LAPLACE TRANSFORMS FURTHER

SUMMARY OF THE LAPLACE TRANFORM

The Laplace Transform of a function $f(t)$, $t \geq 0$ is defined as

$$\mathcal{L}\left[f(t)\right] \equiv \overline{f}(s) \equiv \int_0^{\infty} \mathrm{e}^{-st} f(t)\ dt,$$

where $s \in \mathbb{C}$, with $\mathrm{Re}(s)$ sufficiently large for the integral to converge.

The Laplace Transform is a linear operation

$$\mathcal{L}\left[a f(t) + b g(t)\right] \equiv a\mathcal{L}\left[f(t)\right] + b\mathcal{L}\left[g(t)\right].$$

Laplace Transforms of Common Functions

- $\mathcal{L}\left(t^n\right) = \dfrac{n}{s^{n+1}}$

$$\mathcal{L}(1) = \frac{1}{s}, \quad \mathcal{L}(a) = \frac{a}{s}, \quad \mathcal{L}(t) = \frac{1}{s^2}, \quad \mathcal{L}\left(t^2\right) = \frac{2}{s^3}, \quad \mathcal{L}\left(t^3\right) = \frac{3}{s^4}, \ldots.$$

- $\mathcal{L}\left(\mathrm{e}^{at}\right) = \dfrac{1}{s-a}$, $\mathcal{L}\left(\mathrm{e}^{-at}\right) = \dfrac{1}{s+a}$
- $\mathcal{L}(\cos at) = \dfrac{s}{s^2+a^2}$, $\mathcal{L}(\sin at) = \dfrac{a}{s^2+a^2}$
- $\mathcal{L}(\cosh at) = \dfrac{s}{s^2-a^2}$, $\mathcal{L}(\sinh at) = \dfrac{a}{s^2-a^2}$

Laplace Transforms of Derivatives

- $\mathcal{L}\left[x(t)\right] = \overline{x}(t)$
- $\mathcal{L}\left[\dot{x}(t)\right] = s\overline{x}(t) - x(0)$
- $\mathcal{L}\left[\ddot{x}(t)\right] = s^2\overline{x}(t) - sx(0) - \dot{x}(0)$
- $\mathcal{L}\left[\dddot{x}(t)\right] = s^3\overline{x}(t) - s^2x(0) - s\dot{x}(0) - \ddot{x}(0)$

Laplace Transforms Theorems

- 1st Shift Theorem

$$\mathcal{L}\left[\mathrm{e}^{-at} f(t)\right] = \overline{f}(s+a) \quad \text{or} \quad \mathcal{L}\left[\mathrm{e}^{at} F(t)\right] = \overline{f}(s-a)$$

- 2nd Shift Theorem

$$\mathcal{L}\left[f(t-a)\right] = \mathrm{e}^{-as}\,\overline{f}(s),\ t>a \quad \text{or} \quad \mathcal{L}\left[f(t+a)\right] = \mathrm{e}^{as}\,\overline{f}(s),\ t>-a\,.$$

$$\mathcal{L}\left[\mathrm{H}(t-a)\,f(t-a)\right] = \mathrm{e}^{-as}\,\overline{f}(s) \quad \text{or} \quad \mathcal{L}\left[\mathrm{H}(t+a)\,f(t+a)\right] = \mathrm{e}^{as}\,\overline{f}(s)$$

- Multiplication by t^n

$$\mathcal{L}\left[t^n f(t)\right] = \left(-\frac{d}{ds}\right)^n \left[\overline{f}(s)\right] \quad \text{or} \quad \mathcal{L}\left[t\,f(t)\right] = -\frac{d}{ds}\left[\overline{f}(s)\right]$$

- Division by t

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \overline{f}(\sigma)\ d\sigma$$

provided that $\lim\limits_{t\to 0}\left(\dfrac{f(t)}{t}\right)$ exists and the integral converges.

- Initial/Final value theorem

$$\lim_{t\to 0}\left[f(t)\right] = \lim_{s\to\infty}\left[s\,\overline{f}(s)\right] \quad \text{and} \quad \lim_{t\to\infty}\left[f(t)\right] = \lim_{s\to 0}\left[s\,\overline{f}(s)\right]$$

The Impulse Function / The Dirac Function

1. $\delta(t-c)=\begin{cases}\infty & t=c\\ 0 & t\neq c\end{cases}$, $\delta(t)=\begin{cases}\infty & t=0\\ 0 & t\neq 0\end{cases}$

2. $\displaystyle\int_a^b \delta(t-c)\ dt=\begin{cases}1 & a\le c\le b\\ 0 & \text{otherwise}\end{cases}$

3. $\displaystyle\int_a^b f(t)\delta(t-c)\ dt=\begin{cases} f(a) & a\le c\le b\\ 0 & \text{otherwise}\end{cases}$

4. $\mathcal{L}\left[\delta(t-c)\right]=\mathrm{e}^{-cs}$

5. $\mathcal{L}\left[f(t)\delta(t-c)\right]=f(c)\mathrm{e}^{-cs}$

6. $\dfrac{d}{dt}\left[\mathrm{H}(t-c)\right]=\delta(t-c)$

VARIOUS LAPLACE TRANSFORM QUESTIONS

Question 1

The function $x = x(t)$ is suitably defined for $t \geq 0$.

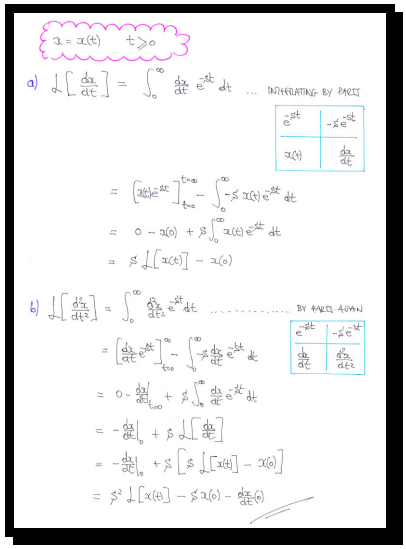
a) Show from first principles that

$$\mathcal{L}\left[\frac{dx}{dt}\right] = s\,\mathcal{L}\left[x(t)\right] - x(0).$$

b) Hence show further that

$$\mathcal{L}\left[\frac{d^2x}{dt^2}\right] = s^2\,\mathcal{L}\left[x(t)\right] - s\,x(0) - \frac{dx}{dt}(0).$$

proof



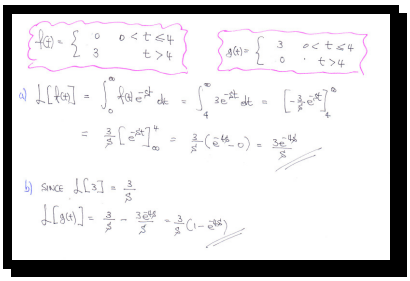
Question 2

$$f(t)\equiv\begin{cases}0 & 0<t\le 4\\ 3 & t>4\end{cases}\quad\text{and}\quad g(t)\equiv\begin{cases}3 & 0<t\le 4\\ 0 & t>4\end{cases}.$$

a) Find the Laplace transform of $f(t)$ from first principles.

b) Hence determine the Laplace transform of $g(t)$.

$$\boxed{\mathcal{L}\left[f(t)\right]=\frac{3\mathrm{e}^{-4s}}{s}},\quad\boxed{\mathcal{L}\left[g(t)\right]=\frac{3}{s}\left(1-\mathrm{e}^{-4s}\right)}$$



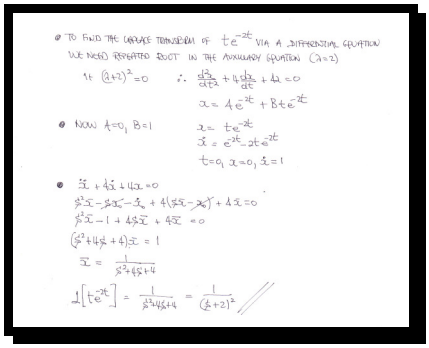
Question 3

By considering a suitable differential equation with appropriate initial conditions show clearly that

$$\mathcal{L}\left(t\mathrm{e}^{-2t}\right)=\frac{1}{(s-2)^2},\ t\ge 0.$$

You may not use integration in this question.

proof



Question 4

Use the differential equation

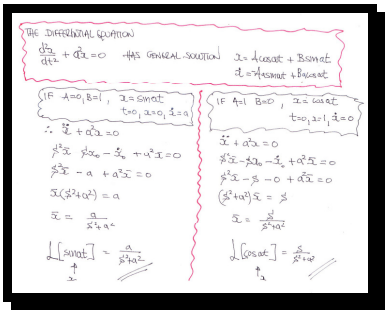
$$\frac{d^2y}{dt^2}+a^2x=0\,,\ t\geq 0\,,$$

with appropriate initial conditions to show that

$$\mathcal{L}\left(\cos at\right)=\frac{s}{s^2+a^2}\quad\text{and}\quad\mathcal{L}\left(\sin at\right)=\frac{a}{s^2+a^2}\,.$$

You may not use integration in this question.

proof



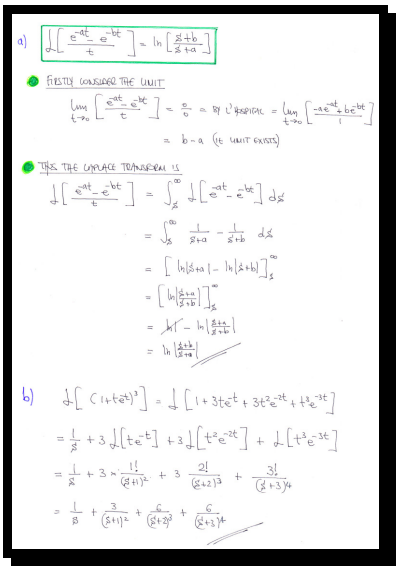
Question 5

Find each of the following Laplace transforms.

a) $\mathcal{L}\left[\dfrac{\mathrm{e}^{-at}-\mathrm{e}^{-bt}}{t}\right]$, $a>0$, $b>0$

b) $\mathcal{L}\left[\left(1+t\,\mathrm{e}^{-t}\right)^3\right]$

$$\mathcal{L}\left[\frac{\mathrm{e}^{-at}-\mathrm{e}^{-bt}}{t}\right]=\ln\left[\frac{s+b}{s+a}\right], \qquad \mathcal{L}\left[\left(1+t\,\mathrm{e}^{-t}\right)^3\right]=\frac{1}{s}+\frac{3}{(s+1)^2}+\frac{6}{(s+2)^3}+\frac{6}{(s+3)^4}$$



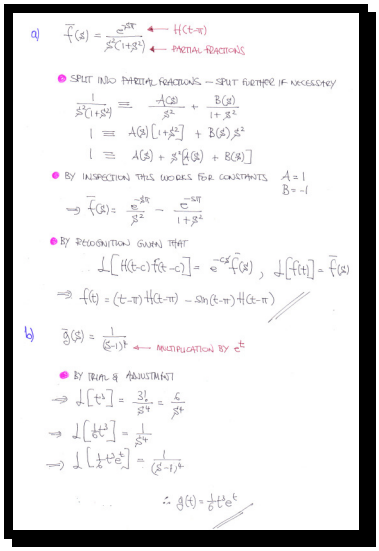
Question 6

Invert each of the following Laplace transforms.

i. $\overline{f}(s)=\dfrac{\mathrm{e}^{-s\pi}}{s^2\left(s^2+1\right)}$

ii. $\overline{g}(s)=\dfrac{1}{(s-1)^4}$

$\boxed{f(t)=t\,\mathrm{H}(t-\pi)-\sin t\,\mathrm{H}(t-\pi)}$, $\boxed{g(t)=\frac{1}{6}t^3\mathrm{e}^t}$



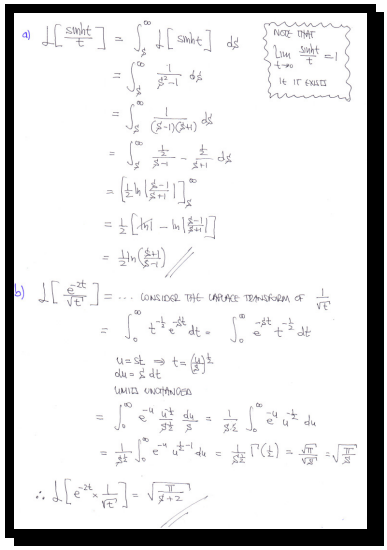
Question 7

Find each of the following Laplace transforms.

c) $\mathcal{L}\left[\dfrac{\sinh t}{t}\right]$

d) $\mathcal{L}\left[\dfrac{\mathrm{e}^{-2t}}{\sqrt{t}}\right]$

$$\boxed{\mathcal{L}\left[\frac{\sinh t}{t}\right]=\frac{1}{2}\ln\left[\frac{s+1}{s-1}\right]}, \quad \boxed{\mathcal{L}\left[\frac{\mathrm{e}^{-2t}}{\sqrt{t}}\right]=\sqrt{\frac{\pi}{s+2}}}$$



Question 8

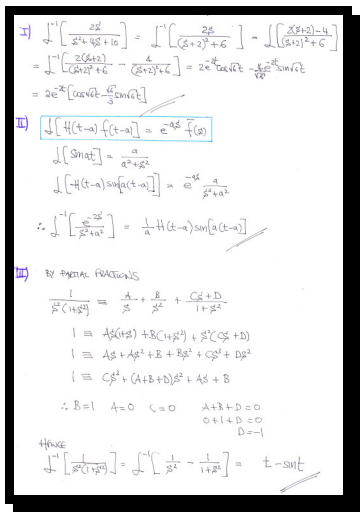
Find the inverse following Laplace transforms of the following functions.

i. $\dfrac{2s}{s^2+4s+10}$.

ii. $\dfrac{\mathrm{e}^{-2s}}{s^2+a^2}$.

iii. $\dfrac{1}{s^2\left(s^2+1\right)}$

$$\boxed{2\mathrm{e}^{-2t}\left[\cos\sqrt{6}\,t-\frac{1}{3}\sqrt{6}\,\mathrm{c}\sin\sqrt{6}\,t\right]},\ \boxed{\frac{1}{a}\mathrm{H}\left(t-a\right)-\sin\left[a\left(t-a\right)\right]},\ \boxed{t-\sin t}$$

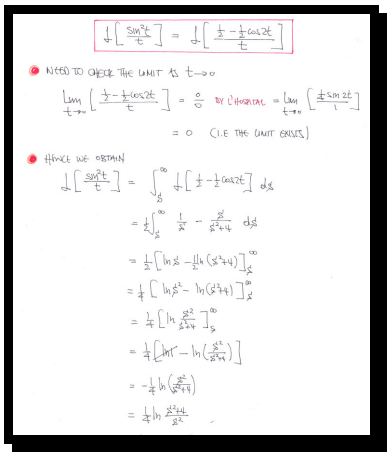


Question 9

Find the following Laplace transform

$$\mathcal{L}\left[\frac{\sin^2 t}{t}\right].$$

$$\boxed{\mathcal{L}\left[\frac{\sin^2 t}{t}\right]=\frac{1}{4}\ln\left[\frac{s^2+4}{s^2}\right]}$$



Question 10

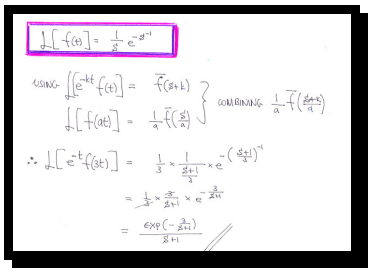
It is given that

$$\mathcal{L}\left[f(t)\right]=\frac{1}{s}\exp\left(-\frac{1}{s}\right),\ t\geq 0.$$

Determine a simplified expression for

$$\mathcal{L}\left[\mathrm{e}^{-t}f(3t)\right].$$

$$\boxed{\mathcal{L}\left[\mathrm{e}^{-t}f(3t)\right]=\frac{1}{s+1}\exp\left(-\frac{3}{s+1}\right)}$$

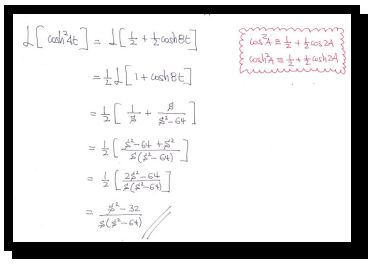


Question 11

Find a simplified expression for

$$\mathcal{L}\left[\cosh^2 4t\right].$$

$$\boxed{\mathcal{L}\left[\cosh^2 4t\right]=\frac{s^2-32}{s\left(s^2-64\right)}}$$



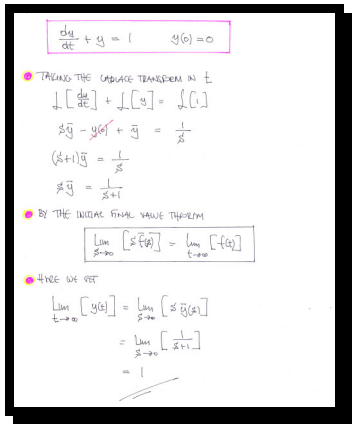
Question 12

The function $y=y(t)$ satisfies the differential equation

$$\frac{dy}{dt}+y=1, \quad t\geq 0, \quad y(0)=0.$$

Use the initial-final value theorem to find $\lim\limits_{t\to\infty}\left[y(t)\right]$.

$$\boxed{\lim_{t\to\infty}\left[y(t)\right]=1}$$



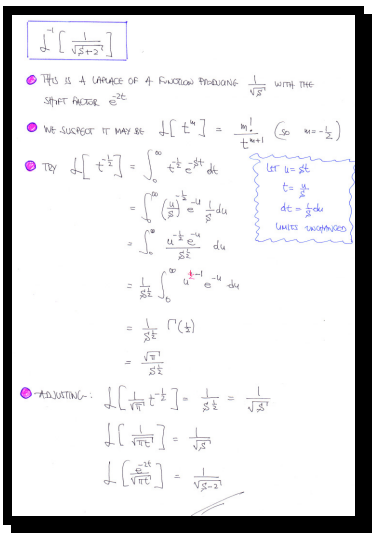
Question 13

The function $y = f(t)$ satisfies

$$\mathcal{L}\left[f(t)\right] = \frac{1}{\sqrt{s+2}}.$$

Determine a simplified expression for $f(t)$.

$$f(t) = \frac{\mathrm{e}^{-2t}}{\sqrt{\pi t}}$$



Question 14

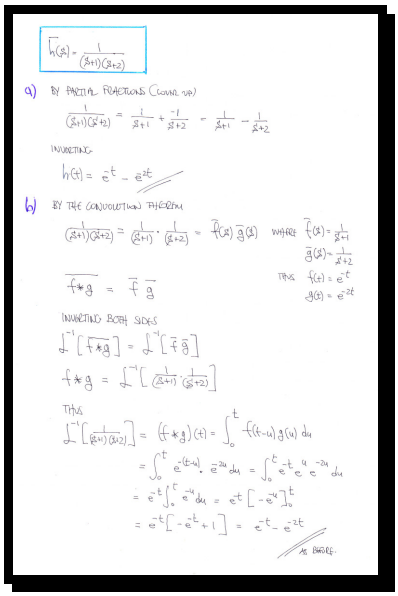
$$\bar{h}(s)=\frac{1}{(s+1)(s+2)}.$$

Invert the above Laplace transform by …

a) … partial fractions

b) … the convolution theorem

$$\boxed{h(t)=\mathrm{e}^{-t}-\mathrm{e}^{-2t}}$$



Question 15

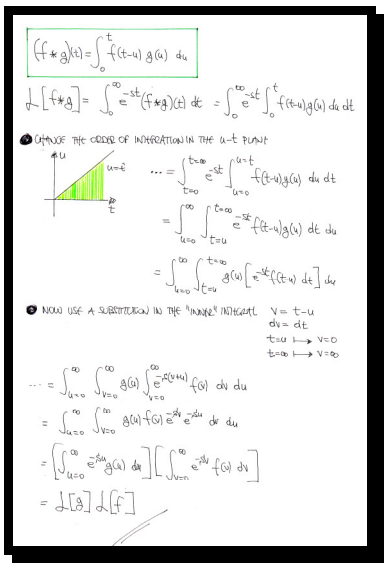
The convolution $[f*g](t)$, of two functions $f(t)$ and $g(t)$ is defined as

$$[f*g](t)=\int_0^t f(t-u)\,g(u)\ du\,.$$

Show that

$$\mathcal{L}\{[f*g](t)\} \ = \ \mathcal{L}[f(t)]\mathcal{L}[g(t)] \ = \ \overline{f}(s)\overline{g}(s)\,.$$

proof



Question 16

Use the differential equation

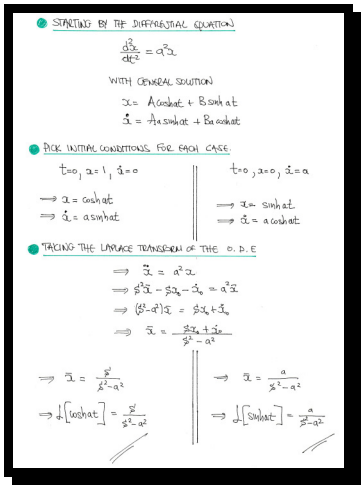
$$\frac{d^2x}{dt^2} = a^2x\,,\ t \geq 0\,,$$

with appropriate initial conditions to show that

$$\mathcal{L}\left(\cosh at\right) = \frac{s}{s^2 - a^2} \quad \text{and} \quad \mathcal{L}\left(\sinh at\right) = \frac{a}{s^2 - a^2}\,.$$

You may not use integration in this question.

, proof



Question 17

The function $y = f(t)$, $t \geq 0$, is twice differentiable.

a) Show from first principles that

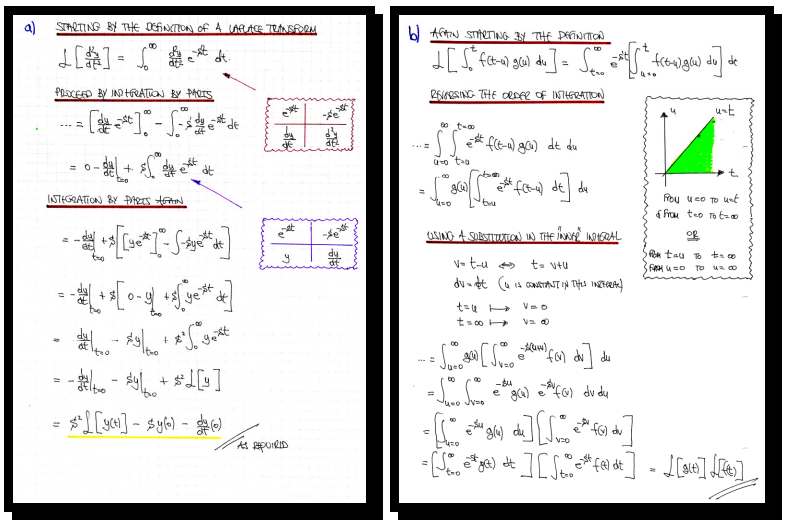
$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s^2\,\mathcal{L}\left[y(t)\right] - s\,y(0) - \frac{dy}{dt}(0)$$

A second function $g(t)$ is defined for $t \geq 0$.

b) Show further that

$$\mathcal{L}\left[\int_0^t f(t-u)\,g(u)\;du\right] = \mathcal{L}\left[f(t)\right]\mathcal{L}\left[g(t)\right].$$

[ ], [proof]



Question 18

$$\mathcal{L}\left[f(t) \right] \equiv \overline{f}(s),\ t \geq 0 .$$

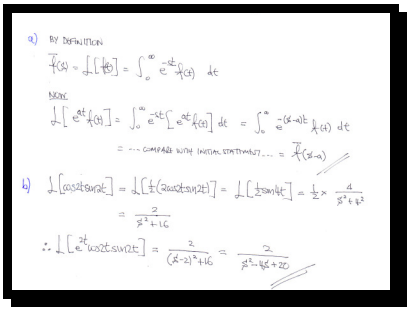
a) Show clearly that

$$\mathcal{L}\left[\mathrm{e}^{at} f(t) \right] \equiv \overline{f}(s-a) .$$

b) Find in its simplest form

$$\mathcal{L}\left[\mathrm{e}^{2t} \cos 2t \sin 2t \right] .$$

$$\boxed{\frac{2}{s^2-4s+20}}$$

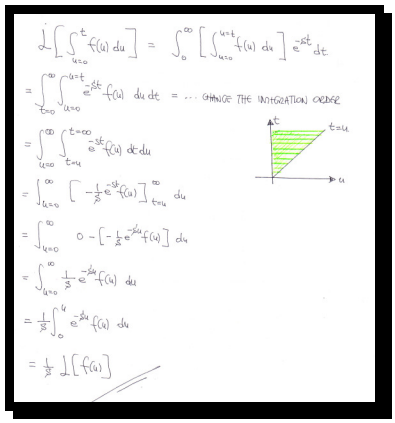


Question 19

Use the definition of a Laplace transform to show that

$$\mathcal{L}\left[\int_0^t f(u)\ du\right]=\frac{1}{s}\mathcal{L}\left[f(u)\right],\ t\geq 0.$$

proof

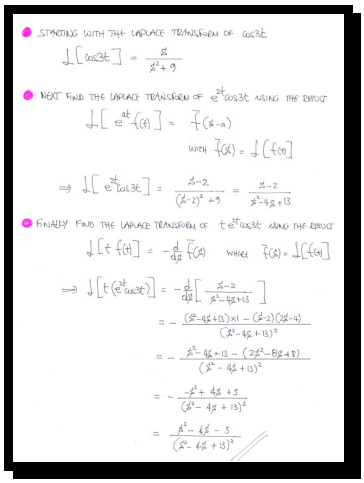


Question 20

Determine a simplified expression for

$$\mathcal{L}\left[t\,\mathrm{e}^{2t}\cos 3t\right].$$

$$\mathcal{L}\left[t\,\mathrm{e}^{2t}\cos 3t\right]=\frac{s^2-4s-5}{\left(s^2-4s+13\right)^2}$$

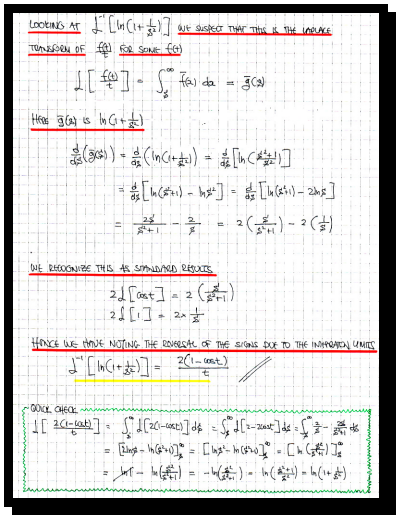


Question 21

Find the following inverse Laplace transform

$$\mathcal{L}^{-1}\left[\ln\left(1+\frac{1}{s^2}\right)\right].$$

$\square$, $\boxed{\mathcal{L}^{-1}\left[\ln\left(1+\frac{1}{s^2}\right)\right]=\frac{2}{t}(1-\cos t)}$



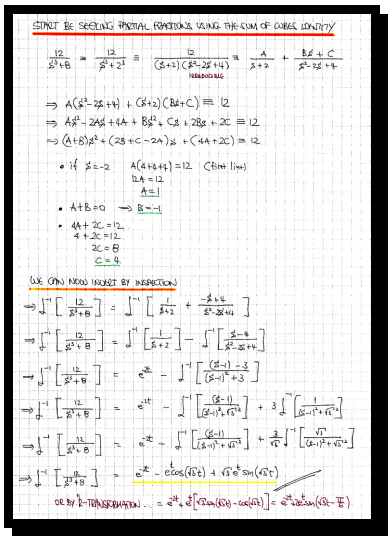
Question 22

Find the following inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{12}{s^3+8}\right].$$

,

$$\mathcal{L}^{-1}\left[\frac{12}{s^3+8}\right]=\mathrm{e}^{-2t}+2\mathrm{e}^{t}\left[\sqrt{3}\sin\left(\sqrt{3}t\right)-\cos\left(\sqrt{3}t\right)\right]=\mathrm{e}^{-2t}+2\mathrm{e}^{t}\sin\left(\sqrt{3}t-\frac{1}{6}\pi\right)$$

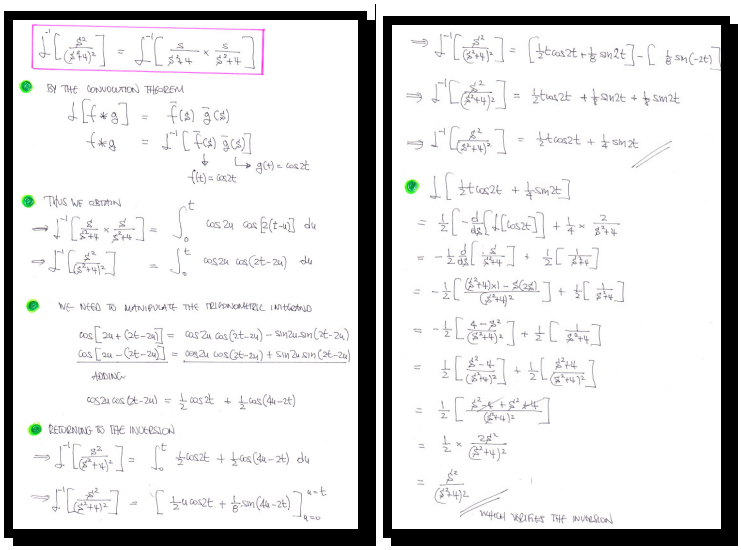


Question 23

Find and verify the following inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{s^2}{\left(s^2+4\right)^2}\right].$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{s^2}{\left(s^2+4\right)^2}\right]=\frac{1}{2}t\cos 2t+\frac{1}{4}\sin 2t}$$

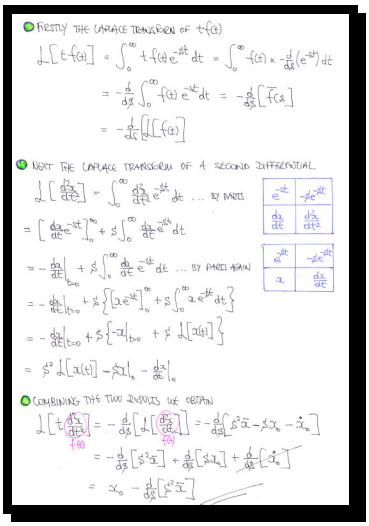


Question 24

Use the definition of a Laplace transform to show that if $x = f(t)$ then

$$\mathcal{L}\left[t^2 \frac{d^2x}{dt^2}\right] = x_0 - \frac{d}{ds}\left[s^2 \mathcal{L}(x)\right], \quad \text{where} \quad x_0 = f(0).$$

proof



Question 25

$$\mathcal{L}\left[f(t)\right]=\overline{f}(s)\equiv\int_0^{\infty}f(t)\mathrm{e}^{-st}\ dt\,,\ t\geq 0\,.$$

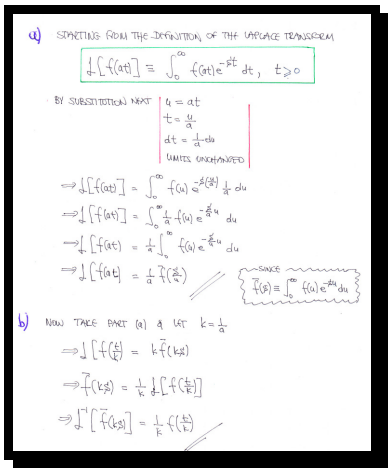
a) Show from the above definition that if a is a non zero constant, then

$$\mathcal{L}\left[f(at)\right]=\frac{1}{a}\overline{f}\left(\frac{s}{a}\right).$$

b) Deduce that if k is a non zero constant, then

$$\mathcal{L}^{-1}\left[\overline{f}(k\,s)\right]=\frac{1}{k}f\left(\frac{t}{k}\right).$$

proof



Question 26

$$\mathcal{L}\left[f(t) \right] \equiv \overline{f}(s),\ t \geq 0 .$$

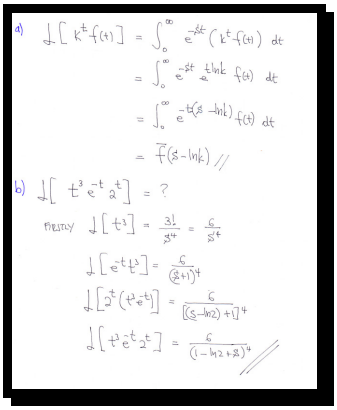
a) Show clearly that

$$\mathcal{L}\left[k^t f(t) \right] \equiv \overline{f}(s - \ln k),\ k > 0 .$$

b) Find in its simplest form

$$\mathcal{L}\left[t^3 \mathrm{e}^{-t} 2^t \right].$$

$$\boxed{\mathcal{L}\left[t^3 \mathrm{e}^{-t} 2^t \right] = \frac{6}{(s+1-\ln 2)^4}}$$

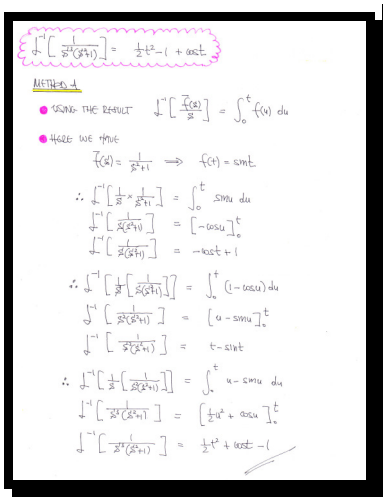


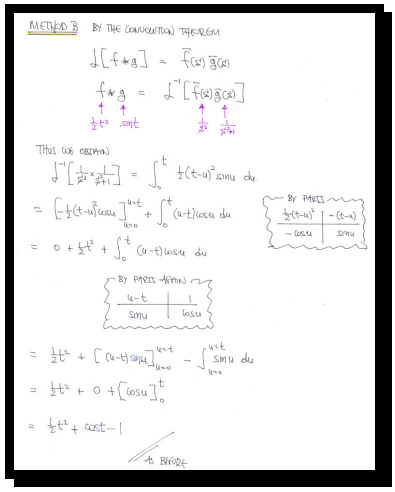
Question 27

Find the following inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{1}{s^3\left(s^2+1\right)}\right].$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{1}{s^3\left(s^2+1\right)}\right]=\frac{1}{2}t^2-1+\cos t}$$



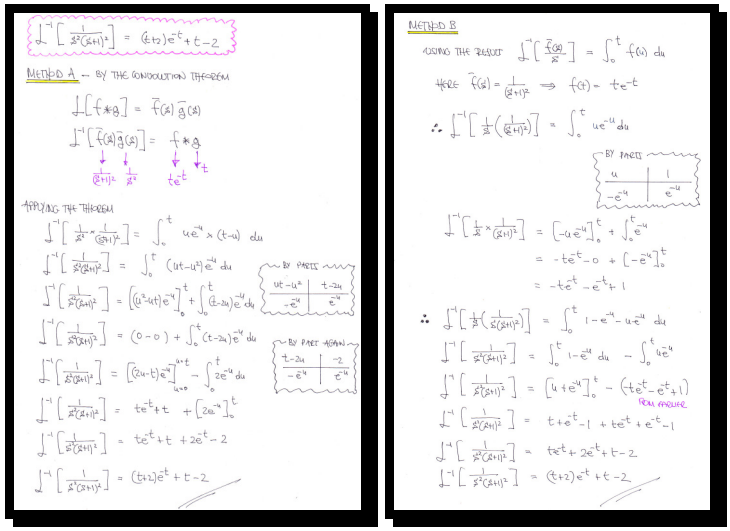


Question 28

Find the following inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right].$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right]=t-2+(t+1)\mathrm{e}^{-t}}$$



Question 29

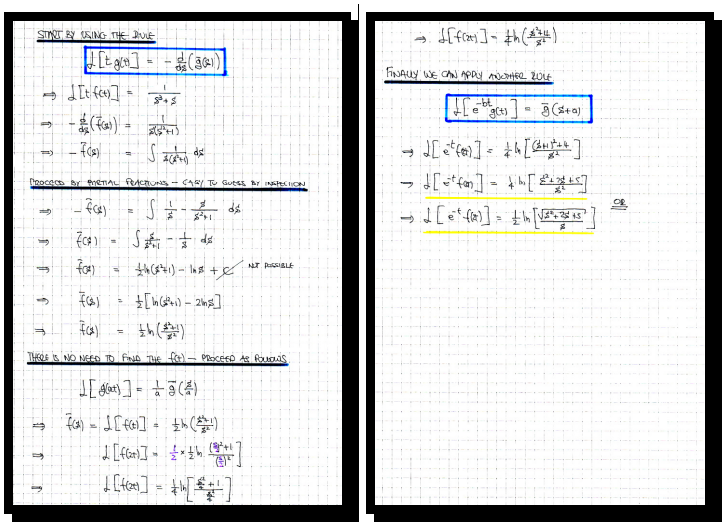
It is given that

$$\mathcal{L}\left[t\,f\left(t\right)\right]=\frac{1}{s^3+s},\ t\geq 0.$$

Determine a simplified expression for

$$\mathcal{L}\left[\mathrm{e}^{-t}\,f\left(2t\right)\right].$$

$\square$, $\mathcal{L}\left[\mathrm{e}^{-t}\,f\left(2t\right)\right]=\frac{1}{2}\ln\left(\frac{\sqrt{s^2+2s+5}}{s+1}\right)$



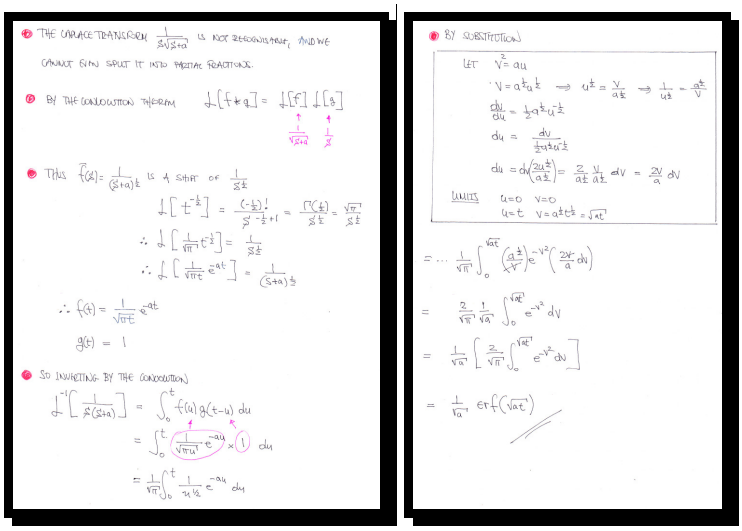
Question 30

Use an appropriate method to show that

$$\mathcal{L}^{-1}\left[\frac{1}{s\sqrt{s+a}}\right] = \frac{1}{\sqrt{a}}\operatorname{erf}\left(\sqrt{at}\right),$$

where a is a positive constant.

proof

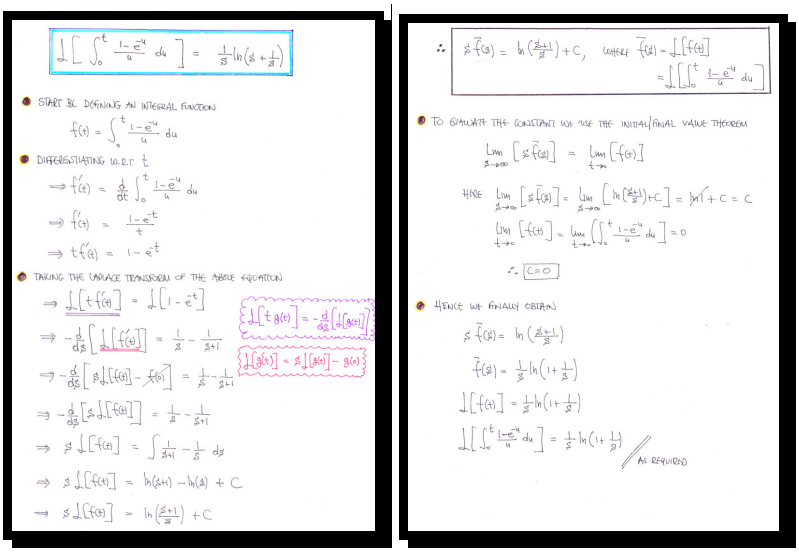


Question 31

Use an appropriate method to show that

$$\mathcal{L}\left[\int_0^t \frac{1-\mathrm{e}^{-u}}{u}\,du\right] = \frac{1}{s}\ln\left(s+\frac{1}{s}\right).$$

proof

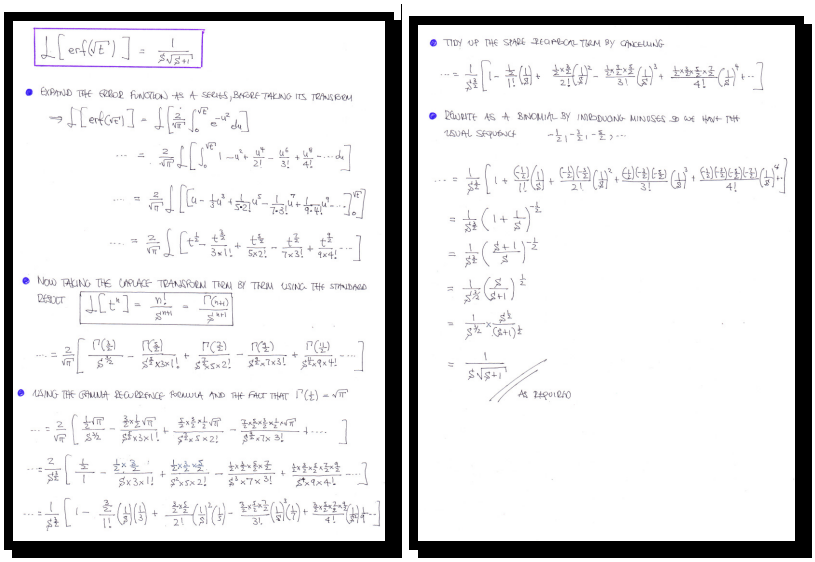


Question 32

Use an appropriate method to show that

$$\mathcal{L}\left[\operatorname{erf}\left(\sqrt{t}\right)\right] = \frac{1}{s\sqrt{s+1}} .$$

proof



Question 33

$$g(t) \equiv \int_0^t f(x)\ dx\,,\ t \geq 0\,.$$

a) Show clearly that

$$\mathcal{L}\big(g(t)\big) = \frac{\overline{f}(s)}{s},$$

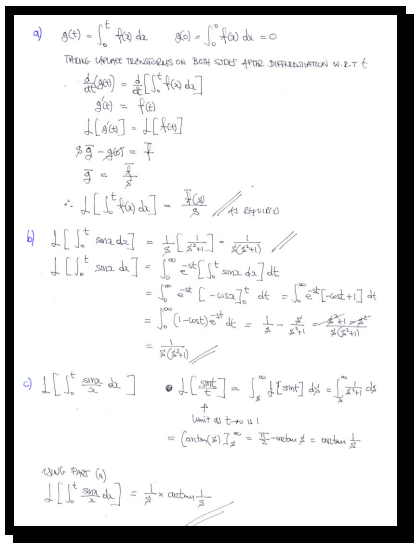
where $\overline{f}(s) = \mathcal{L}\big(f(t)\big)$.

b) Verify the validity of the result of part **(a)** by using $f(x) = \sin x$ and finding $\mathcal{L}\big(g(t)\big)$ by its integral definition.

c) Use the result of part **(a)** to determine

$$\mathcal{L}\left[\int_0^t \frac{\sin x}{x}\ dx\right].$$

$$\boxed{\frac{1}{s}\arctan\left(\frac{1}{s}\right)}$$



Question 34

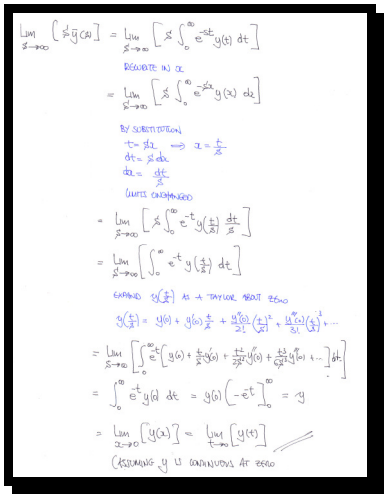
The function $y = y(t)$ is infinitely differentiable and defined for $t \geq 0$.

Show that

$$\lim_{s\to\infty}\left[s\,\overline{y}(s)\right] = \lim_{t\to 0}\left[y(t)\right],$$

where $\overline{y}(s) = \mathcal{L}\left[y(t)\right]$

proof



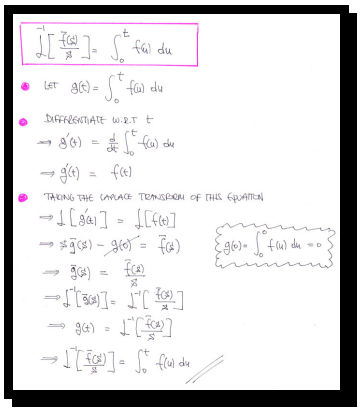
Question 35

The Laplace transform of $f(t)$, $t \geq 0$, is denoted by $\overline{f}(s) = \mathcal{L}\{f(t)\}$.

Show that the inverse Laplace transform of $\dfrac{\overline{f}(s)}{s}$ satisfies

$$\mathcal{L}^{-1}\left\{\frac{\overline{f}(s)}{s}\right\} = \int_0^t f(u)\ du\,.$$

proof



Question 36

$$t\frac{d^2y}{dt^2}+\frac{dy}{dt}+ty\,,\ t>0\,.$$

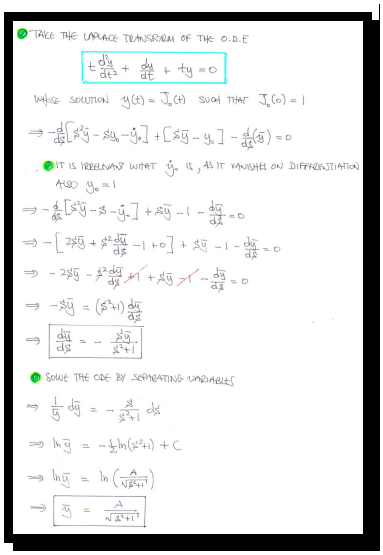
The function $y=J_0(t)$ is a solution of the above differential equation.

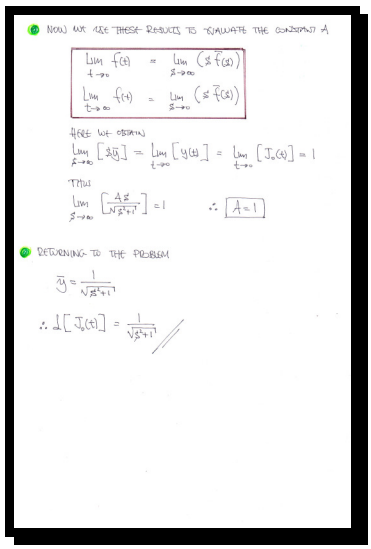
It is further given that $\lim_{t\to 0}\left[J_0(t)\right]=1$.

By taking the Laplace transform of the above differential equation, show that

$$\mathcal{L}\left[J_0(t)\right]=\frac{1}{\sqrt{s^2+1}}\,.$$

proof



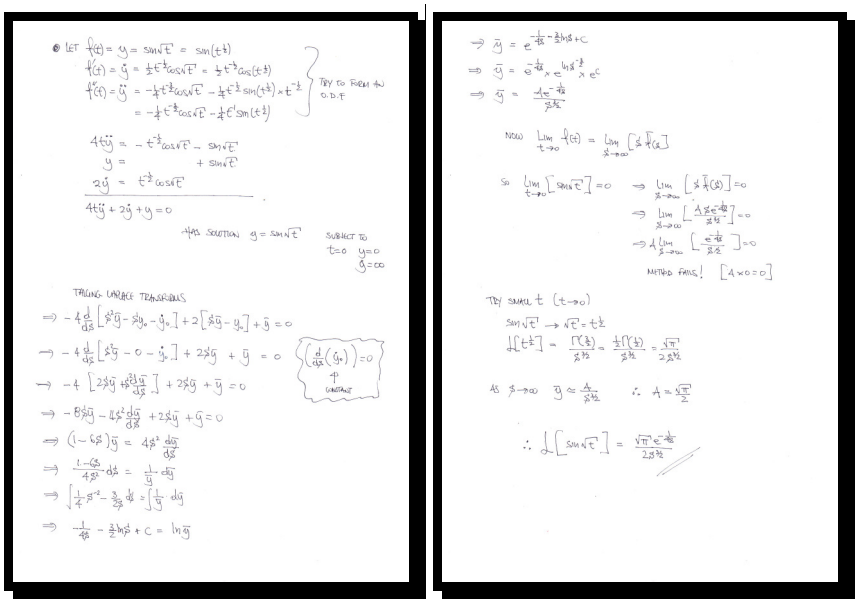


Question 37

By forming and taking the Laplace transform of a suitable second order differential equation, show that

$$\mathcal{L}\left[\sin\sqrt{t}\right]=\frac{\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{4s}}}{2s^{\frac{3}{2}}}.$$

proof



Question 38

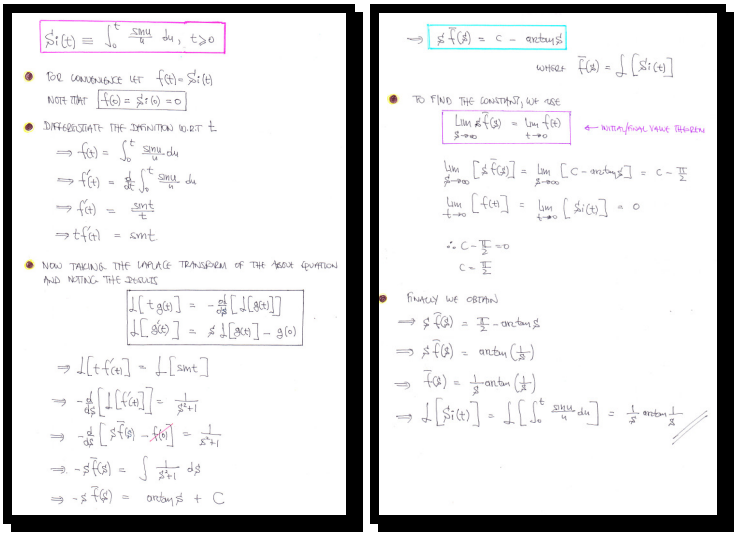
The Sine integral function $\mathrm{Si}(t)$ is defined as

$$\mathrm{Si}(t) \equiv \int_0^t \frac{\sin u}{u}\, du\,,\ t \geq 0\,.$$

Show that

$$\mathcal{L}\left[\mathrm{Si}(t)\right] = \frac{1}{s}\arctan\left(\frac{1}{s}\right).$$

proof



Question 39

Find the following inverse Laplace transform by 3 different methods.

$$\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right],\ a>0.$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=\frac{t\sin at}{2a}}$$



Question 40

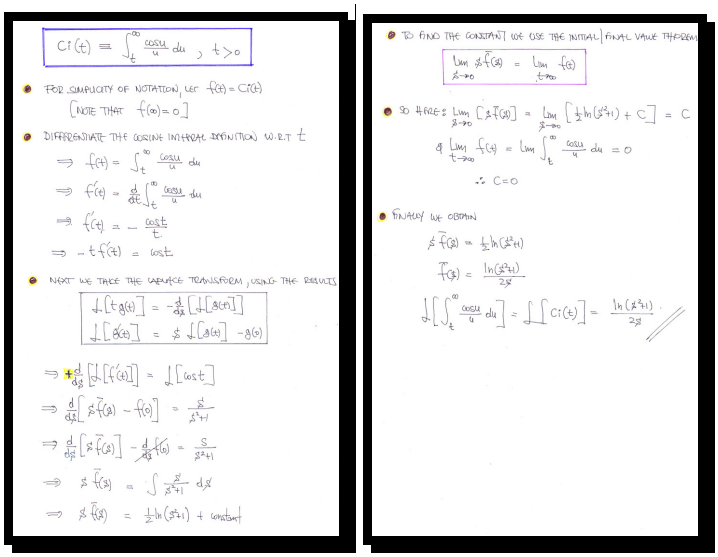
The Cosine integral function $\mathrm{Ci}(t)$ is defined as

$$\mathrm{Ci}(t)\equiv\int_t^{\infty}\frac{\cos u}{u}\,du\,,\ t>0\,.$$

Show that

$$\mathcal{L}\left[\mathrm{Ci}(t)\right]=\frac{\ln\left(s^2+1\right)}{2s}\,.$$

proof



Question 41

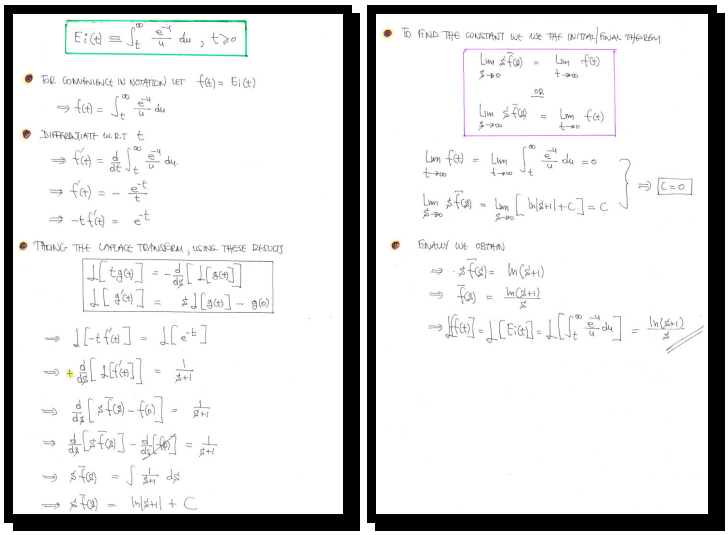
The Exponential integral function $\mathrm{Ei}(t)$ is defined as

$$\mathrm{Ei}(t) \equiv \int_t^{\infty} \frac{\mathrm{e}^{-u}}{u}\, du\,,\ t \geq 0\,.$$

Show that

$$\mathcal{L}\left[\mathrm{Ei}(t)\right] = \frac{\ln(s+1)}{s}\,.$$

proof



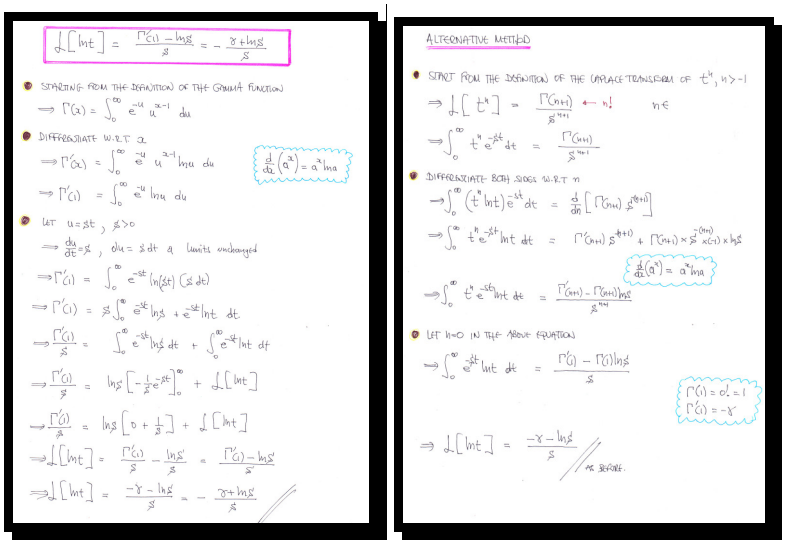
Question 42

By differentiating the integral definition of the Gamma function, $\Gamma(x)$, with respect to x, show that

$$\mathcal{L}[\ln t]=-\frac{\gamma+\ln s}{s}.$$

You may assume that $\Gamma'(1)=-\gamma$.

proof



Question 43

$$\mathcal{L}\left[f(t)\right]=\overline{f}(s)\equiv\int_0^{\infty}f(t)\mathrm{e}^{-st}\ dt\,,\ t\geq 0\,.$$

a) Show from the above definition that if a is a non zero constant, then

$$\mathcal{L}\left[f(at)\right]=\frac{1}{a}\overline{f}\left(\frac{s}{a}\right).$$

b) By taking the Laplace transform of Bessel's equation

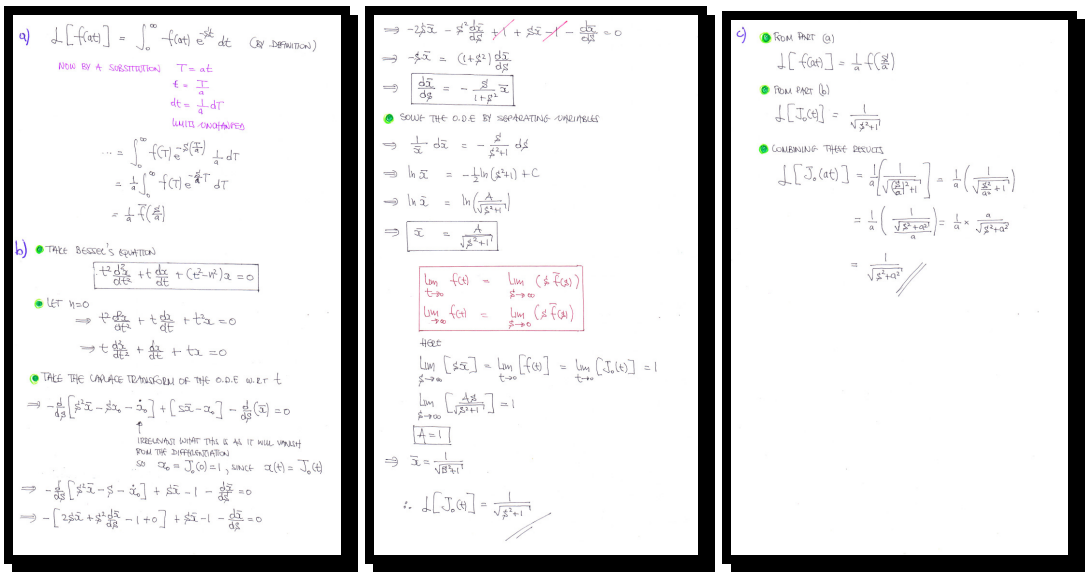
$$t^2\frac{d^2x}{dt^2}+t\frac{dx}{dx}+\left(t^2-n^2\right)x=0\,,\ n\in\mathbb{N}\,,$$

and assuming further that $J_0(0)=1$, show that

$$\mathcal{L}\left[J_0(t)\right]=\frac{1}{\sqrt{s^2+1}}\,.$$

c) Deduce in simplified form the Laplace transform of $J_0(at)$

$$\boxed{\mathcal{L}\left[J_0(at)\right]=\frac{1}{\sqrt{s^2+a^2}}}$$



Question 44

$$\mathcal{L}\left[f(t)\right]=\overline{f}(s)\equiv\int_0^{\infty}f(t)\mathrm{e}^{-st}\ dt\,,\ t\geq 0\,.$$

a) Show from the above definition that if k is a non zero constant, then

$$\mathcal{L}^{-1}\left[\overline{f}(k\,s)\right]=\frac{1}{k}f\left(\frac{t}{k}\right).$$

b) Show further that

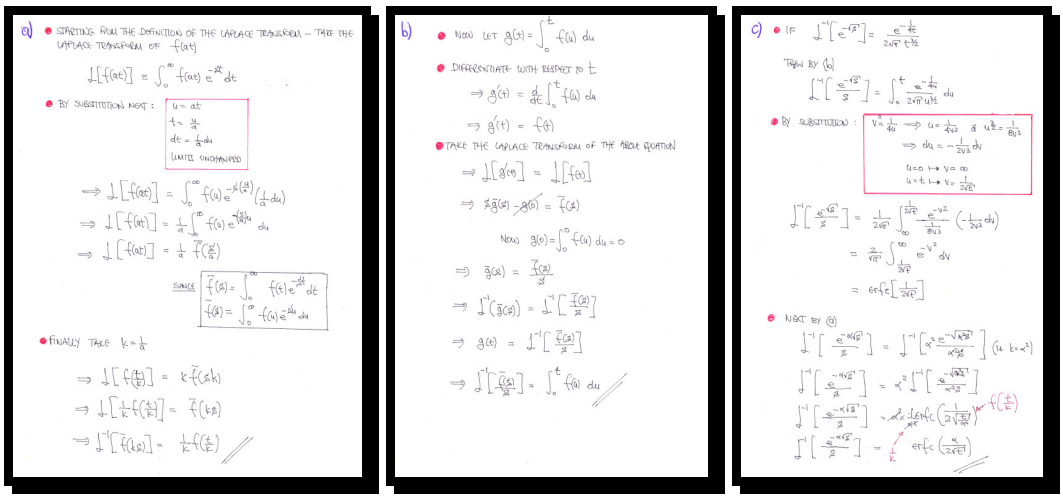
$$\mathcal{L}^{-1}\left[\frac{\overline{f}(s)}{s}\right]=\int_0^{t}f(u)\ du\,.$$

c) Given that $\mathcal{L}^{-1}\left[\mathrm{e}^{-\sqrt{s}}\right]=\dfrac{\mathrm{e}^{-\frac{1}{4t}}}{2t^{\frac{3}{2}}\sqrt{\pi}}$, use parts **(a)** and **(b)** to prove that

$$\mathcal{L}^{-1}\left[\frac{\mathrm{e}^{-\alpha\sqrt{s}}}{s}\right]=\operatorname{erfc}\left(\frac{\alpha}{2\sqrt{t}}\right),$$

where α is a positive constant.

proof



Question 45

The Laplace transform $\overline{y}(s)$, of a function $y = y(t)$, $t \geq 0$ is given by

$$\overline{y}(s) = \mathrm{e}^{-\sqrt{s}}, \quad s > 0.$$

a) Show that $\overline{y}(s)$ satisfies the differential equation

$$4s\,\overline{y}''(s) + 2\,\overline{y}'(s) - \overline{y}(s) = 0.$$

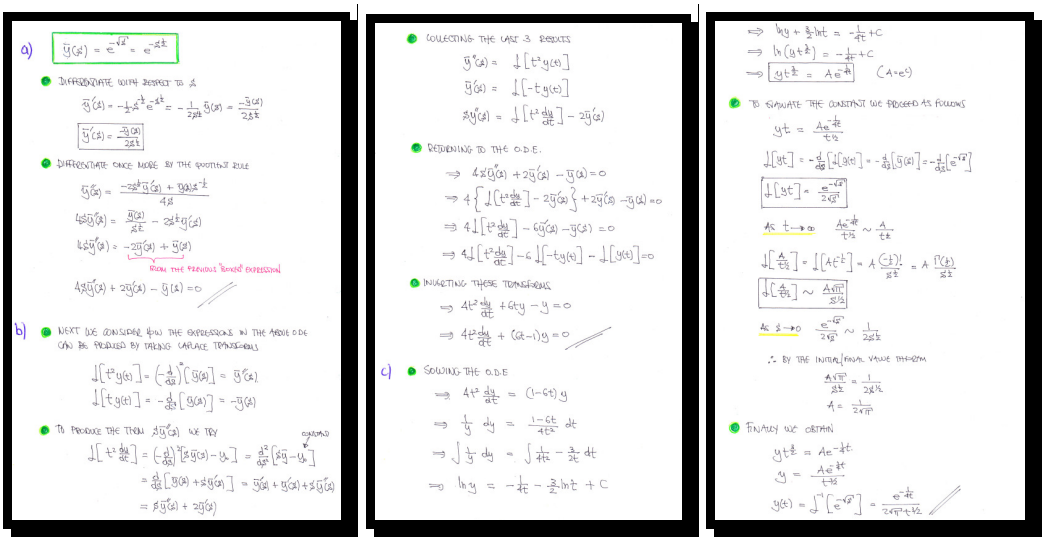
b) Hence show further that

$$4t^2\frac{dy}{dt} + (6t-1)\,y = 0.$$

c) Use parts **(a)** and **(b)** to prove that

$$y(t) = \mathcal{L}^{-1}\left(\mathrm{e}^{-\sqrt{s}}\right) = \frac{\mathrm{e}^{-\frac{1}{4t}}}{2t^{\frac{3}{2}}\sqrt{\pi}}.$$

proof



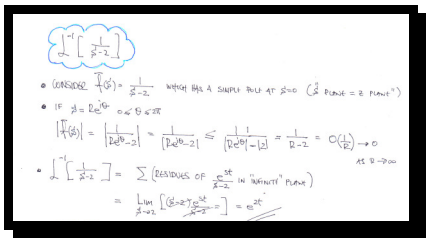
INVERSION BY COMPLEX VARIABLES

Question 1

Use the method of residues to find

$$\mathcal{L}^{-1}\left(\frac{1}{s-2}\right).$$

$$\boxed{\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)=\mathrm{e}^{2t}}$$

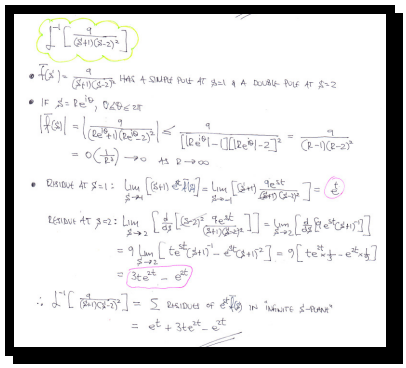


Question 2

Use the method of residues to find

$$\mathcal{L}^{-1}\left[\frac{9}{(s+1)(s-2)^2}\right].$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{9}{(s+1)(s-2)^2}\right]=\mathrm{e}^{t}+(3t-1)\mathrm{e}^{2t}}$$

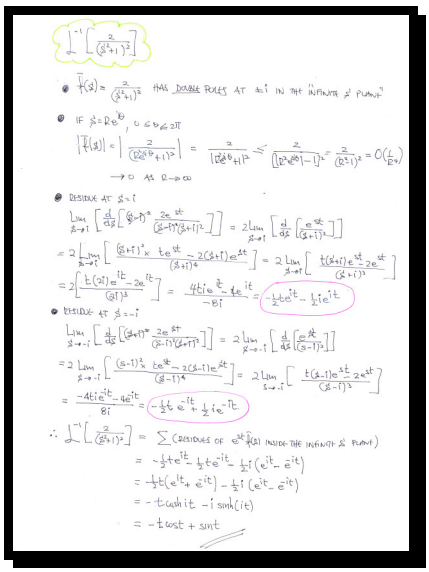


Question 3

Use the method of residues to find

$$\mathcal{L}^{-1}\left[\frac{2}{\left(s^2+1\right)^2}\right].$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{2}{\left(s^2+1\right)^2}\right]=\sin t-t\cos t}$$

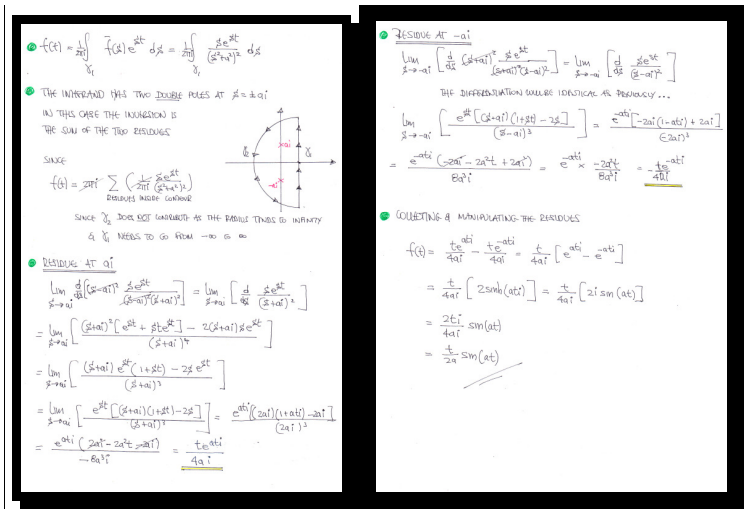


Question 4

Use complex integration to find the following inverse Laplace transform.

$$\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right],\ a>0\,.$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=\frac{t\sin at}{2a}}$$

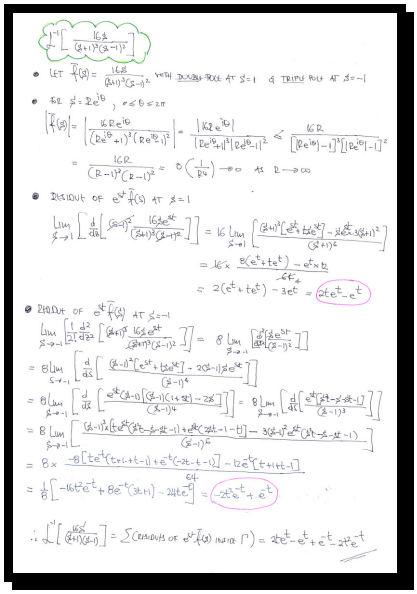


Question 5

Use the method of residues to find

$$\mathcal{L}^{-1}\left[\frac{16s}{(s+1)^3(s-1)^2}\right].$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{16s}{(s+1)^3(s-1)^2}\right]=(2t-1)\mathrm{e}^{t}+\left(1-2t^2\right)\mathrm{e}^{-t}}$$

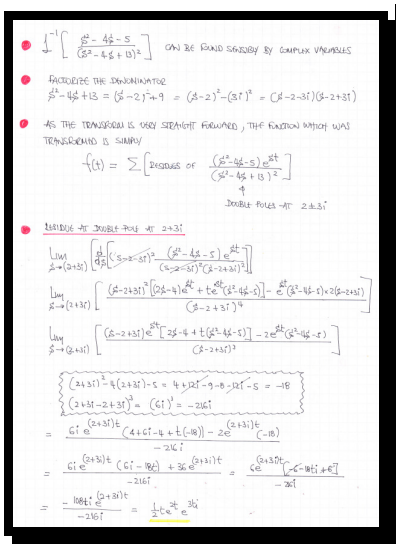


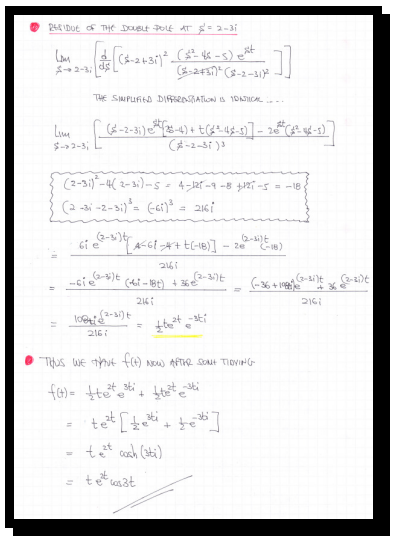
Question 6

Use complex variables to find

$$\mathcal{L}^{-1}\left[\frac{s^2-4s-5}{\left(s^2-4s+13\right)^2}\right].$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{s^2-4s-5}{\left(s^2-4s+13\right)^2}\right]=t\,\mathrm{e}^{2t}\cos 3t}$$





Question 7

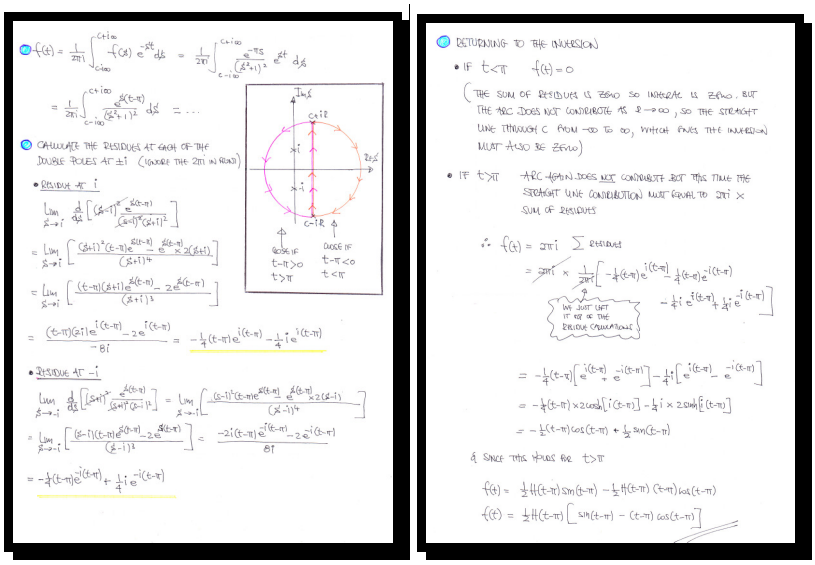
$$\overline{f}(s)=\frac{\mathrm{e}^{-s\pi}}{\left(s^2+1\right)^2}.$$

Use complex variable methods to invert the above Laplace transform.

Use a detailed method, describing briefly each stage in the workings.

Give the final answer in terms of Heaviside functions.

$$\boxed{f(t)=\frac{1}{2}\mathrm{H}(t-\pi)\left[\sin(t-\pi)-(t-\pi)\cos(t-\pi)\right]}$$



Question 8

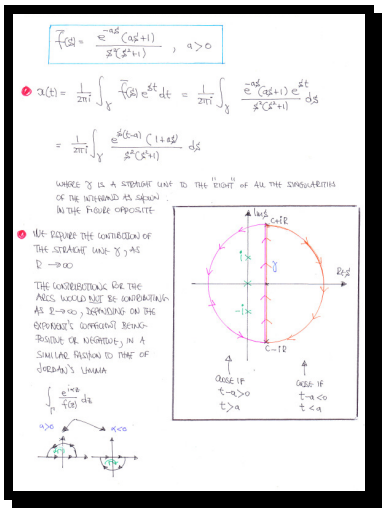
$$\overline{f}(s)=\frac{(as+1)\mathrm{e}^{-as}}{s^2\left(s^2+1\right)},\ a>0.$$

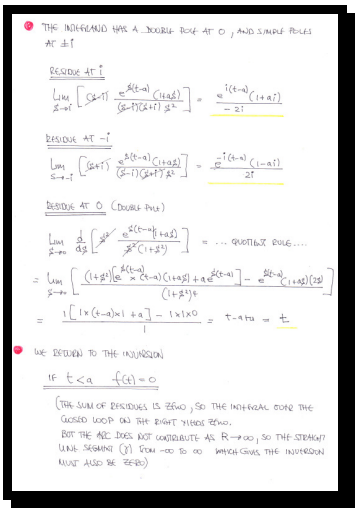
Use complex variable methods to invert the above Laplace transform.

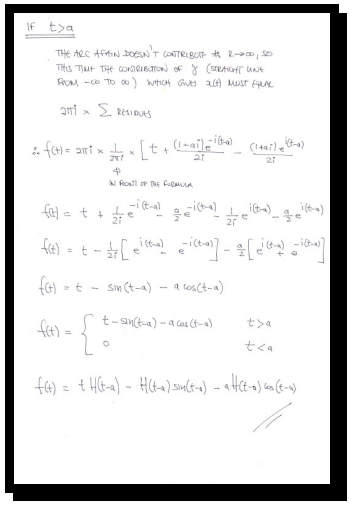
$$\mathcal{L}\left[\overline{f}(s)\right]=t\,\mathrm{H}(t-a)-\mathrm{H}(t-a)\sin(t-a)+a\,\mathrm{H}(t-a)\cos(t-a)$$

Use a detailed method, describing briefly each stage in the workings.

proof







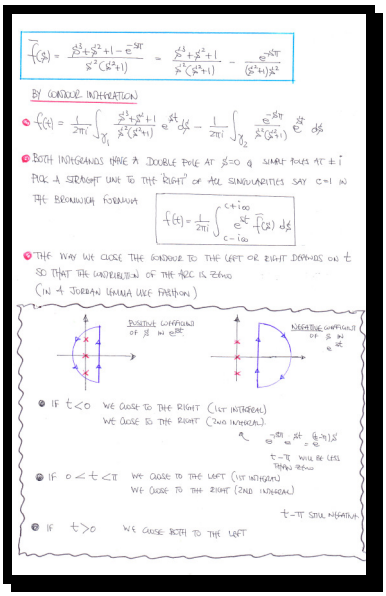
Question 9

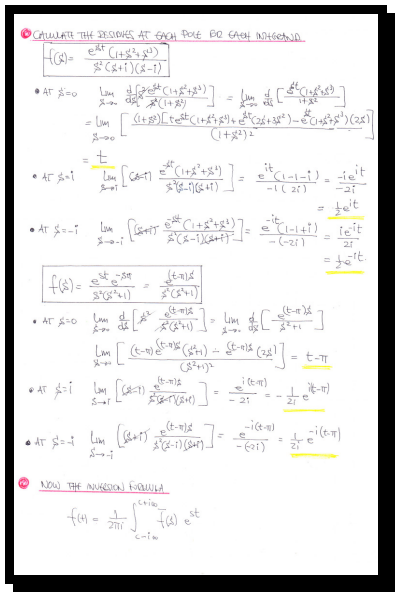
$$\bar{f}(s)=\frac{s^3+s^2+1-\mathrm{e}^{-s\pi}}{s^2\left(s^2+1\right)}.$$

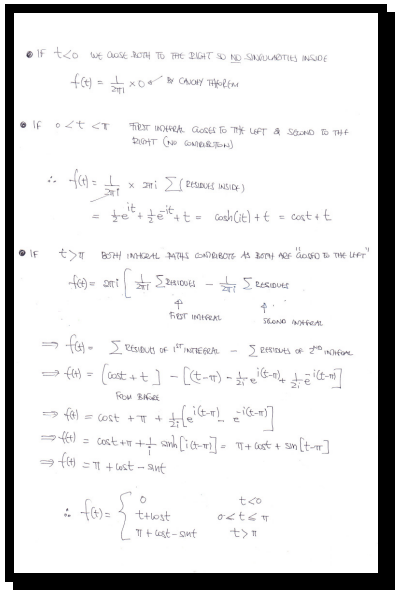
Use complex variable methods to invert the above Laplace transform.

Use a detailed method, describing briefly each stage in the workings.

$$f(t)=\begin{cases}0 & t<0\\ t+\cos t & 0\le t\le\pi\\ \pi+\cos t-\sin t & t>\pi\end{cases}$$







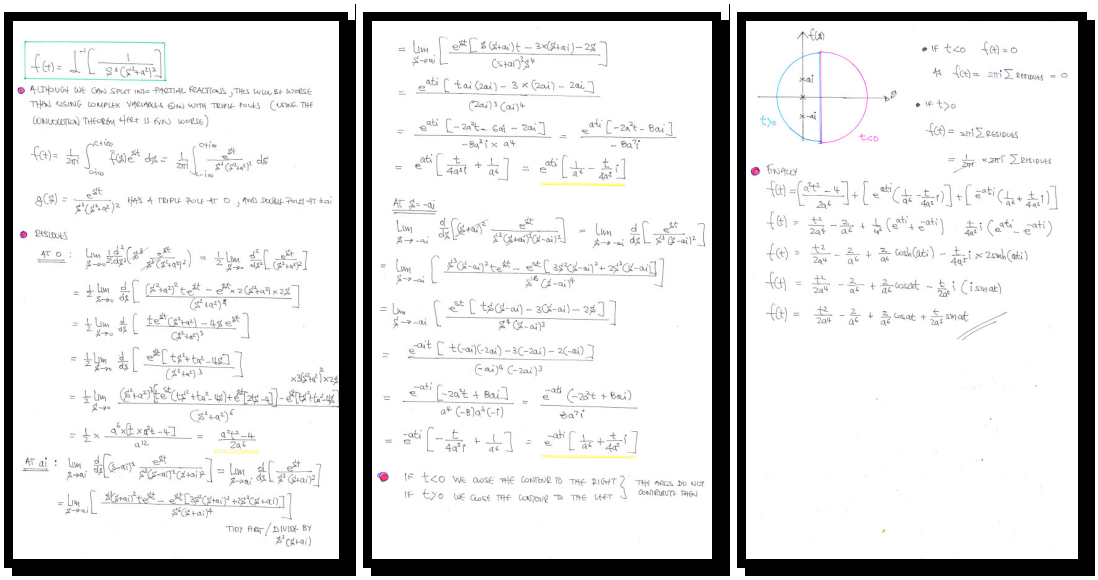
Question 10

Given that a is a positive constant, use complex variable methods to find the following inverse Laplace transform.

$$\mathcal{L}^{-1}\left[\frac{1}{s^3\left(s^2+a^2\right)^2}\right].$$

Use a detailed method, describing briefly each stage in the workings.

$$\boxed{f(t)=\frac{t^2}{2a^4}-\frac{2}{a^6}+\frac{2}{a^6}\cos at+\frac{t}{2a^5}\sin at}$$



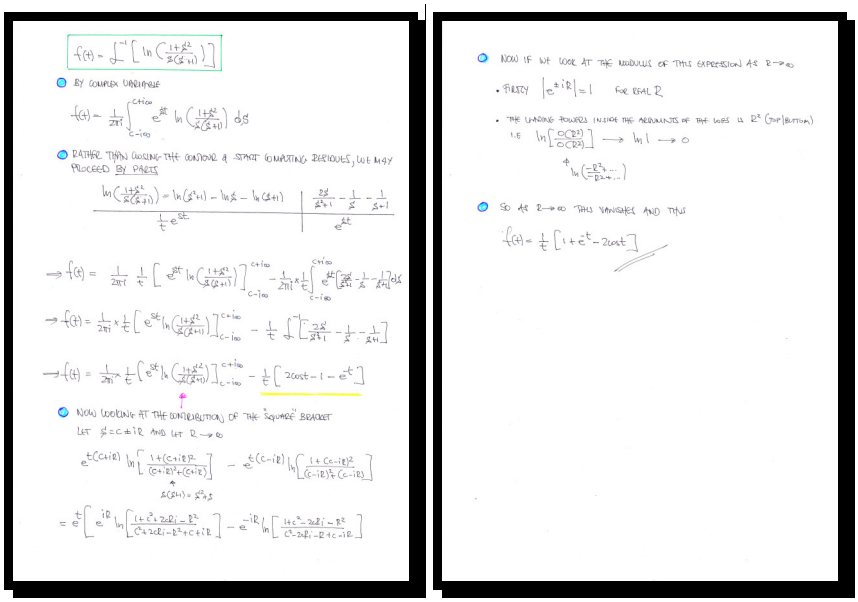
Question 11

Use complex variable methods to find the following inverse Laplace transform.

$$\mathcal{L}^{-1}\left[\ln\left[\frac{1+s^2}{s(s+1)}\right]\right].$$

Use a detailed method, describing briefly each stage in the workings.

$$\boxed{f(t)=\frac{1}{t}\left[1+\mathrm{e}^{-t}-2\cos t\right]}$$



Question 12

The function $y = f(t)$, $t \geq 0$ satisfies

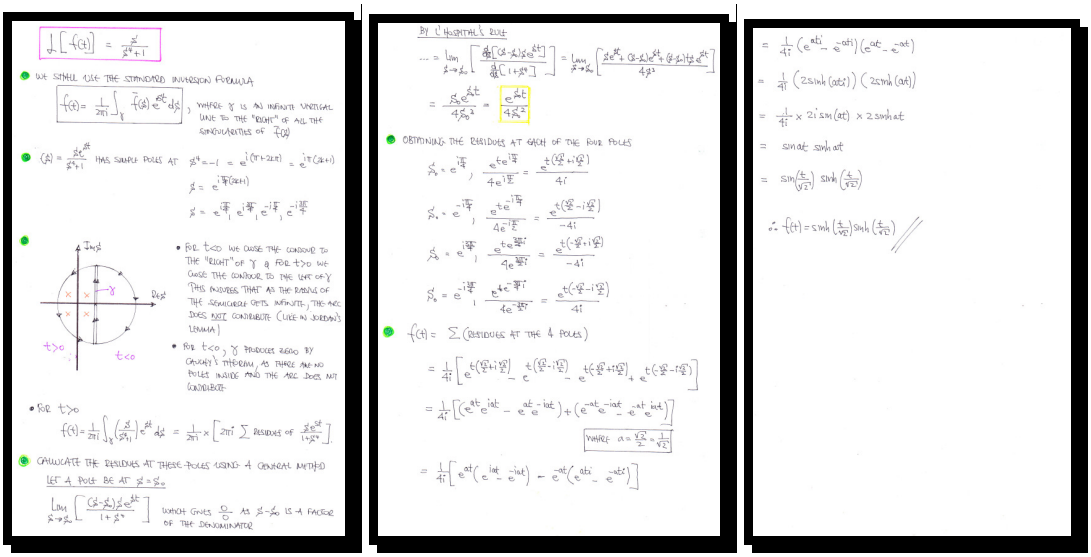
$$\mathcal{L}\left[f(t)\right] = \frac{s}{s^4+1}.$$

Use complex variable methods to show that

$$f(t) = \sin\left(\frac{t}{\sqrt{2}}\right)\sinh\left(\frac{t}{\sqrt{2}}\right).$$

Use a detailed method, describing briefly each stage in the workings.

proof



Question 13

The Bromwich integral for inverting a Laplace transform $\overline{f}(s)$ is given by

$$f(t)=\frac{1}{2\pi \mathrm{i}}\int_{\gamma} \mathrm{e}^{st}\,\overline{f}(s)\,ds\,.$$

a) Describe briefly the contour used in this integral and the general method used to invert the transform.

b) Given that a is a positive constant, show that

$$\mathcal{L}^{-1}\left[\mathrm{e}^{-a\sqrt{s}}\right]=\frac{a}{2t^{\frac{3}{2}}\sqrt{\pi}}\exp\left(-\frac{a^2}{4t}\right).$$

c) Hence find in a simplified form of a convolution integral the following inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{\mathrm{e}^{-a\sqrt{s}}}{\sqrt{s}}\right].$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{\mathrm{e}^{-a\sqrt{s}}}{\sqrt{s}}\right]=\frac{a}{2\pi}\int_0^{\infty}\left[\frac{1}{u^{\frac{3}{2}}\sqrt{t=u}}\right]\exp\left(-\frac{a^2}{4u}\right)\,du}$$

[solution overleaf]

