)$ is given by

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}\left[t^n\ P_n(x)\right]$$

Question 1

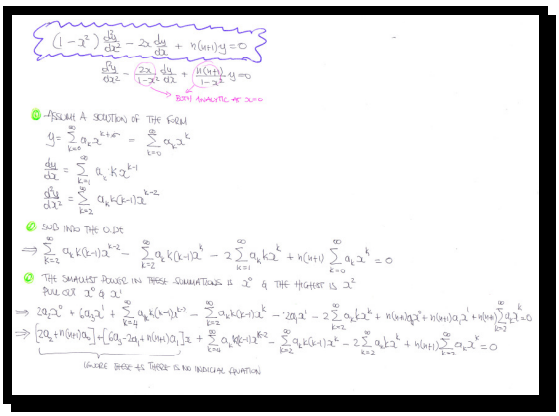
Find the two independent solutions of Legendre's equation

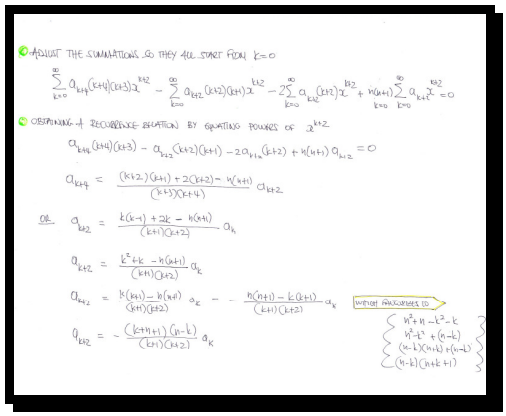
$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0\,,\ n\in\mathbb{R}\,.$$

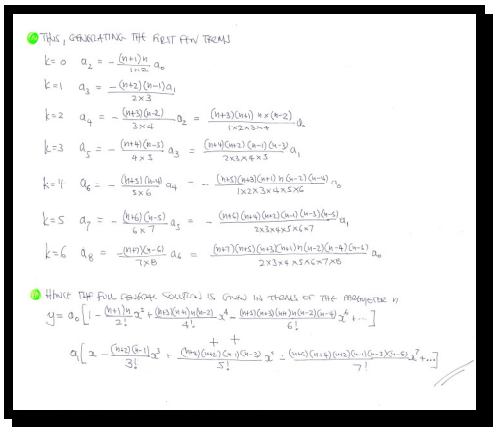
$$y=A\left[1-\frac{(n+1)n}{2!}x^2+\frac{(n+3)(n+1)n(n-2)}{4!}x^4-\frac{(n+5)(n+3)(n+1)n(n-2)(n-4)}{6!}x^6+\ldots\right]$$

$$+$$

$$B\left[x-\frac{(n+2)(n-1)}{3!}x^3+\frac{(n+4)(n+2)(n-1)(n-3)}{3!}x^5-\frac{(n+6)(n+4)(n+2)(n-1)(n-3)(n-5)}{7!}x^7+\ldots\right]$$







Question 2

Legendre's equation is given below

$$\left(1-t^2\right)\frac{d^2w}{dt^2}-2t\frac{dw}{dt}+n\left(n+1\right)w=0\,,\ n\in\mathbb{N}\,.$$

a) By assuming a series solution of the form

$$w\left(t\right)=\sum_{r=0}^{\infty}a_r\,t^r\,,\ a_0\neq 0\,,$$

show by a detailed method that

$$a_{r+2}=-\frac{\left(n-r\right)\left(n+r+1\right)}{\left(r+2\right)\left(r+1\right)}a_r\,.$$

b) By rewriting the recurrence relation of part **(a)** backwards, and taking the value of a_n as

$$a_n=\prod_{m=1}^{n}\frac{\left(2n-2m+1\right)}{n!}\,,$$

show further that the Legendre's polynomials $P_n\left(t\right)$ can be written as

$$P_n\left(t\right)=\sum_{k=0}^{N}\left[\frac{\left(2n-2k\right)!}{2^n\ k!\left(n-k\right)!\left(n-2k\right)!}\left(-1\right)^k\,t^{n-2k}\right],$$

where N is the floor function

$$N=\begin{cases}\frac{1}{2}n & \text{if } n \text{ is even}\\ \frac{1}{2}\left(n-1\right) & \text{if } n \text{ is odd}\end{cases}$$

proof

[solution overleaf]



Question 3

It can be shown that the Legendre's polynomials $P_n(x)$ can be written as

$$P_n(x)=\sum_{k=0}^{N}\left[\frac{(2n-2k)!}{2^n\,k!\,(n-k)!(n-2k)!}(-1)^k\,x^{n-2k}\right],$$

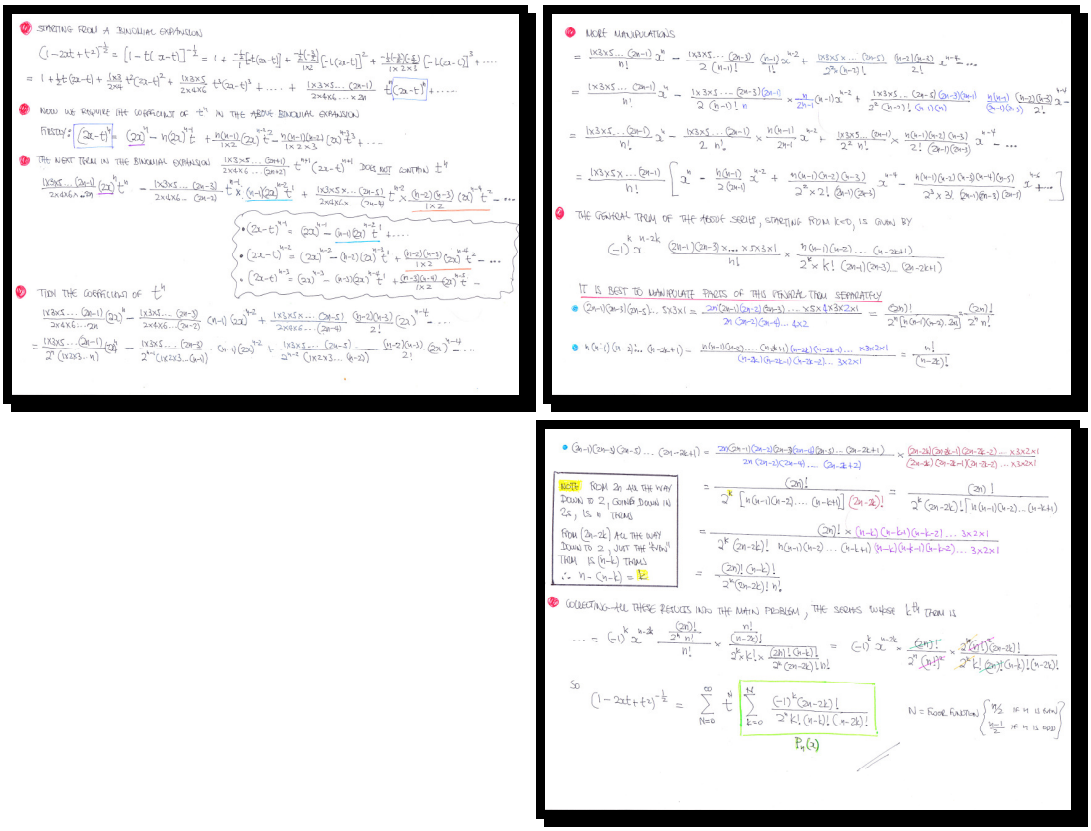
where N is the floor function

$$N=\begin{cases}\frac{1}{2}n & \text{if } n \text{ is even}\\ \frac{1}{2}(n-1) & \text{if } n \text{ is odd}\end{cases}$$

Show that the generating function for $P_n(x)$ satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}\left[t^nP_n(x)\right].$$

proof



Question 4

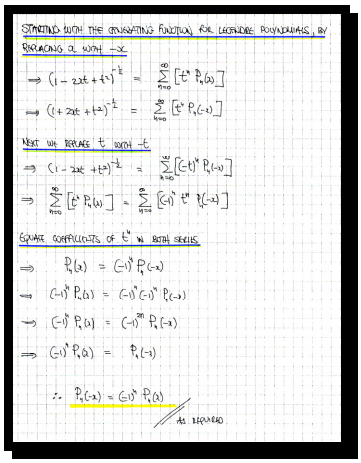
The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

Use this relationship to prove that

$$P_n(-x)=(-1)^n P_n(x).$$

[ ], [proof]



Question 5

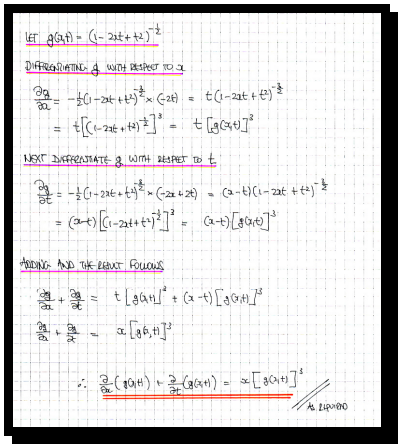
The generating function $g(x,t)$ for the Legendre's polynomials $P_n(x)$, satisfies

$$g(x,t)=\left(1-2xt+t^2\right)^{-\frac{1}{2}} \ = \ \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

Use this relationship to prove that

$$\frac{\partial}{\partial x}\left[g(x,t)\right]+\frac{\partial}{\partial t}\left[g(x,t)\right]=x\left[g(x,t)\right]^3.$$

[ ], [proof]



Question 6

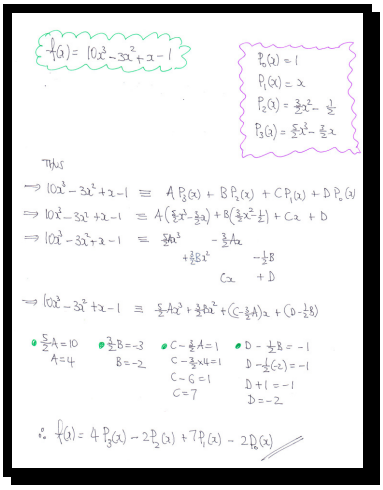
$$f(x) \equiv 10x^3 - 3x^2 + x - 1.$$

Express $f(x)$ as a linear combination of Legendre's polynomials, $P_n(x)$.

You may assume

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$
- $P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$,
- $P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$

$$\boxed{f(x) = 4P_3(x) - 2P_2(x) + 7P_1(x) - 2P_0(x)}$$



Question 7

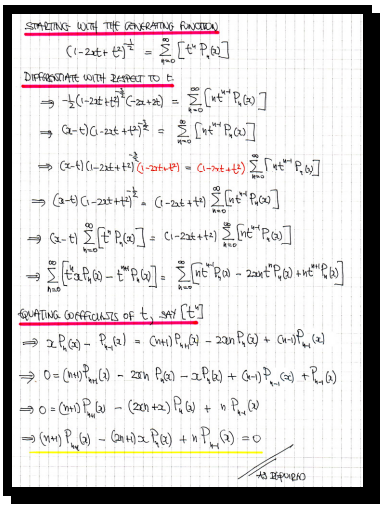
The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

By differentiating the above relationship with respect to t, prove that

$$(2n+1)\,x\,P_n(x)-(n+1)\,P_{n+1}(x)+n\,P_{n-1}(x)=0\,.$$

, proof



Question 8

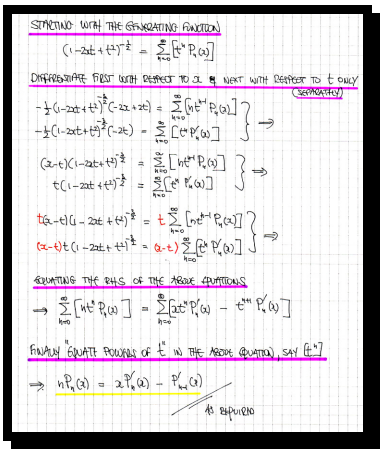
The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

By separately differentiating the above relationship once with respect to t and once with respect to x, prove that

$$n\,P_n(x) = x\,P_n'(x) - P_{n-1}'(x).$$

, proof



Question 9

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

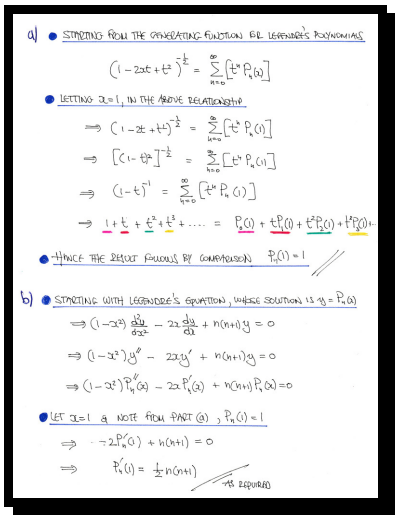
a) Use this result to show that

$$P_n(1)=1.$$

b) By using the result of part **(a)** and Legendre's equation, deduce that

$$P_n'(1)=\frac{1}{2}n(n+1).$$

, proof



Question 10

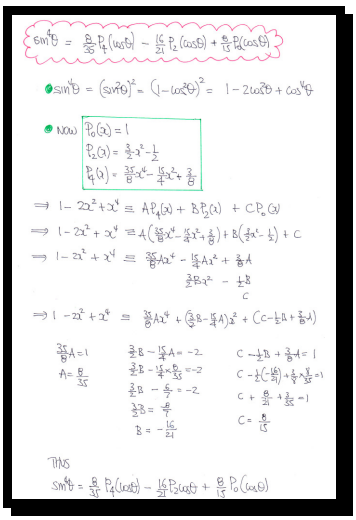
Use trigonometric identities to show that

$$\sin^2\theta=\frac{8}{35}P_4(\cos\theta)-\frac{16}{21}P_2(\cos\theta)+\frac{8}{15}P_0(\cos\theta)$$

You may assume

- $P_0(x)=1$
- $P_1(x)=x$
- $P_2(x)=\frac{1}{2}\left(3x^2-1\right)$
- $P_3(x)=\frac{1}{2}\left(5x^3-3x\right)$,
- $P_4(x)=\frac{1}{8}\left(35x^4-30x^2+3\right)$

proof



Question 11

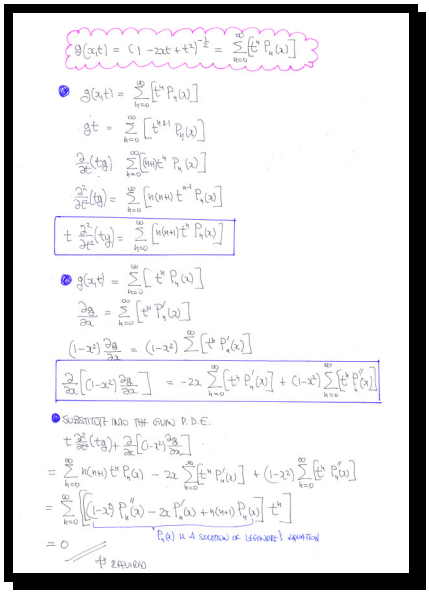
The generating function $g(x,t)$ for the Legendre's polynomials $P_n(x)$, satisfies

$$g(x,t)=\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

Verify that $g=g(x,t)$ is a solution of the differential equation

$$t\frac{\partial^2}{\partial t^2}[t\,g]+\frac{\partial}{\partial x}\left[\left(1-x^2\right)\frac{\partial g}{\partial x}\right]=0\,.$$

proof



Question 12

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

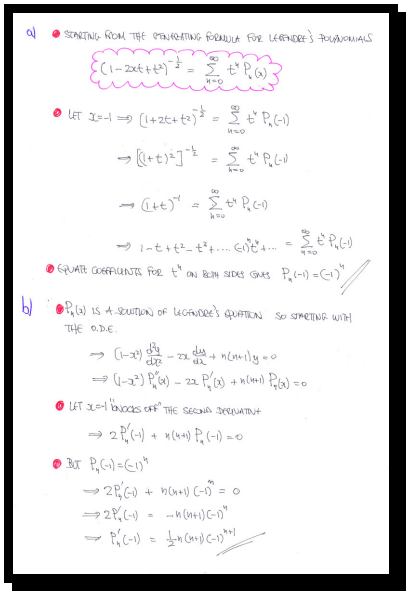
a) Use this result to show that

$$P_n(-1)=(-1)^n.$$

b) By using the result of part **(a)** and Legendre's equation, deduce that

$$P_n'(-1)=\frac{1}{2}n(n+1)(-1)^{n+1}.$$

, proof



Question 13

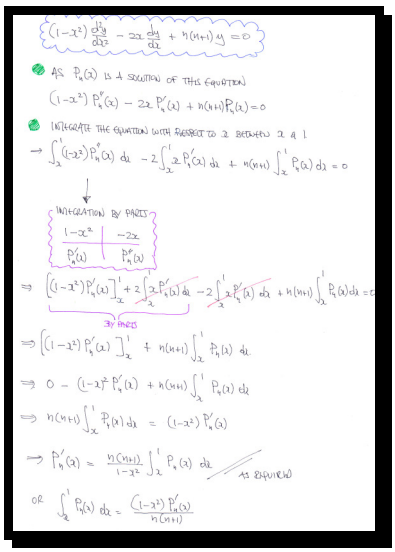
The Legendre's polynomial $P_n(x)$ is a solution of the differential equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)\,y=0\,,\quad n\in\mathbb{R}\,.$$

Show that

$$P_n'(x)=\frac{n(n+1)}{1-x^2}\int_1^x P_n(x)\;dx\,.$$

proof



Question 14

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

a) By differentiating the above relationship with respect to t, prove that

$$(2n+1)\,x\,P_n(x) = (n+1)\,P_{n+1}(x) + n\,P_{n-1}(x).$$

b) By separately differentiating the generating function for the Legendre's polynomials once with respect to t and once with respect to x, prove that

$$n\,P_n(x) = x\,P_n'(x) - P_{n-1}'(x).$$

c) Use parts **(a)** and **(b)** to show that

$$(2n+1)\,P_n(x) = P_{n+1}'(x) + P_{n-1}'(x).$$

d) Use parts **(b)** and **(c)** to deduce that

$$(n+1)\,P_n(x) = P_{n+1}'(x) - x\,P_n'(x).$$

proof



Question 15

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

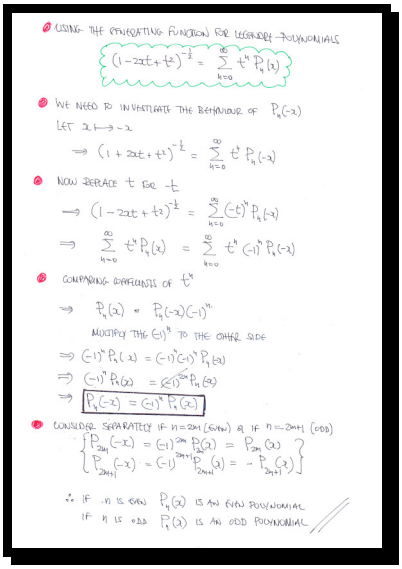
$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

Use this result to show that …

… if n is even, $P_n(x)$ is an even polynomial in x.

… if n is odd, $P_n(x)$ is an odd polynomial in x.

proof



Question 16

The generating function for the Legendre's Polynomials $P_n(x)$, satisfies

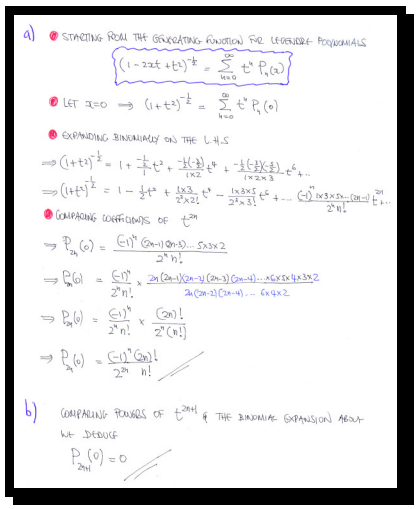
$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

a) Use this result to show that

$$P_{2n}(0)=\frac{(-1)^n(2n)!}{2^{2n}\,n!}.$$

b) Deduce the value of $P_{2n+1}(0)$.

$$\boxed{P_{2n+1}(0)=0}$$



Question 17

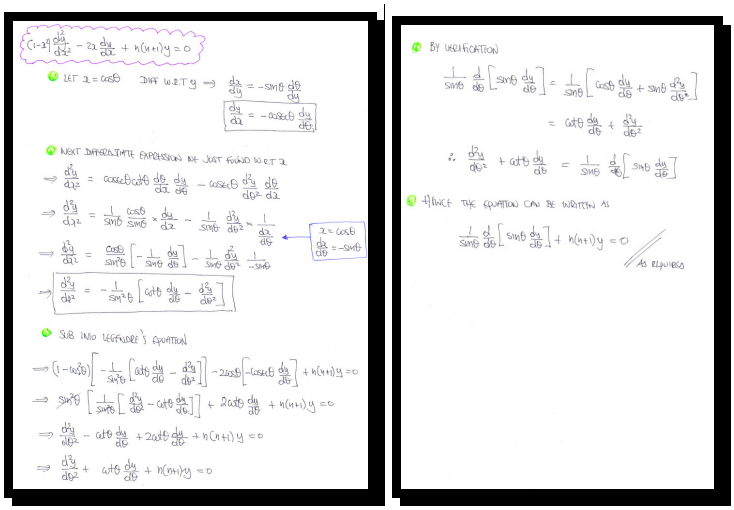
Legendre's equation is given below

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0\,,\quad n\in\mathbb{R}\,.$$

Use the substitution $x=\cos\theta$ to show that

$$\frac{1}{\sin\theta}\,\frac{d}{d\theta}\left[\frac{dy}{d\theta}\sin\theta\right]+n\left(n+1\right)y=0\,.$$

proof

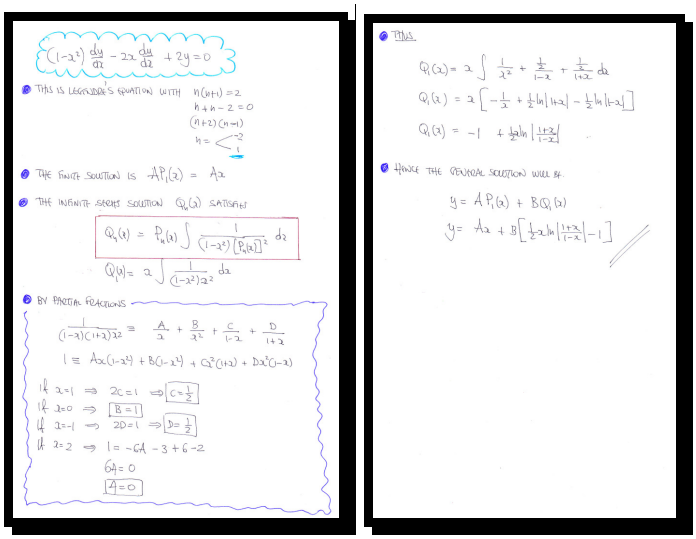


Question 18

Find the two independent solutions of Legendre's equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0\,.$$

$$\boxed{y=Ax+B\left[\frac{1}{x}\ln\left(\frac{1+x}{1-x}\right)-1\right]}$$



Question 19

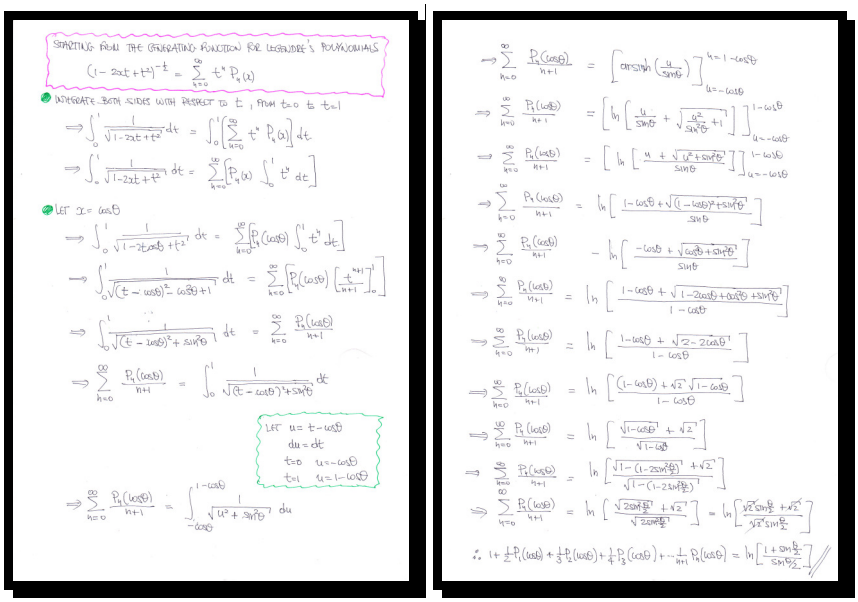
The generating function for the Legendre's Polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

Using this result, and integrating both sides with respect to t, from 0 to 1, show that

$$\sum_{n=0}^{\infty}\left[\frac{P_n(\cos\theta)}{n+1}\right] = \ln\left[1+\operatorname{cosec}\left(\frac{1}{2}\theta\right)\right].$$

proof



Question 20

The generating function g for the Legendre's polynomials $P_n(x)$, satisfies

$$g(x,t)=\left(1-2xt+t^2\right)^{-\frac{1}{2}} \;=\; \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

a) By differentiating g with respect to t, prove that

$$(2n+1)\,x\,P_n(x)=(n+1)\,P_{n+1}(x)+n\,P_{n-1}(x).$$

b) By differentiating g once with respect to t and once with respect to x, prove that

$$n\,P_n(x)=x\,P_n'(x)-P_{n-1}'(x).$$

c) Use parts **(a)** and **(b)** to show that

$$(2n+1)\,P_n(x)=P_{n+1}'(x)+P_{n-1}'(x).$$

d) Use parts **(b)** and **(c)** to deduce that

$$(n+1)\,P_n(x)=P_{n+1}'(x)-x\,P_n'(x).$$

e) Use parts **(b)** and **(d)** to show that

$$\left(1-x^2\right)P_n'(x)=n\left[P_{n-1}(x)-x\,P_n(x)\right].$$

f) Use parts **(a)** and **(e)** to show that

$$\left(1-x^2\right)P_n'(x)=(n+1)\left[xP_n(x)-P_{n+1}(x)\right]$$

proof

[solution overleaf]



Question 21

Find one series solution for the Legendre's equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0\,,\ n\in\mathbb{R}\,,$$

about $x=1$.

$$y=A\sum_{r=0}^{\infty}\left[\frac{(n+r)!}{(n-r)!}\times\frac{1}{(r!)^2}\times\left(\frac{x-1}{2}\right)^2\right]$$

