LAPLACE TRANSFORMS INTRODUCTION

SUMMARY OF THE LAPLACE TRANFORM

The Laplace Transform of a function $f(t)$, $t \geq 0$ is defined as

$$\mathcal{L}\left[f(t)\right] \equiv \overline{f}(s) \equiv \int_0^{\infty} \mathrm{e}^{-st} f(t)\ dt,$$

where $s \in \mathbb{C}$, with $\mathrm{Re}(s)$ sufficiently large for the integral to converge.

The Laplace Transform is a linear operation

$$\mathcal{L}\left[a f(t) + b g(t)\right] \equiv a\mathcal{L}\left[f(t)\right] + b\mathcal{L}\left[g(t)\right].$$

Laplace Transforms of Common Functions

- $\mathcal{L}\left(t^n\right) = \dfrac{n}{s^{n+1}}$

$$\mathcal{L}(1) = \frac{1}{s}, \quad \mathcal{L}(a) = \frac{a}{s}, \quad \mathcal{L}(t) = \frac{1}{s^2}, \quad \mathcal{L}\left(t^2\right) = \frac{2}{s^3}, \quad \mathcal{L}\left(t^3\right) = \frac{3}{s^4}, \ldots.$$

- $\mathcal{L}\left(\mathrm{e}^{at}\right) = \dfrac{1}{s-a}$, $\mathcal{L}\left(\mathrm{e}^{-at}\right) = \dfrac{1}{s+a}$
- $\mathcal{L}(\cos at) = \dfrac{s}{s^2+a^2}$, $\mathcal{L}(\sin at) = \dfrac{a}{s^2+a^2}$
- $\mathcal{L}(\cosh at) = \dfrac{s}{s^2-a^2}$, $\mathcal{L}(\sinh at) = \dfrac{a}{s^2-a^2}$

Laplace Transforms of Derivatives

- $\mathcal{L}\left[x(t)\right] = \overline{x}(t)$
- $\mathcal{L}\left[\dot{x}(t)\right] = s\overline{x}(t) - x(0)$
- $\mathcal{L}\left[\ddot{x}(t)\right] = s^2\overline{x}(t) - sx(0) - \dot{x}(0)$
- $\mathcal{L}\left[\dddot{x}(t)\right] = s^3\overline{x}(t) - s^2x(0) - s\dot{x}(0) - \ddot{x}(0)$

Laplace Transforms Theorems

- 1st Shift Theorem

$$\mathcal{L}\left[\mathrm{e}^{-at} f(t)\right] = \overline{f}(s+a) \quad \text{or} \quad \mathcal{L}\left[\mathrm{e}^{at} F(t)\right] = \overline{f}(s-a)$$

- 2nd Shift Theorem

$$\mathcal{L}\left[f(t-a)\right] = \mathrm{e}^{-as}\,\overline{f}(s),\ t>a \quad \text{or} \quad \mathcal{L}\left[f(t+a)\right] = \mathrm{e}^{as}\,\overline{f}(s),\ t>-a\,.$$

$$\mathcal{L}\left[\mathrm{H}(t-a)\,f(t-a)\right] = \mathrm{e}^{-as}\,\overline{f}(s) \quad \text{or} \quad \mathcal{L}\left[\mathrm{H}(t+a)\,f(t+a)\right] = \mathrm{e}^{as}\,\overline{f}(s)$$

- Multiplication by t^n

$$\mathcal{L}\left[t^n f(t)\right] = \left(-\frac{d}{ds}\right)^n \left[\overline{f}(s)\right] \quad \text{or} \quad \mathcal{L}\left[t\,f(t)\right] = -\frac{d}{ds}\left[\overline{f}(s)\right]$$

- Division by t

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \overline{f}(\sigma)\ d\sigma$$

provided that $\lim\limits_{t\to 0}\left(\dfrac{f(t)}{t}\right)$ exists and the integral converges.

- Initial/Final value theorem

$$\lim_{t\to 0}\left[f(t)\right] = \lim_{s\to\infty}\left[s\,\overline{f}(s)\right] \quad \text{and} \quad \lim_{t\to\infty}\left[f(t)\right] = \lim_{s\to 0}\left[s\,\overline{f}(s)\right]$$

The Impulse Function / The Dirac Function

1. $\delta(t-c)=\begin{cases}\infty & t=c\\ 0 & t\neq c\end{cases}$, $\delta(t)=\begin{cases}\infty & t=0\\ 0 & t\neq 0\end{cases}$

2. $\displaystyle\int_a^b \delta(t-c)\ dt=\begin{cases}1 & a\le c\le b\\ 0 & \text{otherwise}\end{cases}$

3. $\displaystyle\int_a^b f(t)\delta(t-c)\ dt=\begin{cases} f(a) & a\le c\le b\\ 0 & \text{otherwise}\end{cases}$

4. $\mathcal{L}\left[\delta(t-c)\right]=\mathrm{e}^{-cs}$

5. $\mathcal{L}\left[f(t)\delta(t-c)\right]=f(c)\mathrm{e}^{-cs}$

6. $\dfrac{d}{dt}\left[\mathrm{H}(t-c)\right]=\delta(t-c)$

LAPLACE TRANSFORMS FROM FIRST PRINCIPLES

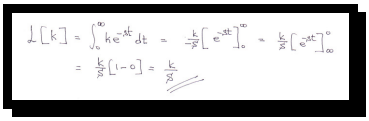
Question 1

Find, from first principles, the Laplace Transform of

$$f(t)=k\ ,\ t\geq 0$$

where k is non zero constant.

$$\boxed{\overline{f}(s)=\frac{a}{s}}$$



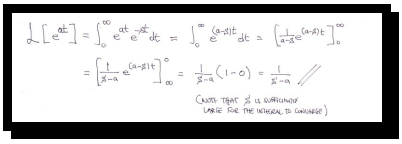
Question 2

Use integration to find the Laplace Transform of

$$f(t)=\mathrm{e}^{at}\ ,\ t\geq 0$$

where a is non zero constant.

$$\boxed{\overline{f}(s)=\frac{1}{s-a}}$$



Question 3

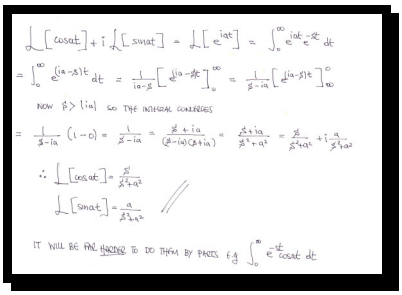
Find, from first principles, the Laplace Transform of

$$f(t)=\cos(at),\quad t\geq 0$$

$$g(t)=\sin(at),\quad t\geq 0$$

where a is non zero constant.

$$\boxed{\overline{f}(s)=\frac{s}{s^2+a^2}},\ \boxed{\overline{g}(s)=\frac{a}{s^2+a^2}}$$



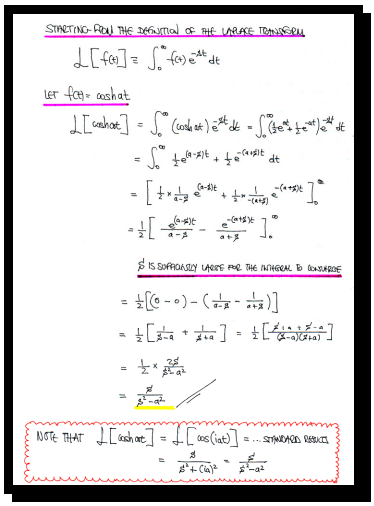
Question 4

Use integration to find the Laplace Transform of

$$f(t)=\cosh(at),\ t\geq 0$$

where a is non zero constant.

$\square$, $\boxed{\overline{f}(s)=\dfrac{s}{s^2-a^2}}$



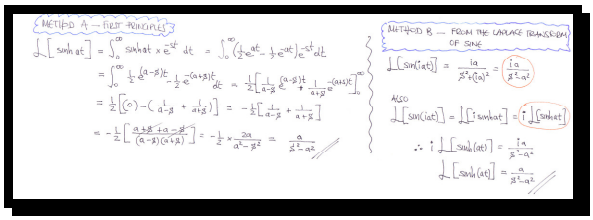
Question 5

Find, from first principles, the Laplace Transform of

$$f(t)=\sinh(at),\quad t\geq 0$$

where a is non zero constant.

$$\boxed{\overline{f}(s)=\frac{a}{s^2-a^2}}$$



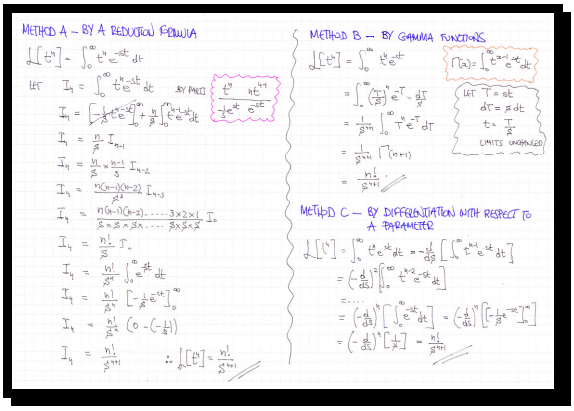
Question 6

Use integration to find the Laplace Transform of

$$f(t)=t^n,\ t\geq 0$$

where $n\neq \ldots -4,-3,-2,-1,0$.

$$\boxed{\overline{f}(s)=\frac{\Gamma(n+1)}{s^{n+1}}=\frac{n!}{s^{n+1}}}$$



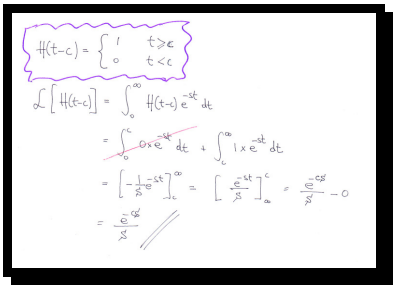
Question 7

The Heaviside function $\mathrm{H}(t)$ is defined as

$$\mathrm{H}(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$$

Determine the Laplace transform of $\mathrm{H}(t-c)$.

$$\mathcal{L}\left(\mathrm{H}(t-c)\right)=\frac{\mathrm{e}^{-cs}}{s}$$



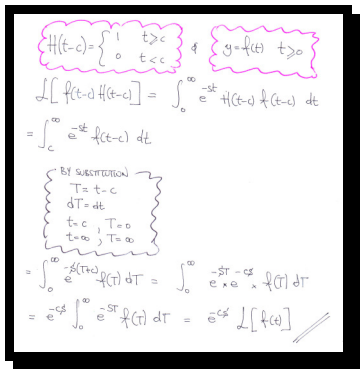
Question 8

The Heaviside step function $\mathrm{H}(t)$ is defined as

$$\mathrm{H}(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$$

Determine the Laplace transform of $\mathrm{H}(t-c)f(t-c)$, where $f(t)$ is a continuous or piecewise continuous function defined for $t\geq 0$.

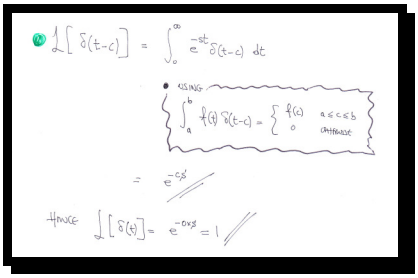
$$\boxed{\mathcal{L}\left(\mathrm{H}(t-c)f(t-c)\right)=\mathrm{e}^{-cs}\,\mathcal{L}\left(f(t)\right)}$$



Question 9

Find the Laplace transform of $\delta(t-c)$, where c is a positive constant, and hence state the Laplace transform of $\delta(t)$.

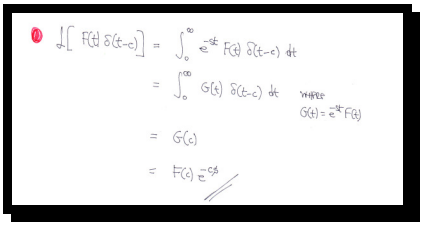
$$\boxed{\mathcal{L}\left[\delta(t-c)\right]=\mathrm{e}^{-cs}},\quad \boxed{\mathcal{L}\left[\delta(t)\right]=1}$$



Question 10

Given that $F(t)$ is a piecewise continuous function defined for $t \geq 0$, find the Laplace transform of $F(t)\,\delta(t-c)$, where c is a positive constant.

$$\boxed{\mathcal{L}\left[F(t)\,\delta(t-c)\right] = F(c)\mathrm{e}^{-cs}}$$



LAPLACE TRANSFORM GENERAL PRACTICE

Question 1

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left(t^3+2\mathrm{e}^{-2t}\right)$

b) $\mathcal{L}\left(\mathrm{e}^{-2t}\cosh 3t\right)$

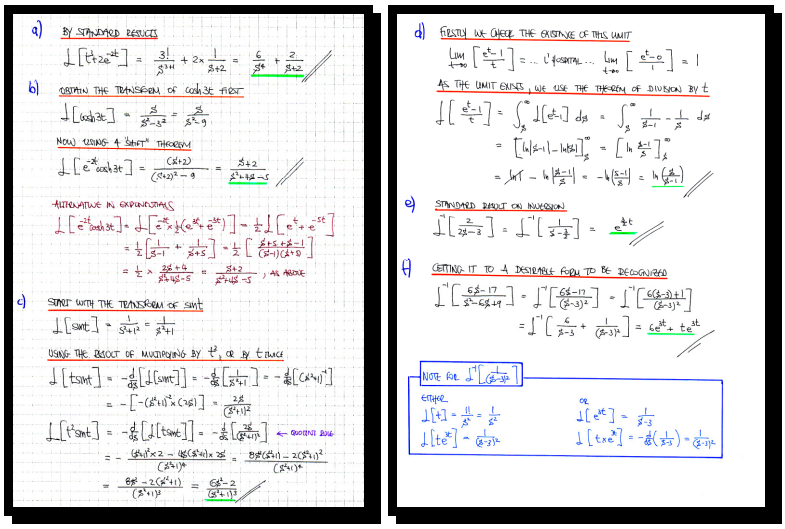
c) $\mathcal{L}\left(t^2\sin t\right)$

d) $\mathcal{L}\left(\dfrac{\mathrm{e}^t-1}{t}\right)$

e) $\mathcal{L}^{-1}\left(\dfrac{2}{2s-3}\right)$

f) $\mathcal{L}^{-1}\left(\dfrac{6s-17}{s^2-6s+9}\right)$

$\square$, $\boxed{\dfrac{6}{s^4}+\dfrac{2}{s+2}}$, $\boxed{\dfrac{s+2}{s^2+4s-5}}$, $\boxed{\dfrac{6s^2-2}{\left(s^2+1\right)^3}}$, $\boxed{\ln\left(\dfrac{s}{s-1}\right)}$, $\boxed{\mathrm{e}^{\frac{1}{2}t}}$, $\boxed{6\mathrm{e}^{3t}+t\mathrm{e}^{3t}}$



Question 2

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left(3\cos 2t - 2\sinh 3t\right)$

b) $\mathcal{L}\left(2\mathrm{e}^{-3t}\cosh 4t\right)$

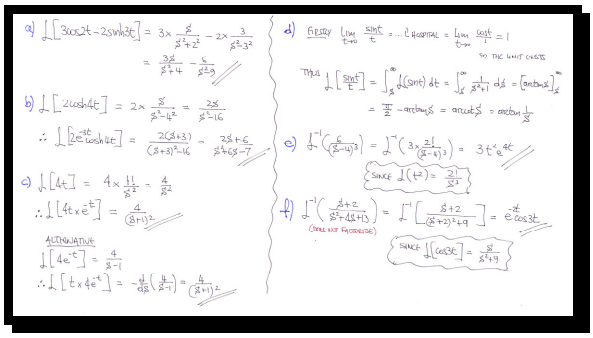
c) $\mathcal{L}\left(4t\,\mathrm{e}^{-t}\right)$

d) $\mathcal{L}\left(\dfrac{\sin t}{t}\right)$

e) $\mathcal{L}^{-1}\left[\dfrac{6}{(s-4)^3}\right]$

f) $\mathcal{L}^{-1}\left(\dfrac{s+2}{s^2+4s+13}\right)$

$\boxed{\dfrac{3s}{s^2+4}-\dfrac{6}{s^2-9}}$, $\boxed{\dfrac{2s+6}{s^2+6s-7}}$, $\boxed{\dfrac{4}{(s+1)^2}}$, $\boxed{\arctan\left(\dfrac{1}{s}\right)}$, $\boxed{3t^2\,\mathrm{e}^{4t}}$, $\boxed{\mathrm{e}^{-2t}\cos 3t}$



Question 3

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left(\mathrm{e}^{3t}+3\sin 2t\right)$

b) $\mathcal{L}\left(3\mathrm{e}^{3t}\sin 2t\right)$

c) $\mathcal{L}\left(t\cosh 2t\right)$

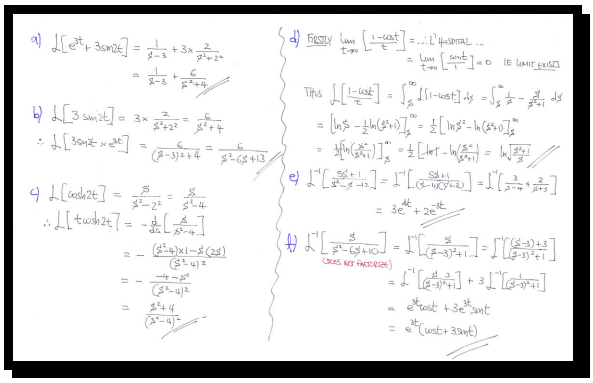
d) $\mathcal{L}\left(\dfrac{1-\cos t}{t}\right)$

e) $\mathcal{L}^{-1}\left(\dfrac{5s+1}{s^2-s-12}\right)$

f) $\mathcal{L}^{-1}\left(\dfrac{s}{s^2-6s+10}\right)$

$\boxed{\dfrac{1}{s-3}+\dfrac{6}{s^2+4}}$, $\boxed{\dfrac{6}{s^2-6s+13}}$, $\boxed{\dfrac{s^2+4}{\left(s^2-4\right)^2}}$, $\boxed{\ln\sqrt{\dfrac{s^2+1}{s^2}}}$, $\boxed{3\mathrm{e}^{4t}+2\mathrm{e}^{-3t}}$,

$\boxed{\mathrm{e}^{3t}\left(\cos t+3\sin t\right)}$



Question 4

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left(\dfrac{2t^4+5t^2}{t}\right)$

b) $\mathcal{L}\left(\mathrm{e}^{2t}\cos t\right)$

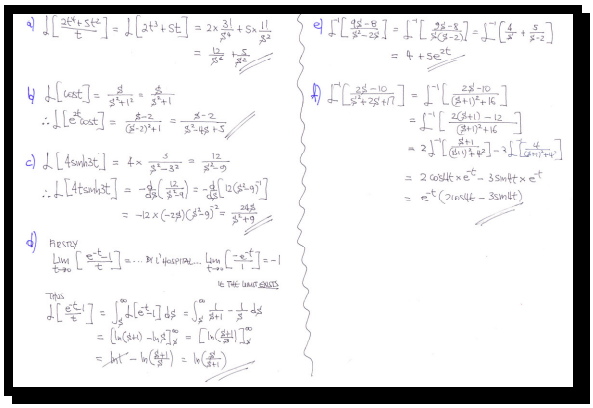
c) $\mathcal{L}\left(4t\sinh 3t\right)$

d) $\mathcal{L}\left(\dfrac{\mathrm{e}^{-t}-1}{t}\right)$

e) $\mathcal{L}^{-1}\left(\dfrac{9s-8}{s^2-2s}\right)$

f) $\mathcal{L}^{-1}\left(\dfrac{2s-10}{s^2+2s+17}\right)$

$\boxed{\dfrac{12}{s^4}+\dfrac{5}{s^2}}$, $\boxed{\dfrac{2s+6}{s^2+6s-7}}$, $\boxed{\dfrac{24s}{s^2+9}}$, $\boxed{\ln\left(\dfrac{s}{s-1}\right)}$, $\boxed{4+5\mathrm{e}^{2t}}$, $\boxed{\mathrm{e}^{-t}\left(2\cos 4t-3\sin 4t\right)}$



Question 5

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left(2t^2-5\right)$

b) $\mathcal{L}\left(\mathrm{e}^t \sinh 2t\right)$

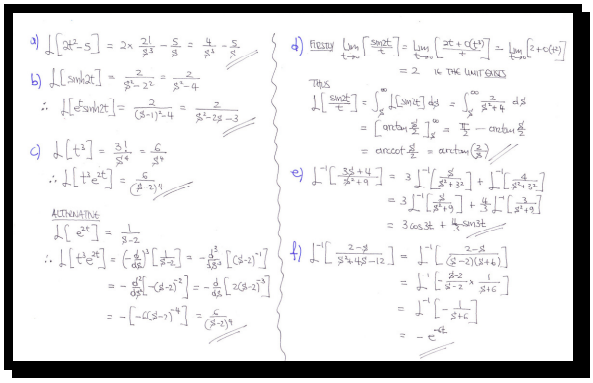
c) $\mathcal{L}\left(t^3 \mathrm{e}^{2t}\right)$

d) $\mathcal{L}\left(\dfrac{\sin 2t}{t}\right)$

e) $\mathcal{L}^{-1}\left(\dfrac{3s+4}{s^2+9}\right)$

f) $\mathcal{L}^{-1}\left(\dfrac{2-s}{s^2+4s-12}\right)$

$\boxed{\dfrac{4}{s^3}-\dfrac{5}{s}}$, $\boxed{\dfrac{2}{s^2-2s-3}}$, $\boxed{\dfrac{6}{(s-2)^4}}$, $\boxed{\arctan\left(\dfrac{2}{s}\right)}$, $\boxed{3\cos 3t+\dfrac{4}{3}\sin 3t}$, $\boxed{-\mathrm{e}^{-6t}}$



Question 6

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\big[(t+2)(t+3)\big]$

b) $\mathcal{L}\left(\mathrm{e}^{4t}\sin 2t\right)$

c) $\mathcal{L}\left[8t\cosh\left(\frac{1}{2}t\right)\right]$

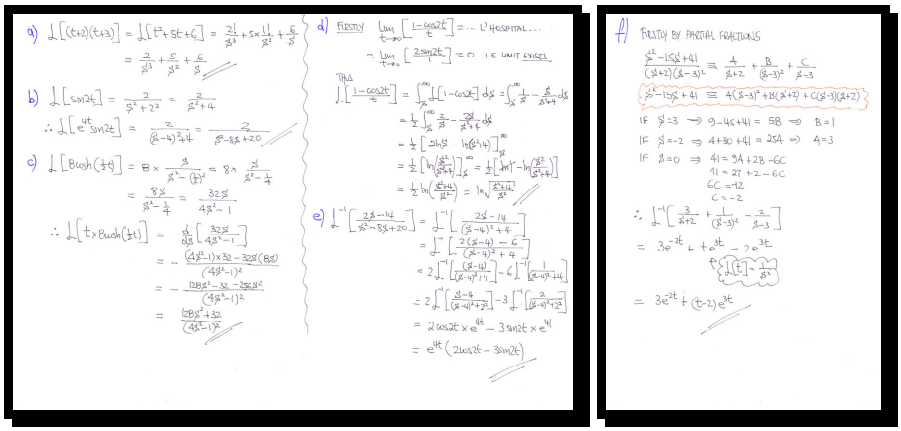
d) $\mathcal{L}\left(\frac{1-\cos 2t}{t}\right)$

e) $\mathcal{L}^{-1}\left(\frac{2s-14}{s^2-8s+20}\right)$

f) $\mathcal{L}^{-1}\left[\frac{s^2-15s+41}{(s+2)(s-3)^2}\right]$

$\boxed{\frac{2}{s^3}+\frac{5}{s^2}+\frac{6}{s}}$, $\boxed{\frac{6}{s^2-6s+13}}$, $\boxed{\frac{128s^2+32}{\left(4s^2-1\right)^2}}$, $\boxed{\ln\sqrt{\frac{s^2+4}{s^2}}}$, $\boxed{\mathrm{e}^{4t}\left(2\cos 2t-3\sin 2t\right)}$,

$\boxed{3\mathrm{e}^{-2t}+(t-2)\mathrm{e}^{3t}}$



Question 7

Determine each of the following inverse Laplace transforms, showing, if appropriate, the techniques used.

a) $\mathcal{L}^{-1}\left[\dfrac{4s^2-5s+6}{(s+1)\left(s^2+4\right)}\right]$

b) $\mathcal{L}^{-1}\left[\dfrac{3\left(s^2+3\right)}{s^4-81}\right]$

c) $\mathcal{L}^{-1}\left[\dfrac{s^2+4}{\left(s^2-4\right)^2}\right]$

d) $\mathcal{L}^{-1}\left[\dfrac{6s^2-2}{\left(s^2+1\right)^3}\right]$

$\boxed{3\mathrm{e}^{-t}+\cos 2t-3\sin 2t}$, $\boxed{\frac{1}{3}\left(\sin 3t+2\sinh 3t\right)}$, $\boxed{t\cosh 2t}$, $\boxed{t^2\sin t}$



Question 8

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left(\cos 6t\right)$

b) $\mathcal{L}\left(t^5 \mathrm{e}^{2t}\right)$

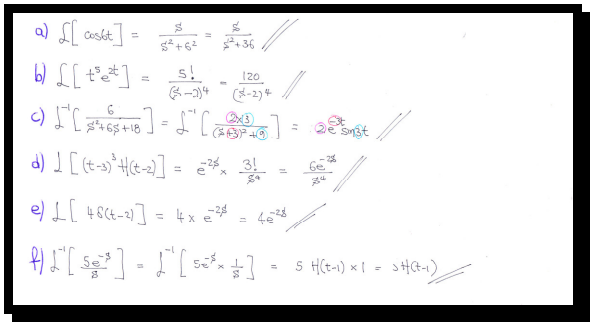
c) $\mathcal{L}^{-1}\left(\dfrac{6}{s^2+6s+18}\right)$

d) $\mathcal{L}\left[\left(t-3\right)^3 \mathrm{H}\left(t-2\right)\right]$

e) $\mathcal{L}\left[4\delta\left(t-2\right)\right]$

f) $\mathcal{L}^{-1}\left(\dfrac{5\mathrm{e}^{-s}}{s}\right)$

$$\boxed{\frac{s}{s^2+36}}, \quad \boxed{\frac{120}{(s-2)^4}}, \quad \boxed{2\mathrm{e}^{-3t}\sin 3t}, \quad \boxed{\frac{6\mathrm{e}^{-5s}}{s^4}}, \quad \boxed{4\mathrm{e}^{-2s}}, \quad \boxed{5\mathrm{H}\left(t-1\right)}$$



Question 9

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left(\mathrm{e}^{3t}\cosh 4t\right)$

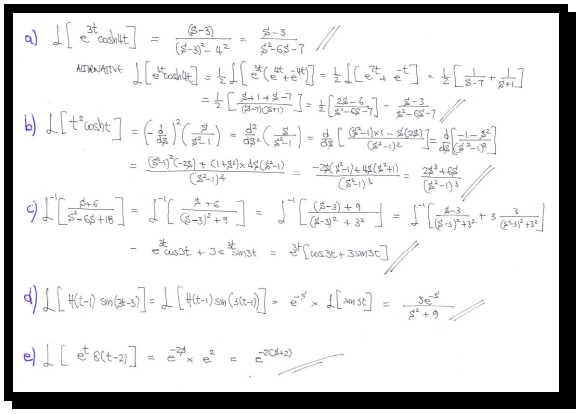
b) $\mathcal{L}\left(t^2\cosh t\right)$

c) $\mathcal{L}^{-1}\left(\dfrac{s+6}{s^2-6s+18}\right)$

d) $\mathcal{L}\left[\mathrm{H}(t-1)\sin(3t-3)\right]$

e) $\mathcal{L}\left[\mathrm{e}^{t}\,\delta(t-2)\right]$

$$\boxed{\frac{s-3}{s^2-6s-7}},\ \boxed{\frac{2s^3+6s}{\left(s^2-1\right)^3}},\ \boxed{\mathrm{e}^{3t}\left(\cos 3t+3\sin 3t\right)},\ \boxed{\frac{3\mathrm{e}^{-s}}{s^2+9}},\ \boxed{\mathrm{e}^{-2(s+2)}}$$



Question 10

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left(t^2 \mathrm{e}^{-\frac{1}{2}t} \right)$

b) $\mathcal{L}^{-1}\left(\dfrac{6s+1}{9s^2+1} \right)$

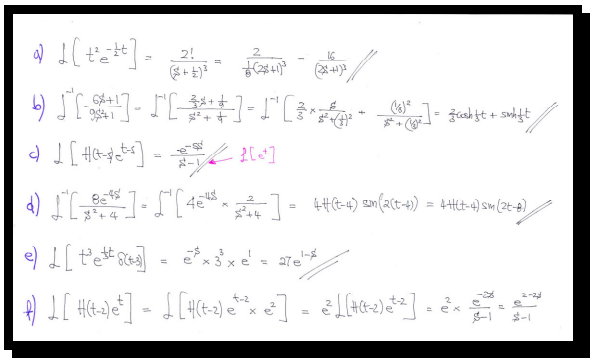
c) $\mathcal{L}\left[\mathrm{e}^{t-5} \mathrm{H}(t-5) \right]$

d) $\mathcal{L}^{-1}\left(\dfrac{8\mathrm{e}^{-4s}}{s^2+4} \right)$

e) $\mathcal{L}\left[t^3 \mathrm{e}^{\frac{1}{3}t} \delta(t-3) \right]$

f) $\mathcal{L}\left[\mathrm{e}^{t} \mathrm{H}(t-2) \right]$

$\boxed{\dfrac{16}{(2s+1)^3}}$, $\boxed{\dfrac{2}{3}\cosh\left(\dfrac{1}{3}t\right)+\sinh\left(\dfrac{1}{3}t\right)}$, $\boxed{\dfrac{\mathrm{e}^{-5s}}{s-1}}$, $\boxed{4\mathrm{H}(t-4)\sin(2t-8)}$, $\boxed{4\mathrm{e}^{-2s}}$, $\boxed{\dfrac{\mathrm{e}^{2-2s}}{s-1}}$



Question 11

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left[t\sin\left(\frac{1}{2}t\right)\right]$

b) $\mathcal{L}^{-1}\left[\frac{1}{(s-2)^6}\right]$

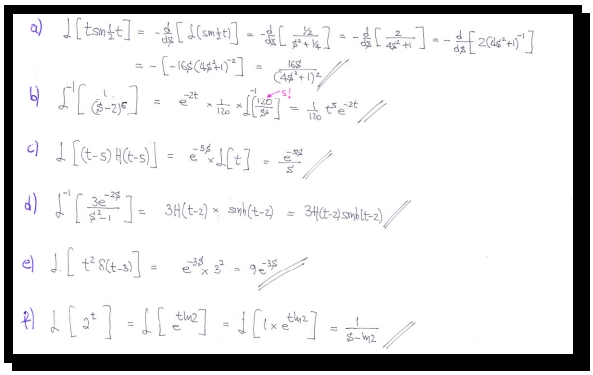
c) $\mathcal{L}\left[(t-5)\mathrm{H}(t-5)\right]$

d) $\mathcal{L}^{-1}\left[\frac{3\mathrm{e}^{-2s}}{s^2-1}\right]$

e) $\mathcal{L}\left[t^2\,\delta(t-2)\right]$

f) $\mathcal{L}\left(2^t\right)$

$$\boxed{\frac{16s}{\left(4s^2+1\right)^2}},\ \boxed{\frac{t^5\,\mathrm{e}^{-2t}}{120}},\ \boxed{\frac{\mathrm{e}^{-5s}}{s}},\ \boxed{3\mathrm{H}(t-2)\sinh(t-2)},\ \boxed{9\mathrm{e}^{-3s}},\ \boxed{\frac{1}{s-\ln 2}}$$



Question 12

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

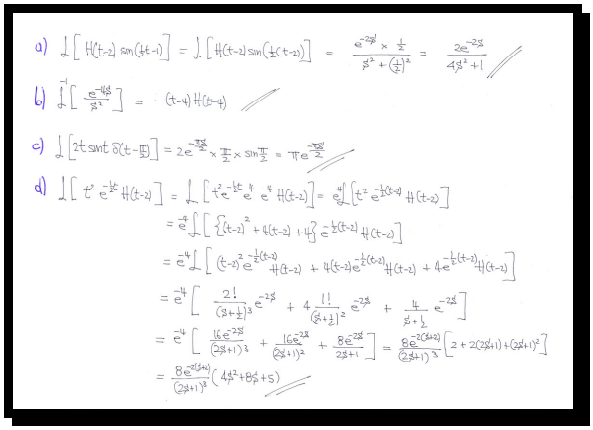
a) $\mathcal{L}\left[\mathrm{H}(t-2)\sin\left(\frac{1}{2}t-1\right)\right]$

b) $\mathcal{L}^{-1}\left[\dfrac{\mathrm{e}^{-4s}}{s^2}\right]$

c) $\mathcal{L}\left[2t\sin t\ \delta\left(t-\dfrac{\pi}{2}\right)\right]$

d) $\mathcal{L}\left[\mathrm{t}^2\,\mathrm{e}^{-\frac{1}{2}t}\,\mathrm{H}(t-2)\right]$

$\boxed{\dfrac{2\mathrm{e}^{-2s}}{4s^2+1}}$, $\boxed{(t-4)\,\mathrm{H}(t-4)}$, $\boxed{\pi\,\mathrm{e}^{-\frac{1}{2}\pi s}}$, $\boxed{\dfrac{8\mathrm{e}^{-2(s+2)}}{(2s+1)^3}\left[4s^2+8s+5\right]}$



HEAVISIDE FUNCTION

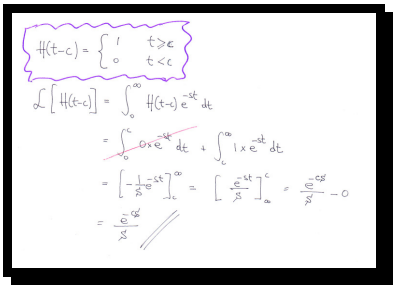
Question 1

The Heaviside function $\mathrm{H}(t)$ is defined as

$$\mathrm{H}(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$$

Determine the Laplace transform of $\mathrm{H}(t-c)$.

$$\mathcal{L}\left(\mathrm{H}(t-c)\right)=\frac{\mathrm{e}^{-cs}}{s}$$



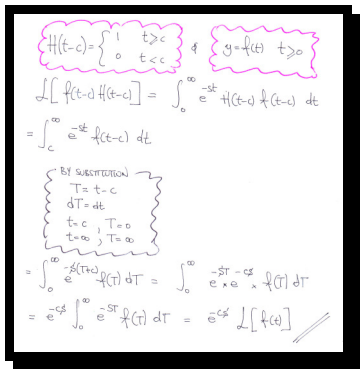
Question 2

The Heaviside step function $\mathrm{H}(t)$ is defined as

$$\mathrm{H}(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$$

Determine the Laplace transform of $\mathrm{H}(t-c)\,f(t-c)$, where $f(t)$ is a continuous or piecewise continuous function defined for $t\geq 0$.

$$\boxed{\mathcal{L}\left(\mathrm{H}(t-c)\,f(t-c)\right)=\mathrm{e}^{-cs}\,\mathcal{L}\left(f(t)\right)}$$



Question 3

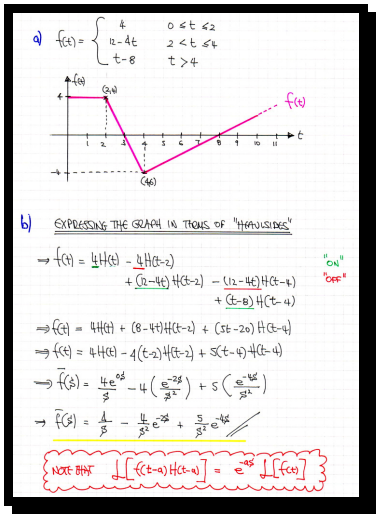
The piecewise continuous function $f(t)$ is defined as

$$f(t)=\begin{cases} 4 & 0\le t\le 2 \\ 12-4t & 2<t\le 4 \\ t-8 & t>4 \end{cases}$$

a) Sketch the graph of $f(t)$.

b) Express $f(t)$ in terms of the Heaviside step function, and hence find the Laplace transform of $f(t)$.

$$f(t)=4\mathrm{H}(t)-4(t-2)\mathrm{H}(t-2)+5(t-4)\mathrm{H}(t-4), \quad \mathcal{L}\{f(t)\}=\frac{8}{s}-\frac{4\mathrm{e}^{-2s}}{s^2}+\frac{5\mathrm{e}^{-4s}}{s^2}$$



Question 4

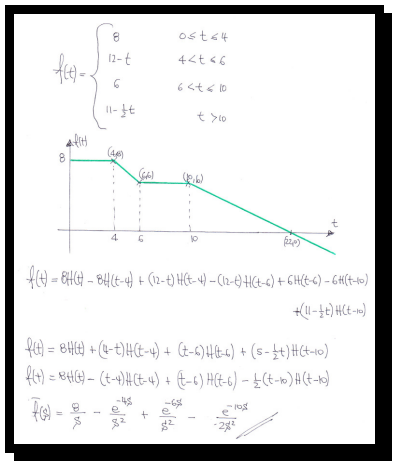
The piecewise continuous function $f(t)$ is defined as

$$f(t)=\begin{cases} 8 & 0\le t\le 4 \\ 12-t & 4<t\le 6 \\ 6 & 6<t\le 10 \\ 11-\frac{1}{2}t & t>10 \end{cases}$$

Express $f(t)$ in terms of the Heaviside step function, and hence find the Laplace transform of $f(t)$.

$$\boxed{f(t)=8\mathrm{H}(t)-(t-4)\mathrm{H}(t-4)+(t-4)\mathrm{H}(t-6)-\frac{1}{2}(t-10)\mathrm{H}(t-10)},$$

$$\boxed{\mathcal{L}\{f(t)\}=\frac{8}{s}-\frac{\mathrm{e}^{-4s}}{s^2}+\frac{\mathrm{e}^{-6s}}{s^2}-\frac{\mathrm{e}^{-10s}}{2s^2}}$$



Question 5

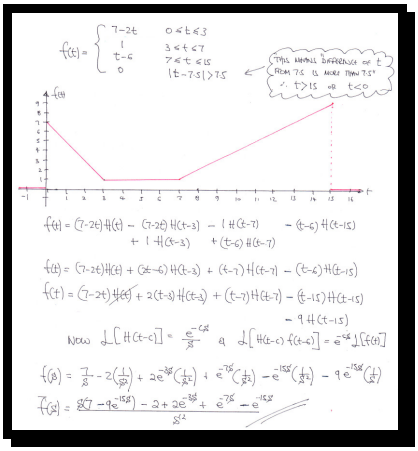
The piecewise continuous function $f(t)$ is defined as

$$f(t)=\begin{cases} 7-2t & 0<t\le 3 \\ 1 & 3<t\le 7 \\ t-6 & 7<t\le 15 \\ 0 & |t-7.5|>7.5 \end{cases}$$

Express $f(t)$ in terms of the Heaviside step function, and hence find the Laplace transform of $f(t)$.

$$\boxed{f(t)=(7-2t)\mathrm{H}(t)+2(t-3)\mathrm{H}(t-3)+(t-7)\mathrm{H}(t-7)-(t-15)\mathrm{H}(t-15)-9\mathrm{H}(t-15)},$$

$$\boxed{\mathcal{L}\{f(t)\}=\frac{s\left(7-9\mathrm{e}^{-15s}\right)-2+2\mathrm{e}^{-3s}+\mathrm{e}^{-7s}-\mathrm{e}^{-15s}}{s^2}}$$



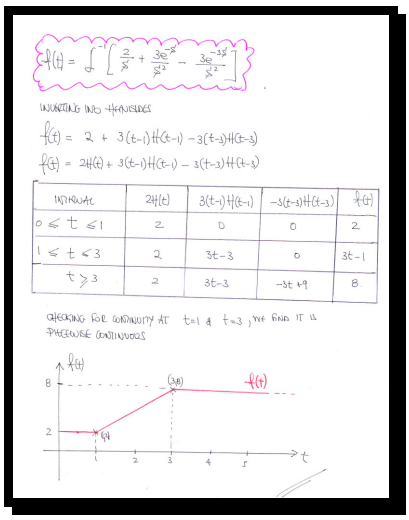
Question 6

The piecewise continuous function $f(t)$ is defined as

$$f(t)=\mathcal{L}^{-1}\left(\frac{2}{s}+\frac{3\mathrm{e}^{-s}}{s^2}-\frac{3\mathrm{e}^{-3s}}{s^2}\right).$$

Sketch the graph of $f(t)$.

graph



Question 7

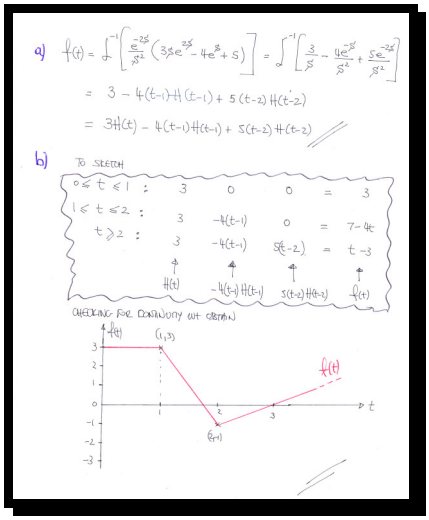
The piecewise continuous function $f(t)$ is defined as

$$f(t)=\mathcal{L}^{-1}\left[\frac{\mathrm{e}^{-2s}}{s^2}\left(3s\,\mathrm{e}^{2s}-4\mathrm{e}^{s}+5\right)\right].$$

a) Determine an expression for $f(t)$.

b) Sketch the graph of $f(t)$.

$$\boxed{f(t)=3\mathrm{H}(t)-4(t-1)\mathrm{H}(t-1)+5(t-2)\mathrm{H}(t-2)}$$



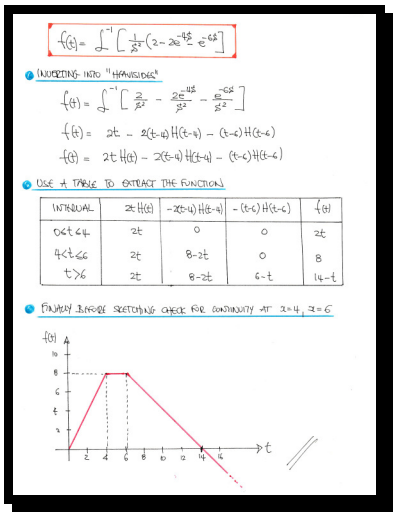
Question 8

The piecewise continuous function $f(t)$ is defined as

$$f(t)=\mathcal{L}^{-1}\left[\frac{1}{s^2}\left(2-2\mathrm{e}^{-4s}-\mathrm{e}^{-6s}\right)\right].$$

Sketch the graph of $f(t)$.

[ ], [graph]



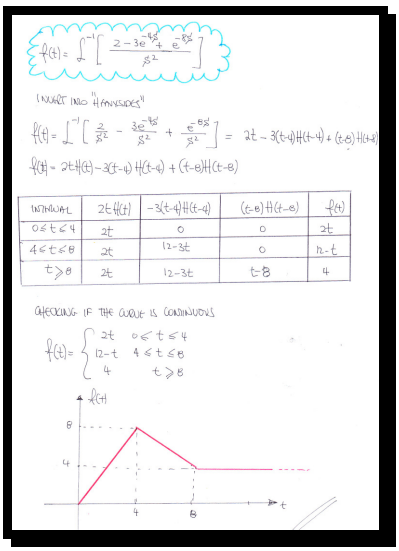
Question 9

The piecewise continuous function $f(t)$ is defined as

$$f(t)=\mathcal{L}^{-1}\left[\frac{2-3\mathrm{e}^{-4s}+\mathrm{e}^{-8s}}{s^2}\right].$$

Sketch the graph of $f(t)$.

graph



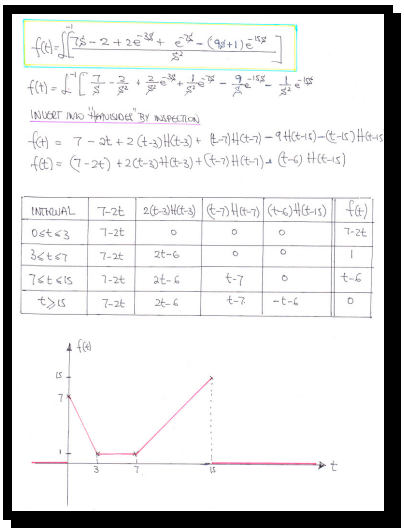
Question 10

The piecewise continuous function $f(t)$ is defined as

$$f(t)=\mathcal{L}^{-1}\left[\frac{s\left(7-9\mathrm{e}^{-15s}\right)-2+2\mathrm{e}^{-3s}+\mathrm{e}^{-7s}-\mathrm{e}^{-15s}}{s^2}\right].$$

Sketch the graph of $f(t)$.

graph



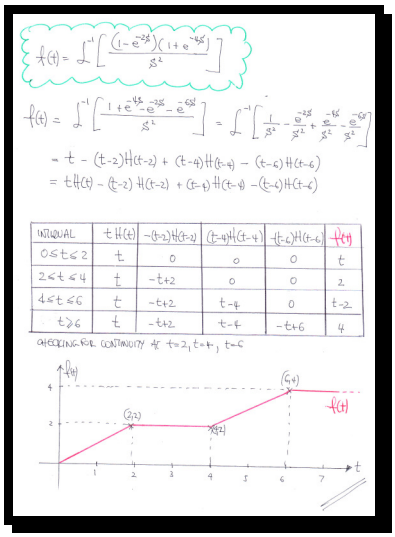
Question 11

The piecewise continuous function $f(t)$ is defined as

$$f(t)=\mathcal{L}^{-1}\left[\frac{\left(1-\mathrm{e}^{-2s}\right)\left(1+\mathrm{e}^{-4s}\right)}{s^2}\right].$$

Sketch the graph of $f(t)$.

graph



PERIODIC FUNCTIONS

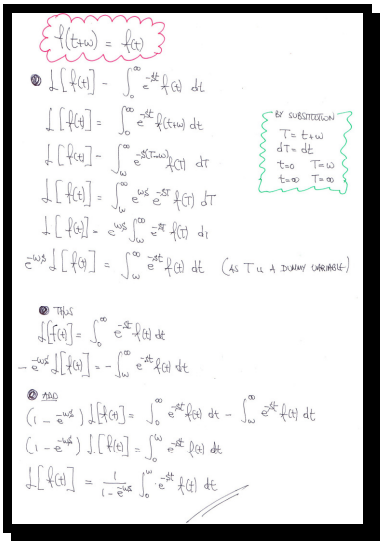
Question 1

The piecewise continuous function $f(t)$ is defined for $t \geq 0$ and further satisfies $f(t+\omega) = f(t)$.

Show from the definition of a Laplace transform, that

$$\mathcal{L}\left[f(t) \right] = \frac{1}{1-\mathrm{e}^{-\omega s}} \int_0^{\omega} \mathrm{e}^{-st} f(t) \, dt .$$

proof

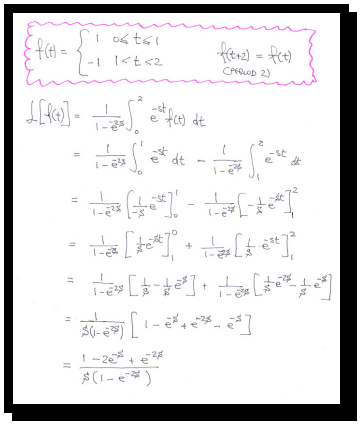


Question 2

$$f(t)=\begin{cases} 1 & 0\le t\le 1 \\ -1 & 1<t<2 \end{cases} \qquad \text{and} \qquad f(t+2)=f(t),\ t\ge 0.$$

Determine the Laplace transform of $f(t)$.

$$\mathcal{L}\left(f(t)\right)=\frac{1-2\mathrm{e}^{-s}+\mathrm{e}^{-2s}}{s\left(1+\mathrm{e}^{-s}\right)}$$

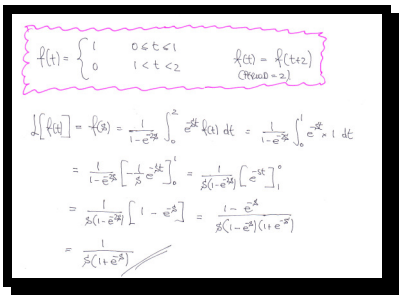


Question 3

$$f(t)=\begin{cases}1 & 0\le t\le 1\\ 0 & 1<t<2\end{cases}\qquad\text{and}\qquad f(t+2)=f(t),\ t\ge 0.$$

Determine the Laplace transform of $f(t)$.

$$\boxed{\mathcal{L}\{f(t)\}=\frac{1}{s\left(1+\mathrm{e}^{-s}\right)}}$$

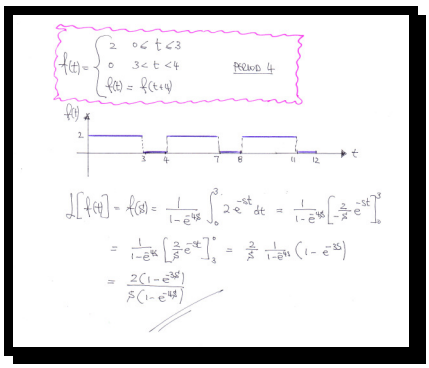


Question 4

$$f(t)=\begin{cases}2 & 0\le t\le 3\\ 0 & 3<t<4\end{cases}\qquad\text{and}\qquad f(t+4)=f(t),\ t\ge 0.$$

Determine the Laplace transform of $f(t)$.

$$\boxed{\mathcal{L}\{f(t)\}=\frac{2\left(1-\mathrm{e}^{-3s}\right)}{s\left(1-\mathrm{e}^{-4s}\right)}}$$

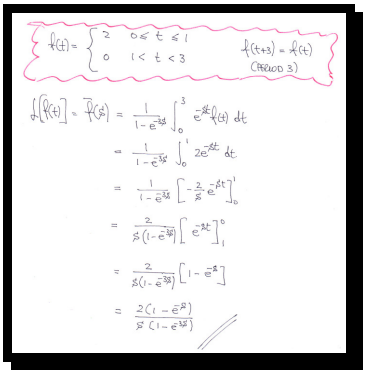


Question 5

$$f(t)=\begin{cases}2 & 0\le t\le 1\\ 0 & 1<t<3\end{cases} \qquad \text{and} \qquad f(t+4)=f(t),\ t\ge 0.$$

Determine the Laplace transform of $f(t)$.

$$\mathcal{L}\left(f(t)\right)=\frac{2\left(1-\mathrm{e}^{-s}\right)}{s\left(1-\mathrm{e}^{-3s}\right)}$$

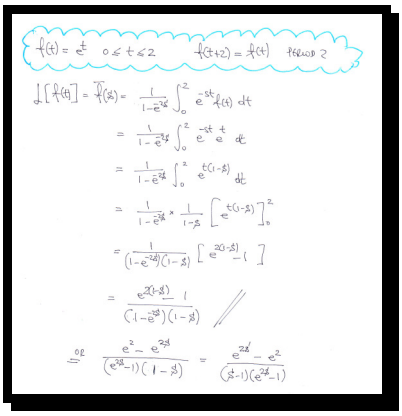


Question 6

$$f(t)=\mathrm{e}^{t},\ t\geq 0 \qquad \text{and} \qquad f(t+2)=f(t).$$

Determine the Laplace transform of $f(t)$.

$$\boxed{\mathcal{L}\left(f(t)\right)=\frac{\mathrm{e}^{2(1-s)}-1}{(1-s)\left(1-\mathrm{e}^{-2s}\right)}=\frac{\mathrm{e}^{2s}-\mathrm{e}^{2}}{(s-1)\left(\mathrm{e}^{2s}-1\right)}}$$



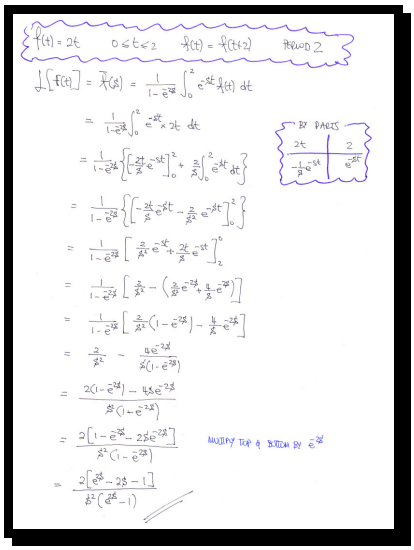
Question 7

$$f(t)=2t\,,\ t\geq 0 \qquad \text{and} \qquad f(t+2)=f(t)\,.$$

Show that the Laplace transform of $f(t)$ is

$$\frac{2\left(\mathrm{e}^{2s}-2s-1\right)}{s^2\left(\mathrm{e}^{2s}-1\right)}$$

proof



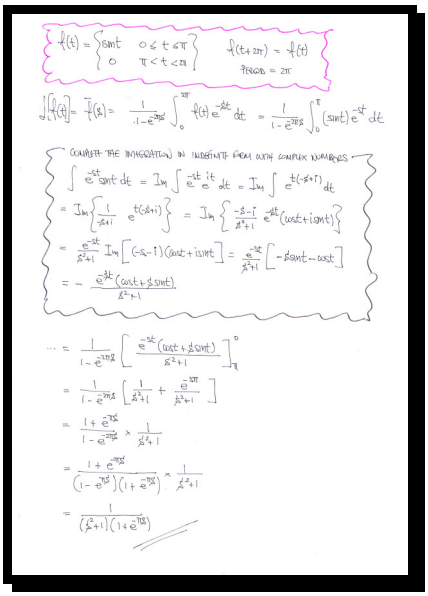
Question 8

$$f(t)=\begin{cases}\sin t & 0\le t\le \pi \\ 0 & \pi < t < 2\pi\end{cases} \qquad \text{and} \qquad f(t+2\pi)=f(t),\ t\ge 0.$$

Show that the Laplace transform of $f(t)$ is

$$\frac{1}{\left(s^2+1\right)\left(1+\mathrm{e}^{-\pi s}\right)}$$

proof



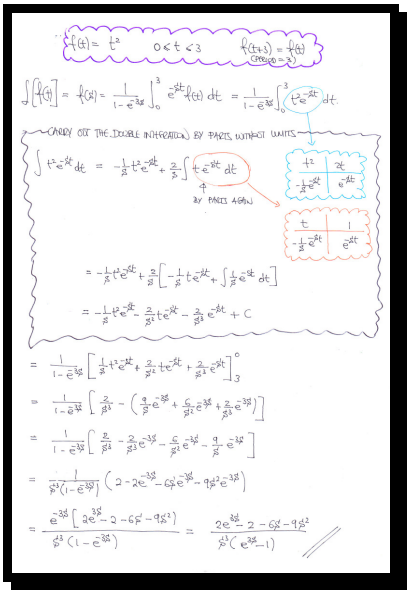
Question 9

$$f(t)=t^2,\ t\geq 0 \qquad \text{and} \qquad f(t+3)=f(t).$$

Show that the Laplace transform of $f(t)$ is

$$\frac{2\mathrm{e}^{3s}-2-6s-9s^2}{s^3\left(\mathrm{e}^{3s}-1\right)}$$

proof



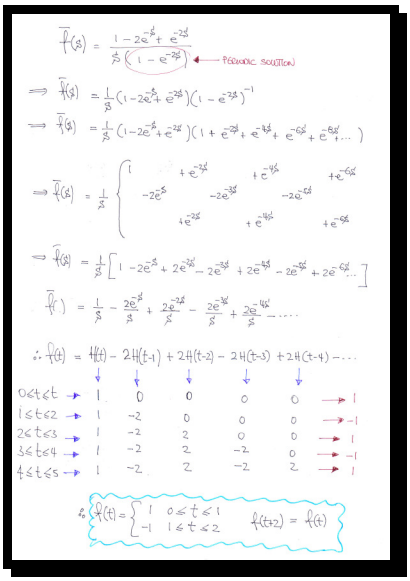
Question 10

The piecewise continuous function $f(t)$ is defined as

$$f(t)=\mathcal{L}^{-1}\left[\frac{1-2\mathrm{e}^{-s}+\mathrm{e}^{-2s}}{s\left(1-\mathrm{e}^{-2s}\right)}\right].$$

Find an expression for $f(t)$.

$$\boxed{f(t)=\begin{cases}1 & 0\le t\le 1\\ -1 & 1<t<2\end{cases}\qquad f(t+2)=f(t)}$$



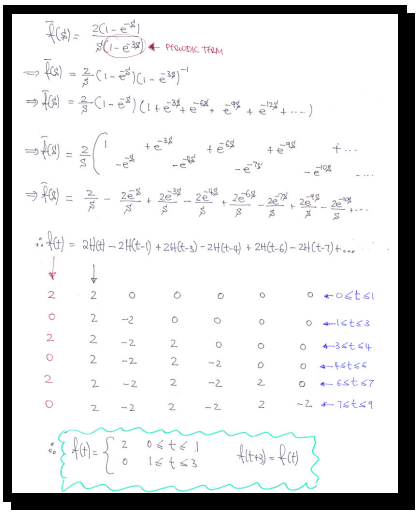
Question 11

The piecewise continuous function $f(t)$ is defined as

$$f(t)=\mathcal{L}^{-1}\left[\frac{2\left(1-\mathrm{e}^{-s}\right)}{s\left(1-\mathrm{e}^{-3s}\right)}\right].$$

Find an expression for $f(t)$.

$$\boxed{f(t)=\begin{cases}2 & 0\le t\le 1\\ 0 & 1<t<3\end{cases}\qquad f(t+3)=f(t)}$$



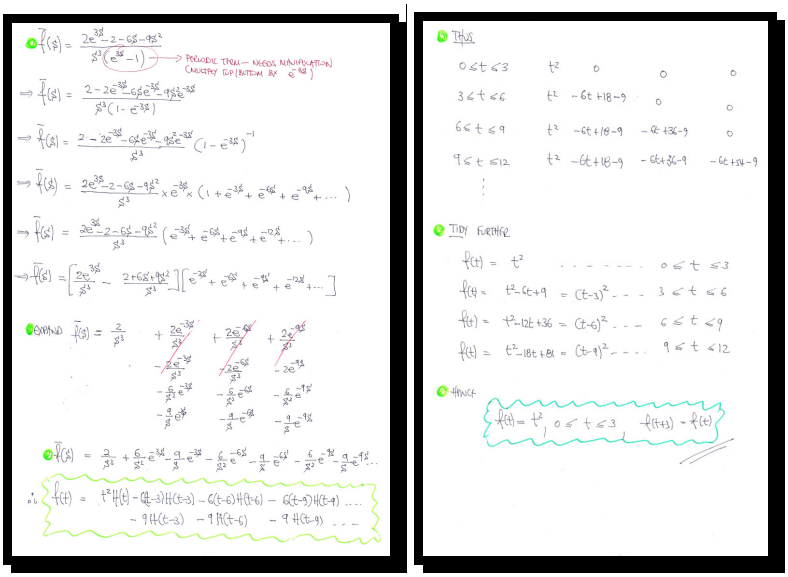
Question 12

The piecewise continuous function $f(t)$ is defined as

$$f(t)=\mathcal{L}^{-1}\left[\frac{2\mathrm{e}^{3s}-2-6s-9s^2}{s^3\left(\mathrm{e}^{3s}-1\right)}\right].$$

Find an expression for $f(t)$.

$$\boxed{f(t)=t^2 \quad 0\le t\le 3 \quad f(t+3)=f(t)}$$



SOLVING SIMPLE O.D.E.s

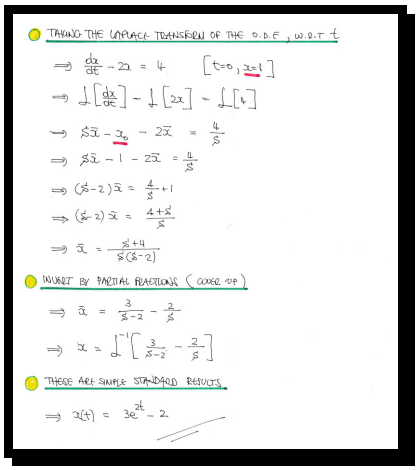
Question 1

Use Laplace transforms to solve the differential equation

$$\frac{dx}{dt}-2x=4\,,\ t\geq 0\,,$$

subject to the initial condition $x=1$ at $t=0$.

$\boxed{\phantom{x=3e^{2t}-2}}$, $\boxed{x=3\mathrm{e}^{2t}-2}$



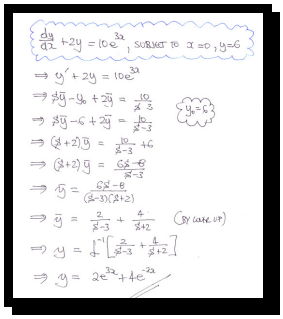
Question 2

Use Laplace transforms to solve the differential equation

$$\frac{dy}{dx}+2y=10\mathrm{e}^{3x}\,,\ x\geq 0\,,$$

subject to the boundary condition $y=6$ at $x=0$.

$\boxed{y=2\mathrm{e}^{3x}+4\mathrm{e}^{-2x}}$



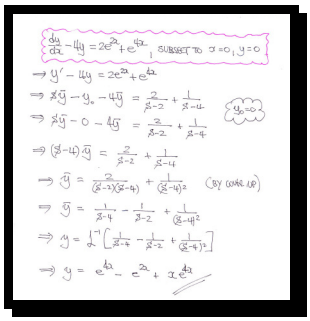
Question 3

Use Laplace transforms to solve the differential equation

$$\frac{dy}{dx}-4y=2\mathrm{e}^{2x}+\mathrm{e}^{4x},\ x\geq 0,$$

subject to the boundary condition $y=0$ at $x=0$.

$$\boxed{y=x\mathrm{e}^{4x}+\mathrm{e}^{4x}-2\mathrm{e}^{2x}}$$



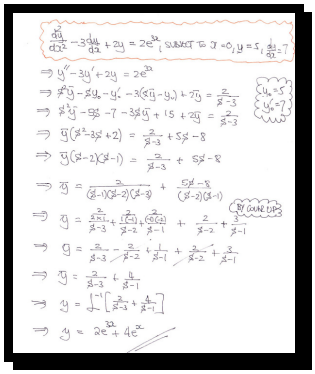
Question 4

Use Laplace transforms to solve the differential equation

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=2\mathrm{e}^{3x},\ x\geq 0,$$

subject to the boundary conditions $y=5$, $\frac{dy}{dx}=7$ at $x=0$.

$$\boxed{y=2\mathrm{e}^{3x}+4\mathrm{e}^{x}}$$



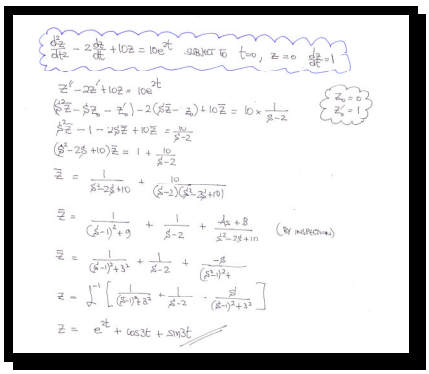
Question 5

Use Laplace transforms to solve the differential equation

$$\frac{d^2z}{dt^2}-2\frac{dz}{dt}+10z=10\mathrm{e}^{2t},$$

subject to the initial conditions $z=0$, $\dfrac{dz}{dt}=1$ at $t=0$.

$$\boxed{y=\mathrm{e}^{2t}+\cos 3t+\sin 3t}$$



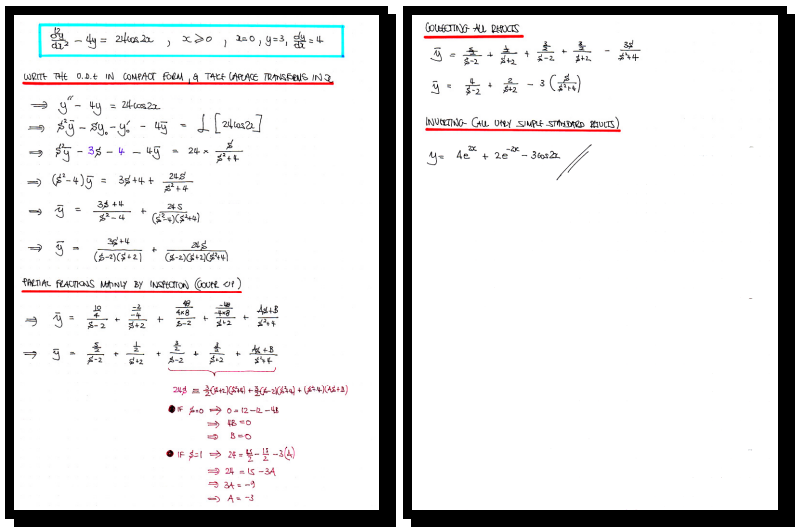
Question 6

Use Laplace transforms to solve the differential equation

$$\frac{d^2y}{dx^2}-4y=24\cos 2x\,,\ x\geq 0\,,$$

subject to the boundary conditions $y=3\,,\ \dfrac{dy}{dx}=4$ at $x=0\,.$

$\boxed{}\,,\ \boxed{y=4\mathrm{e}^{2x}+2\mathrm{e}^{-2x}-3\cos 2x}$



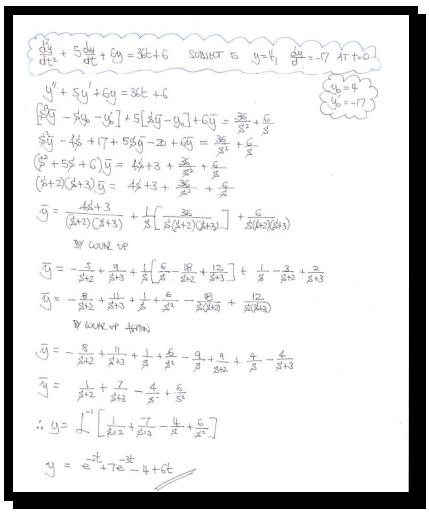
Question 7

Use Laplace transforms to solve the differential equation

$$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=36t+6,$$

subject to the initial conditions $y=4$, $\dfrac{dy}{dt}=-17$ at $t=0$.

$$\boxed{y=\mathrm{e}^{-2t}+7\mathrm{e}^{-3t}+6t-4}$$

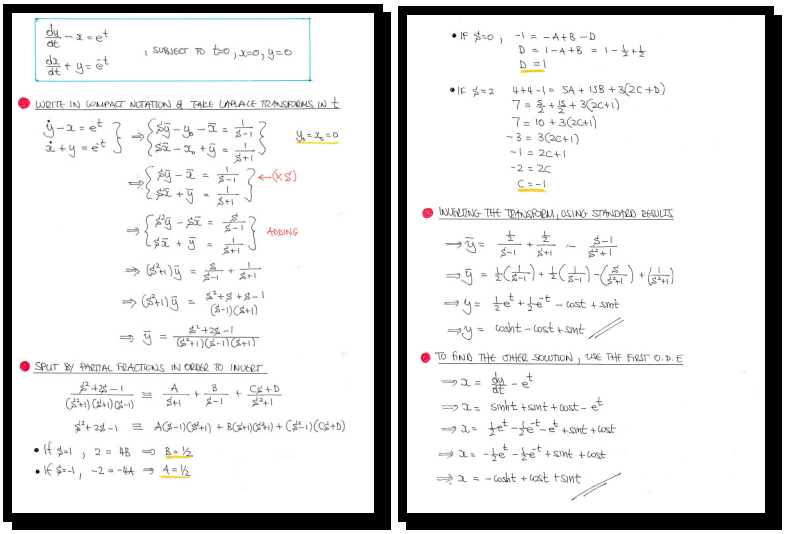


Question 8

$$\frac{dx}{dt}+y=\mathrm{e}^{-t} \qquad \text{and} \qquad \frac{dy}{dt}-x=\mathrm{e}^{t}\,.$$

Use Laplace transformations to solve the above simultaneous differential equations, subject to the initial conditions $x=0$, $y=0$ at $t=0$.

$\boxed{}$, $\boxed{x=-\cosh t+\sin t+\cos t,\quad y=\cosh t+\sin t-\cos t}$

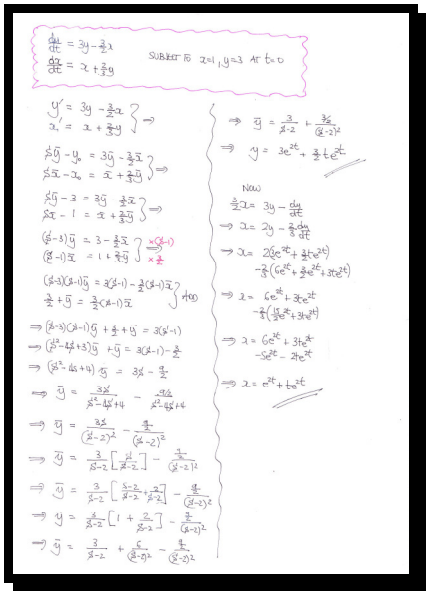


Question 9

$$\frac{dx}{dt}=x+\frac{2}{3}y \qquad \text{and} \qquad \frac{dy}{dt}=3y-\frac{3}{2}x\,.$$

Use Laplace transformations to solve the above simultaneous differential equations, subject to the initial conditions $x=1$, $y=3$ at $t=0$.

$$\boxed{x=\mathrm{e}^{2t}+t\mathrm{e}^{2t},\quad y=3\mathrm{e}^{2t}+\frac{3}{2}t\mathrm{e}^{2t}}$$



Question 10

$$\frac{d^2x}{dt^2}=15\frac{dy}{dt}-9y+22\mathrm{e}^{t} \qquad \text{and} \qquad \frac{d^2y}{dt^2}=2x+\mathrm{e}^{3t}.$$

The functions $x=f(t)$ and $y=g(t)$ satisfy the above simultaneous differential equations, subject to the initial conditions

$$x=2,\quad y=-3,\quad \frac{dx}{dt}=10,\quad \frac{dy}{dt}=-1 \quad \text{at} \quad t=0.$$

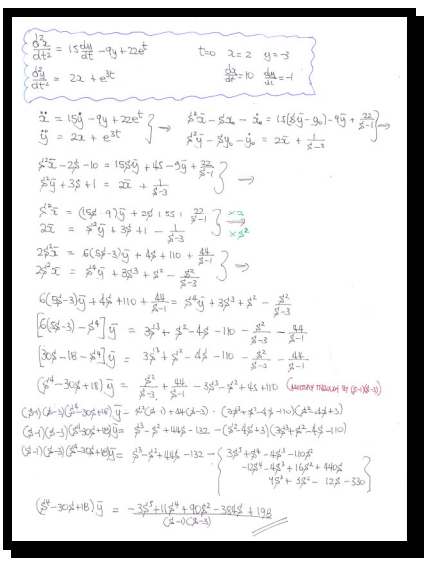
a) By using Laplace transforms, show that

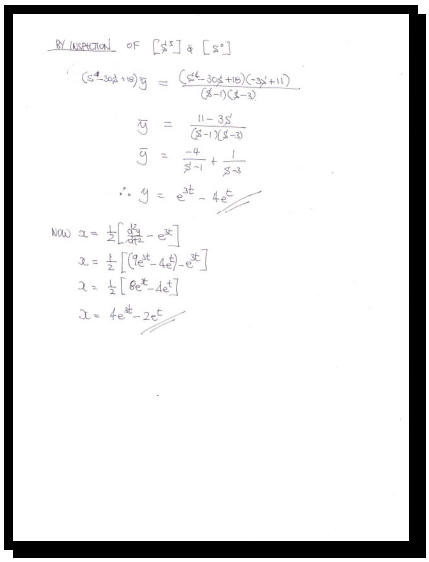
$$\left(s^4-30s+18\right)\overline{y}=\frac{-3s^5+11s^4+90s^2-384s+198}{(s-1)(s-3)},$$

where $\overline{y}=\mathcal{L}\left[g(t)\right]$.

b) Given further that $s^4-30s+18$ is a factor of $-3s^5+11s^4+90s^2-384s+198$, find expressions for x and y.

$$\boxed{x=4\mathrm{e}^{3t}-2\mathrm{e}^{t},\quad y=\mathrm{e}^{3t}-4\mathrm{e}^{t}}$$





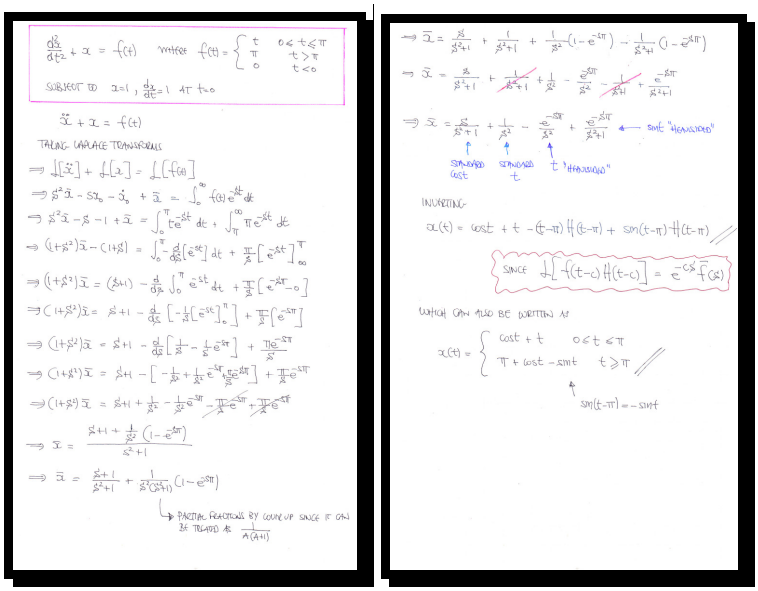
Question 11

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2}+x=f\left(t\right),$$

given further that $x=1$, $\dfrac{dx}{dt}=1$ at $t=0$, and $f\left(t\right)=\begin{cases}0 & t<0\\ t & 0\le t\le \pi\\ \pi & t>\pi\end{cases}$

$$\boxed{x=t+\cos t-\left(t-\pi\right)\mathrm{H}\left(t-\pi\right)+\sin\left(t-\pi\right)\mathrm{H}\left(t-\pi\right)}$$



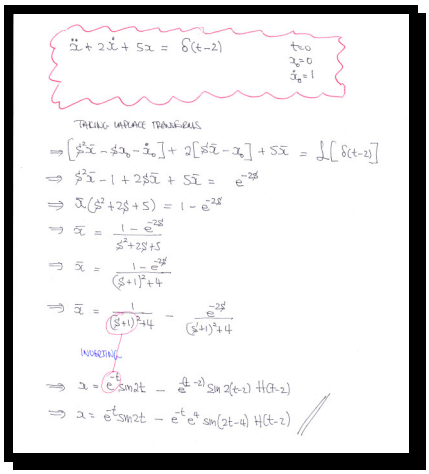
Question 12

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=\delta(t-2),$$

given further that $x=0$, $\dfrac{dx}{dt}=1$ at $t=0$.

$$\boxed{x=\mathrm{e}^{-t}\left[\sin 2t-\mathrm{e}^{4}\sin(2t-4)\mathrm{H}(t-2)\right]}$$



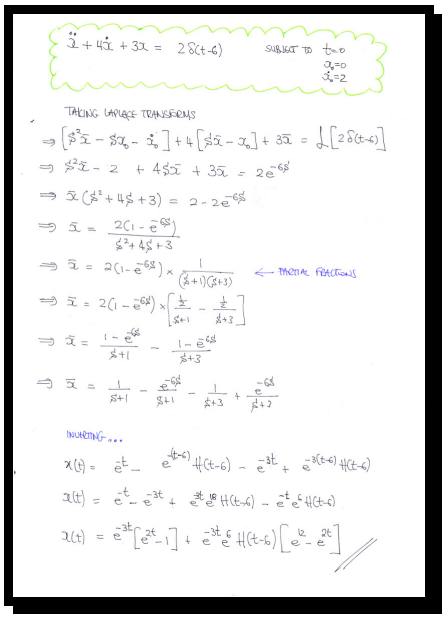
Question 13

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+3x=2\delta(t-6),$$

given further that $x=0$, $\frac{dx}{dt}=2$ at $t=0$.

$$\boxed{x=\mathrm{e}^{-3t}\left[\mathrm{e}^{2t}-1\right]+\mathrm{e}^{-3t}\,\mathrm{e}^{6}\left[\mathrm{e}^{12}-\mathrm{e}^{2t}\right]\mathrm{H}(t-6)}$$



Question 14

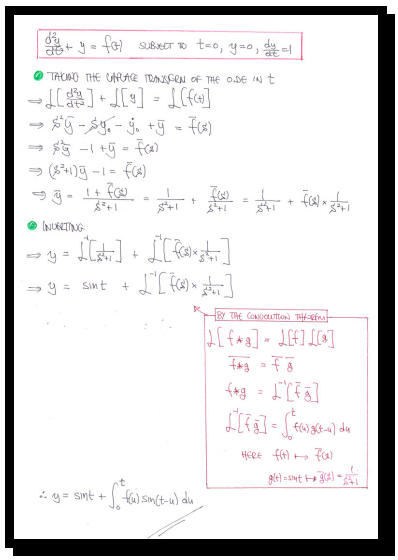
Use Laplace transforms to solve the differential equation

$$\frac{d^2y}{dt^2}+y=f\left(t\right),$$

given further that $y=0$, $\frac{dy}{dt}=1$ at $t=0$, and $f\left(t\right)$ is a known function which has a Laplace transform.

You may leave the final answer containing a convolution type integral.

$$\boxed{y=\sin t+\int_0^t f\left(u\right)\sin\left(t-u\right)\,du}$$



Question 15

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+2x=f(t).$$

a) Use Laplace transforms to solve the above differential equation, given further that $x=0$, $\frac{dx}{dt}=0$ at $t=0$, and $f(t)$ is a known function which has a Laplace transform.

You may leave the answer containing a convolution type integral.

b) If $f(t)=\mathrm{e}^{2t}$ find $x=x(t)$ explicitly.

$$\boxed{x=\int_0^t f(t-u)\,\mathrm{e}^{-u}\sin u\ du}\ ,\qquad \boxed{x=-\frac{1}{10}\mathrm{e}^{-t}\left[3\sin t+\cos t\right]+\frac{1}{10}\mathrm{e}^{2t}}$$

