LAPLACE TRANSFORMS APPLICATIONS

SUMMARY OF THE LAPLACE TRANFORM

The Laplace Transform of a function $f(t)$, $t \geq 0$ is defined as

$$\mathcal{L}\left[f(t)\right] \equiv \overline{f}(s) \equiv \int_0^{\infty} \mathrm{e}^{-st} f(t)\ dt,$$

where $s \in \mathbb{C}$, with $\mathrm{Re}(s)$ sufficiently large for the integral to converge.

The Laplace Transform is a linear operation

$$\mathcal{L}\left[a f(t) + b g(t)\right] \equiv a\mathcal{L}\left[f(t)\right] + b\mathcal{L}\left[g(t)\right].$$

Laplace Transforms of Common Functions

- $\mathcal{L}\left(t^n\right) = \dfrac{n}{s^{n+1}}$

$$\mathcal{L}(1) = \frac{1}{s}, \quad \mathcal{L}(a) = \frac{a}{s}, \quad \mathcal{L}(t) = \frac{1}{s^2}, \quad \mathcal{L}\left(t^2\right) = \frac{2}{s^3}, \quad \mathcal{L}\left(t^3\right) = \frac{3}{s^4}, \ldots.$$

- $\mathcal{L}\left(\mathrm{e}^{at}\right) = \dfrac{1}{s-a}$, $\mathcal{L}\left(\mathrm{e}^{-at}\right) = \dfrac{1}{s+a}$
- $\mathcal{L}(\cos at) = \dfrac{s}{s^2+a^2}$, $\mathcal{L}(\sin at) = \dfrac{a}{s^2+a^2}$
- $\mathcal{L}(\cosh at) = \dfrac{s}{s^2-a^2}$, $\mathcal{L}(\sinh at) = \dfrac{a}{s^2-a^2}$

Laplace Transforms of Derivatives

- $\mathcal{L}\left[x(t)\right] = \overline{x}(t)$
- $\mathcal{L}\left[\dot{x}(t)\right] = s\overline{x}(t) - x(0)$
- $\mathcal{L}\left[\ddot{x}(t)\right] = s^2\overline{x}(t) - sx(0) - \dot{x}(0)$
- $\mathcal{L}\left[\dddot{x}(t)\right] = s^3\overline{x}(t) - s^2x(0) - s\dot{x}(0) - \ddot{x}(0)$

Laplace Transforms Theorems

- 1st Shift Theorem

$$\mathcal{L}\left[\mathrm{e}^{-at} f(t)\right] = \overline{f}(s+a) \quad \text{or} \quad \mathcal{L}\left[\mathrm{e}^{at} F(t)\right] = \overline{f}(s-a)$$

- 2nd Shift Theorem

$$\mathcal{L}\left[f(t-a)\right] = \mathrm{e}^{-as}\,\overline{f}(s),\ t>a \quad \text{or} \quad \mathcal{L}\left[f(t+a)\right] = \mathrm{e}^{as}\,\overline{f}(s),\ t>-a\,.$$

$$\mathcal{L}\left[\mathrm{H}(t-a)\,f(t-a)\right] = \mathrm{e}^{-as}\,\overline{f}(s) \quad \text{or} \quad \mathcal{L}\left[\mathrm{H}(t+a)\,f(t+a)\right] = \mathrm{e}^{as}\,\overline{f}(s)$$

- Multiplication by t^n

$$\mathcal{L}\left[t^n f(t)\right] = \left(-\frac{d}{ds}\right)^n \left[\overline{f}(s)\right] \quad \text{or} \quad \mathcal{L}\left[t\,f(t)\right] = -\frac{d}{ds}\left[\overline{f}(s)\right]$$

- Division by t

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \overline{f}(\sigma)\ d\sigma$$

provided that $\lim\limits_{t\to 0}\left(\dfrac{f(t)}{t}\right)$ exists and the integral converges.

- Initial/Final value theorem

$$\lim_{t\to 0}\left[f(t)\right] = \lim_{s\to\infty}\left[s\,\overline{f}(s)\right] \quad \text{and} \quad \lim_{t\to\infty}\left[f(t)\right] = \lim_{s\to 0}\left[s\,\overline{f}(s)\right]$$

The Impulse Function / The Dirac Function

1. $\delta(t-c)=\begin{cases}\infty & t=c\\ 0 & t\neq c\end{cases}$, $\delta(t)=\begin{cases}\infty & t=0\\ 0 & t\neq 0\end{cases}$

2. $\displaystyle\int_a^b \delta(t-c)\ dt=\begin{cases}1 & a\le c\le b\\ 0 & \text{otherwise}\end{cases}$

3. $\displaystyle\int_a^b f(t)\delta(t-c)\ dt=\begin{cases} f(a) & a\le c\le b\\ 0 & \text{otherwise}\end{cases}$

4. $\mathcal{L}\left[\delta(t-c)\right]=\mathrm{e}^{-cs}$

5. $\mathcal{L}\left[f(t)\delta(t-c)\right]=f(c)\mathrm{e}^{-cs}$

6. $\dfrac{d}{dt}\left[\mathrm{H}(t-c)\right]=\delta(t-c)$

SOLVING SIMPLE O.D.E.s

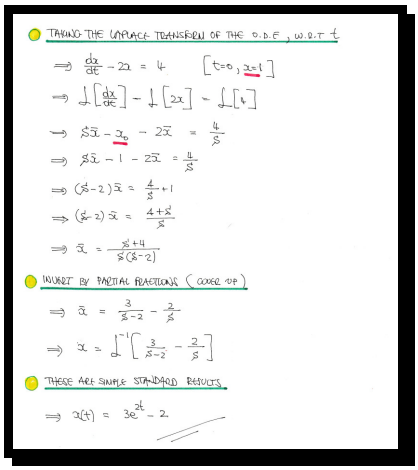
Question 1

Use Laplace transforms to solve the differential equation

$$\frac{dx}{dt}-2x=4\,,\ t\geq 0\,,$$

subject to the initial condition $x=1$ at $t=0$.

$\boxed{\phantom{x=3e^{2t}-2}}$, $\boxed{x=3\mathrm{e}^{2t}-2}$



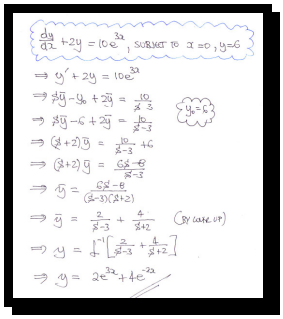
Question 2

Use Laplace transforms to solve the differential equation

$$\frac{dy}{dx}+2y=10\mathrm{e}^{3x}\,,\ x\geq 0\,,$$

subject to the boundary condition $y=6$ at $x=0$.

$\boxed{y=2\mathrm{e}^{3x}+4\mathrm{e}^{-2x}}$



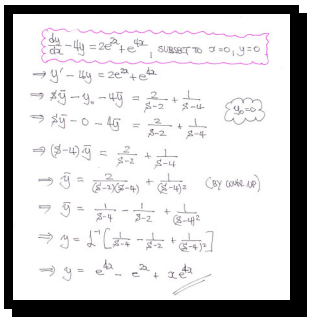
Question 3

Use Laplace transforms to solve the differential equation

$$\frac{dy}{dx}-4y=2\mathrm{e}^{2x}+\mathrm{e}^{4x},\ x\geq 0,$$

subject to the boundary condition $y=0$ at $x=0$.

$$\boxed{y=x\mathrm{e}^{4x}+\mathrm{e}^{4x}-2\mathrm{e}^{2x}}$$



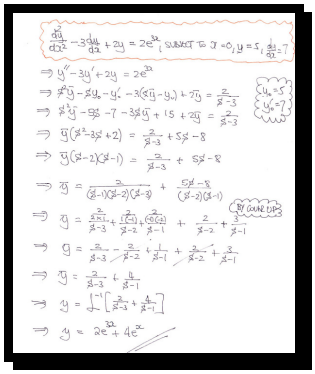
Question 4

Use Laplace transforms to solve the differential equation

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=2\mathrm{e}^{3x},\ x\geq 0,$$

subject to the boundary conditions $y=5$, $\frac{dy}{dx}=7$ at $x=0$.

$$\boxed{y=2\mathrm{e}^{3x}+4\mathrm{e}^{x}}$$



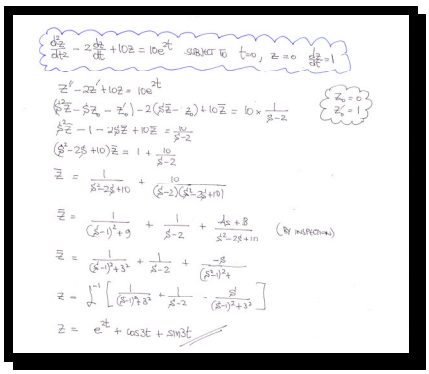
Question 5

Use Laplace transforms to solve the differential equation

$$\frac{d^2z}{dt^2}-2\frac{dz}{dt}+10z=10\mathrm{e}^{2t},$$

subject to the initial conditions $z=0$, $\dfrac{dz}{dt}=1$ at $t=0$.

$$\boxed{y=\mathrm{e}^{2t}+\cos 3t+\sin 3t}$$



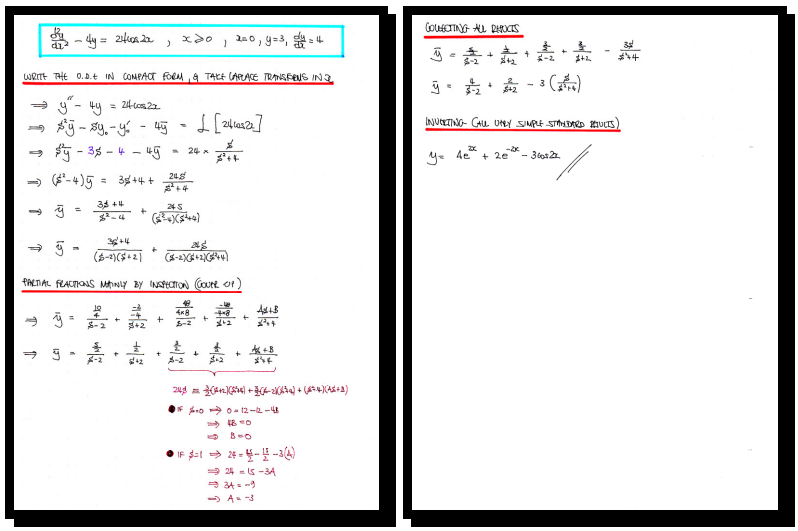
Question 6

Use Laplace transforms to solve the differential equation

$$\frac{d^2y}{dx^2}-4y=24\cos 2x\,,\ x\geq 0\,,$$

subject to the boundary conditions $y=3\,,\ \dfrac{dy}{dx}=4$ at $x=0\,.$

$\boxed{}\,,\ \boxed{y=4\mathrm{e}^{2x}+2\mathrm{e}^{-2x}-3\cos 2x}$



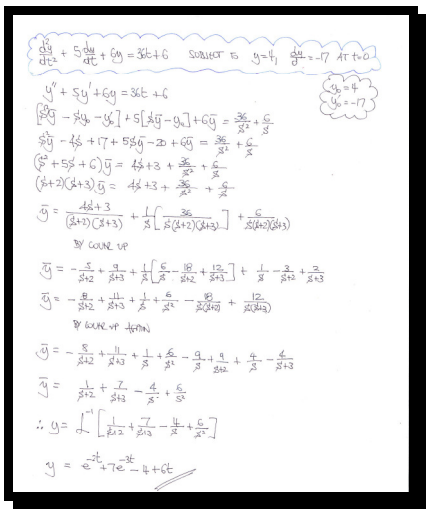
Question 7

Use Laplace transforms to solve the differential equation

$$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=36t+6,$$

subject to the initial conditions $y=4$, $\dfrac{dy}{dt}=-17$ at $t=0$.

$$\boxed{y=\mathrm{e}^{-2t}+7\mathrm{e}^{-3t}+6t-4}$$

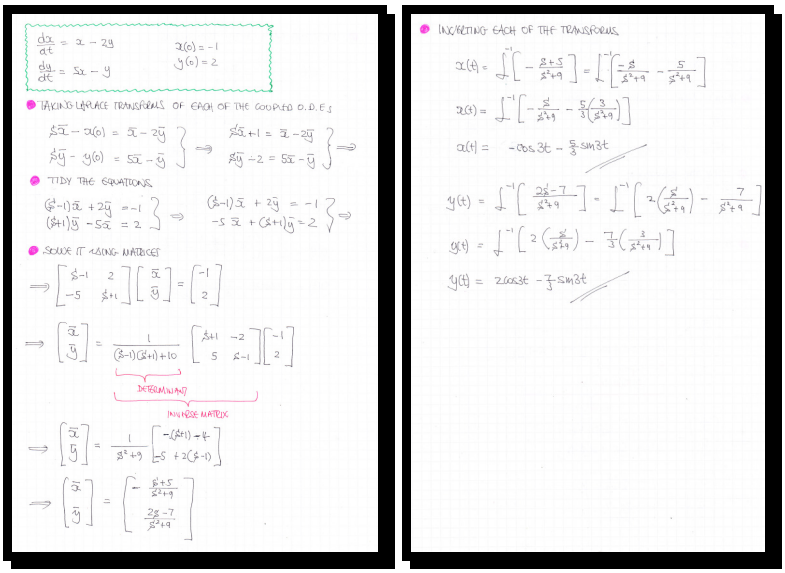


Question 8

By using Laplace transforms, or otherwise, solve the following simultaneous differential equations, subject to the initial conditions $x=-1$, $y=2$ at $t=0$.

$$\frac{dx}{dt}=x-2y \qquad \text{and} \qquad \frac{dy}{dt}=5x-y\,.$$

$$\boxed{x=-\cos 3t-\frac{5}{3}\sin 3t,\quad y=2\cos 3t-\frac{7}{3}\sin 3t}$$

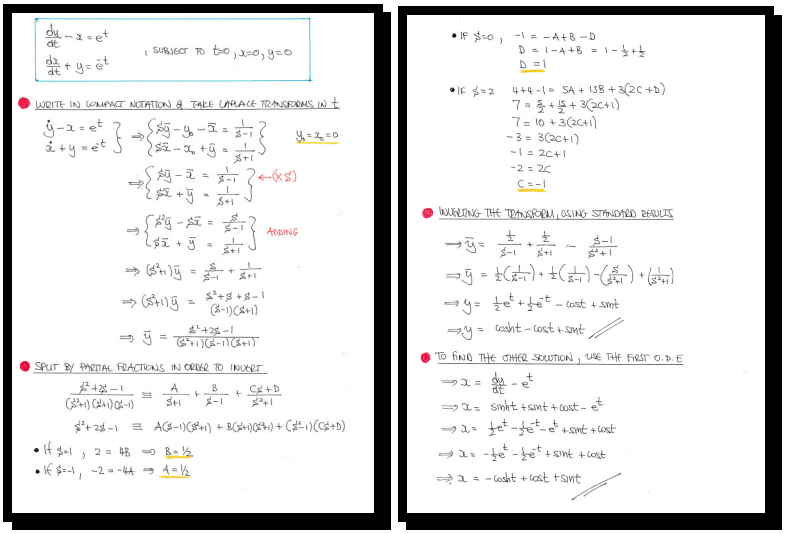


Question 9

$$\frac{dx}{dt}+y=\mathrm{e}^{-t} \qquad \text{and} \qquad \frac{dy}{dt}-x=\mathrm{e}^{t}\,.$$

Use Laplace transformations to solve the above simultaneous differential equations, subject to the initial conditions $x=0$, $y=0$ at $t=0$.

$\boxed{}$, $\boxed{x=-\cosh t+\sin t+\cos t,\quad y=\cosh t+\sin t-\cos t}$

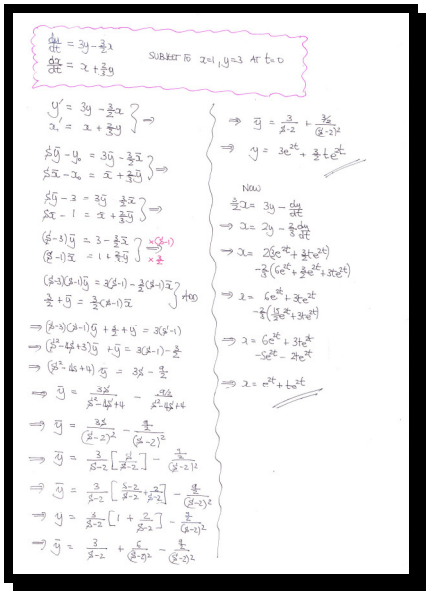


Question 10

$$\frac{dx}{dt}=x+\frac{2}{3}y \qquad \text{and} \qquad \frac{dy}{dt}=3y-\frac{3}{2}x\,.$$

Use Laplace transformations to solve the above simultaneous differential equations, subject to the initial conditions $x=1$, $y=3$ at $t=0$.

$$\boxed{x=\mathrm{e}^{2t}+t\mathrm{e}^{2t},\quad y=3\mathrm{e}^{2t}+\frac{3}{2}t\mathrm{e}^{2t}}$$

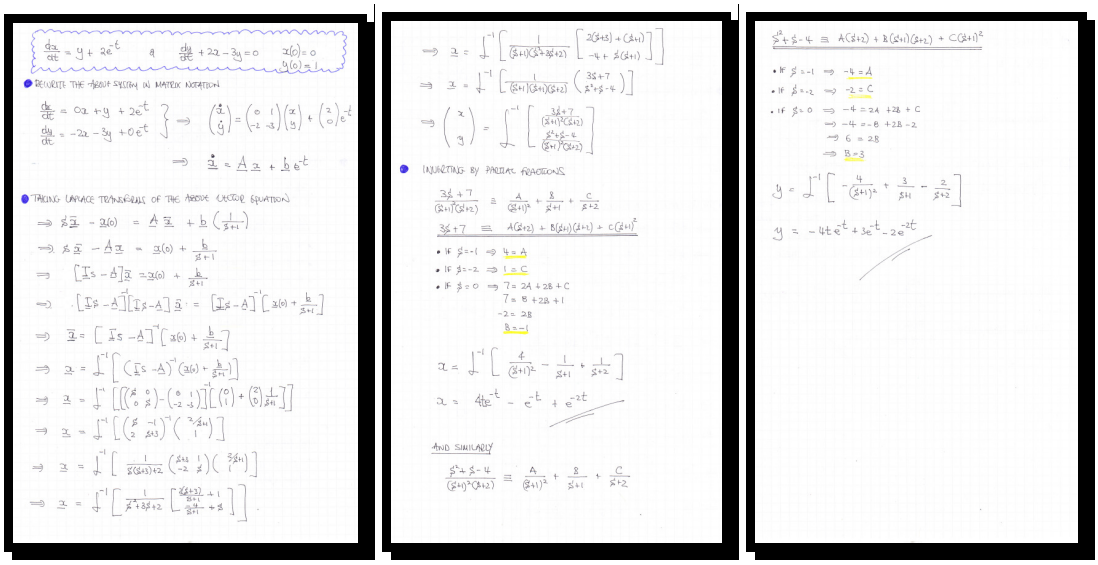


Question 11

$$\frac{dx}{dt} = y + 2\mathrm{e}^{-t} \qquad \text{and} \qquad \frac{dy}{dt} + 2x - 3y = 0 .$$

Use Laplace transformations to solve the above simultaneous differential equations, subject to the initial conditions $x = 0$, $y = 1$ at $t = 0$.

$$\boxed{x = 4t\,\mathrm{e}^{-t} - \mathrm{e}^{-t} + \mathrm{e}^{-2t}, \quad y = -4t\,\mathrm{e}^{-t} + 3\mathrm{e}^{-t} - 2\mathrm{e}^{-2t}}$$



Question 12

$$\frac{d^2x}{dt^2}=15\frac{dy}{dt}-9y+22\mathrm{e}^{t} \qquad \text{and} \qquad \frac{d^2y}{dt^2}=2x+\mathrm{e}^{3t}.$$

The functions $x=f(t)$ and $y=g(t)$ satisfy the above simultaneous differential equations, subject to the initial conditions

$$x=2,\quad y=-3,\quad \frac{dx}{dt}=10,\quad \frac{dy}{dt}=-1 \quad \text{at} \quad t=0.$$

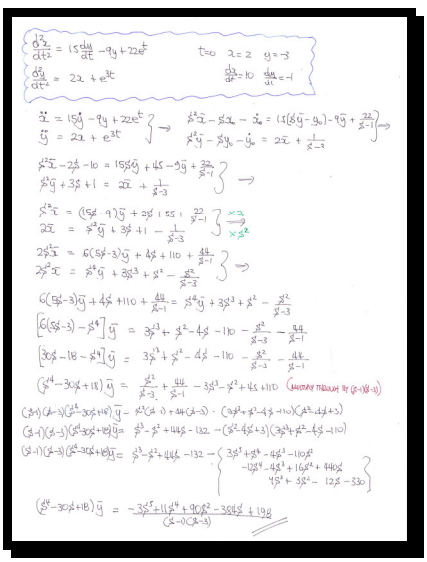
a) By using Laplace transforms, show that

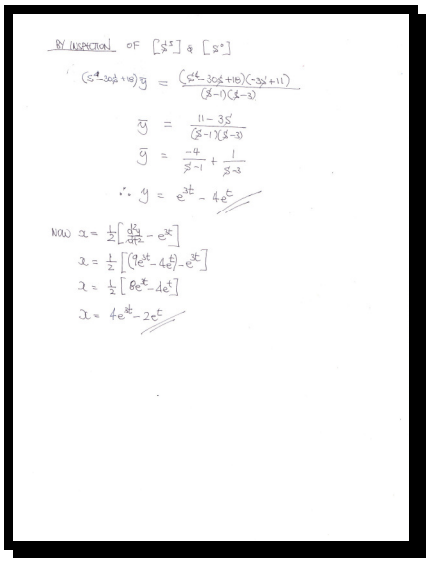
$$\left(s^4-30s+18\right)\overline{y}=\frac{-3s^5+11s^4+90s^2-384s+198}{(s-1)(s-3)},$$

where $\overline{y}=\mathcal{L}\left[g(t)\right]$.

b) Given further that $s^4-30s+18$ is a factor of $-3s^5+11s^4+90s^2-384s+198$, find expressions for x and y.

$$\boxed{x=4\mathrm{e}^{3t}-2\mathrm{e}^{t},\quad y=\mathrm{e}^{3t}-4\mathrm{e}^{t}}$$





Question 13

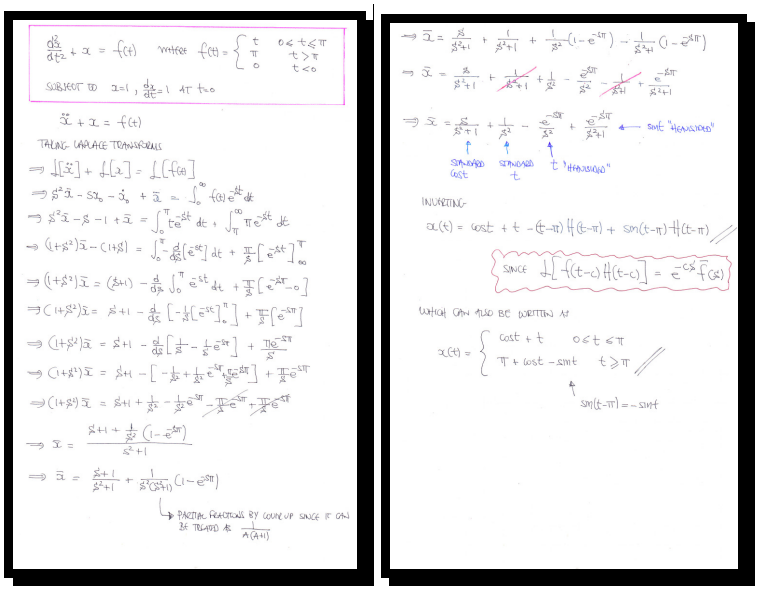
Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2}+x=f\left(t\right),$$

given further that $x=1$, $\dfrac{dx}{dt}=1$ at $t=0$, and

$$f\left(t\right)=\begin{cases}0 & t<0\\ t & 0\le t\le \pi\\ \pi & t>\pi\end{cases}$$

$$\boxed{x=t+\cos t-\left(t-\pi\right)\mathrm{H}\left(t-\pi\right)+\sin\left(t-\pi\right)\mathrm{H}\left(t-\pi\right)}$$



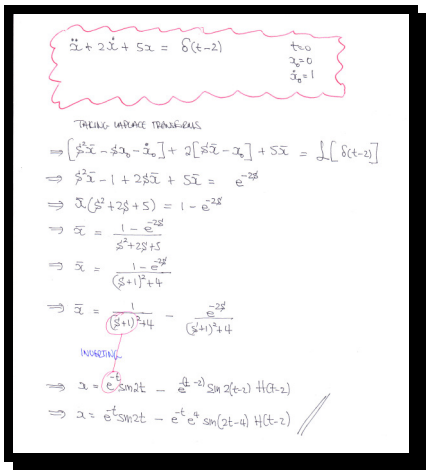
Question 14

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=\delta(t-2),$$

given further that $x=0$, $\dfrac{dx}{dt}=1$ at $t=0$.

$$\boxed{x=\mathrm{e}^{-t}\left[\sin 2t-\mathrm{e}^{4}\sin(2t-4)\mathrm{H}(t-2)\right]}$$



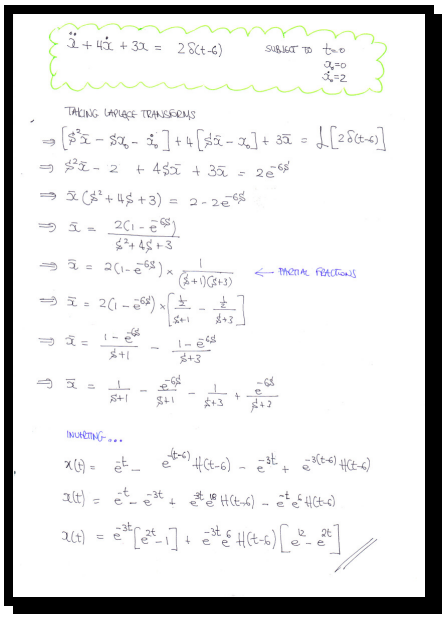
Question 15

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+3x=2\delta(t-6),$$

given further that $x=0$, $\frac{dx}{dt}=2$ at $t=0$.

$$\boxed{x=\mathrm{e}^{-3t}\left[\mathrm{e}^{2t}-1\right]+\mathrm{e}^{-3t}\,\mathrm{e}^{6}\left[\mathrm{e}^{12}-\mathrm{e}^{2t}\right]\mathrm{H}(t-6)}$$



Question 16

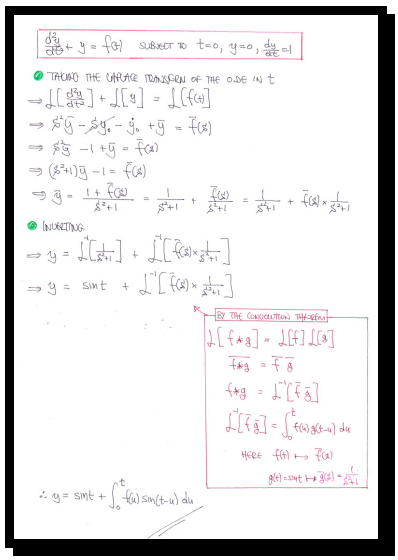
Use Laplace transforms to solve the differential equation

$$\frac{d^2y}{dt^2}+y=f\left(t\right),$$

given further that $y=0$, $\frac{dy}{dt}=1$ at $t=0$, and $f\left(t\right)$ is a known function which has a Laplace transform.

You may leave the final answer containing a convolution type integral.

$$\boxed{y=\sin t+\int_0^t f\left(u\right)\sin\left(t-u\right)\,du}$$



Question 17

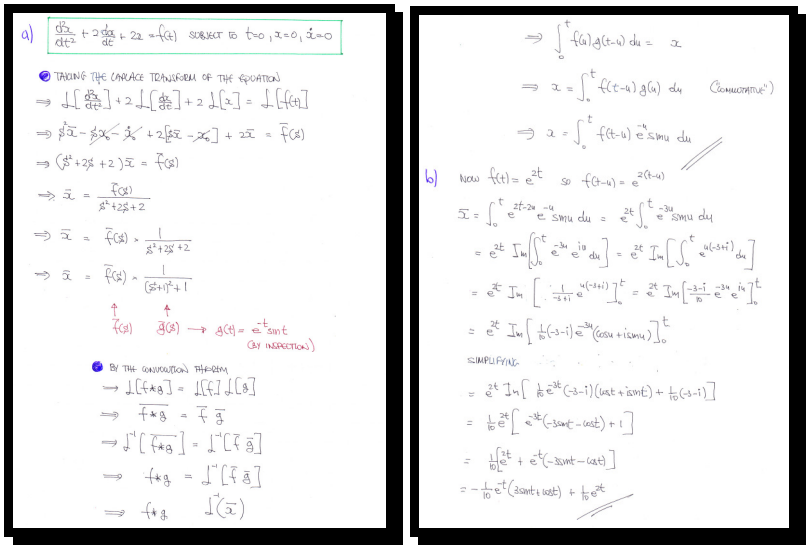
$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+2x=f(t).$$

a) Use Laplace transforms to solve the above differential equation, given further that $x=0$, $\frac{dx}{dt}=0$ at $t=0$, and $f(t)$ is a known function which has a Laplace transform.

You may leave the answer containing a convolution type integral.

b) If $f(t)=\mathrm{e}^{2t}$ find $x=x(t)$ explicitly.

$$\boxed{x=\int_0^t f(t-u)\,\mathrm{e}^{-u}\sin u\ du}\ ,\qquad \boxed{x=-\frac{1}{10}\mathrm{e}^{-t}\left[3\sin t+\cos t\right]+\frac{1}{10}\mathrm{e}^{2t}}$$



Question 18

Use Laplace transforms to solve the differential equation

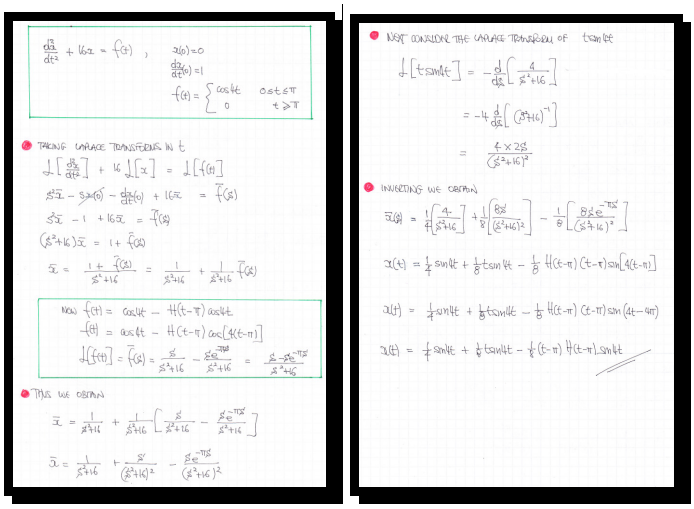
$$\frac{d^2x}{dt^2}+16x=f\left(t\right),$$

given further that $x=0$, $\frac{dx}{dt}=1$ at $t=0$, and

$$f\left(t\right)=\begin{cases}\cos 4t & 0\leqslant t\leqslant \pi \\ 0 & t>\pi \end{cases}$$

[*You may find the Laplace transform of* $t\sin 4t$ *useful in this question.*]

$$\boxed{x\left(t\right)=\frac{1}{4}\sin 4t+\frac{1}{8}t\sin 4t-\frac{1}{8}\left(t-\pi\right)\mathrm{H}\left(t-\pi\right)\sin 4t}$$



Question 19

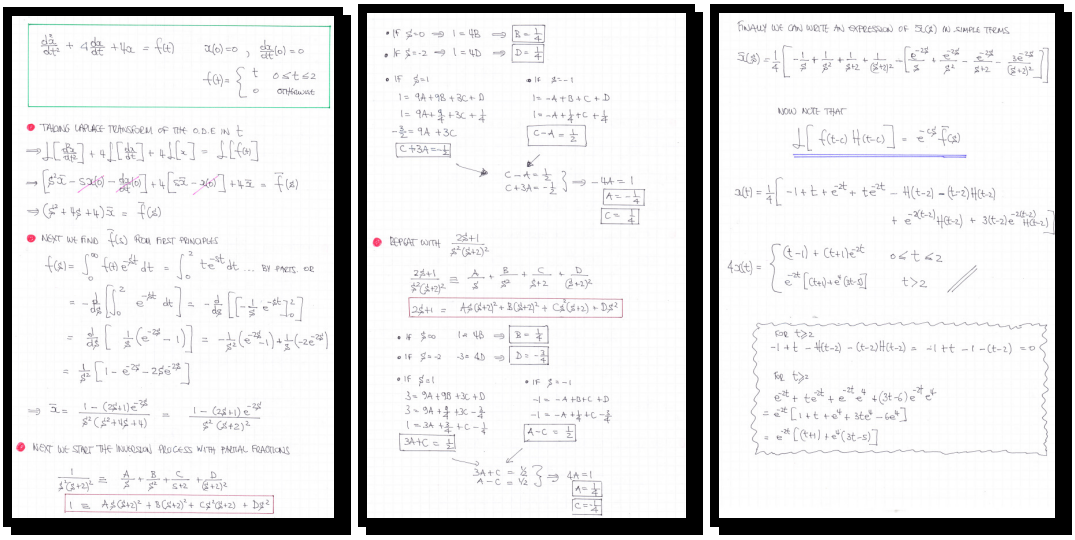
Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+4x=f(t),$$

given further that $x=0$, $\frac{dx}{dt}=0$ at $t=0$, and

$$f(t)=\begin{cases}0 & t<0\\ t & 0\le t\le 2\\ 0 & t>2\end{cases}$$

$$\boxed{x(t)=\begin{cases}0 & t<0\\ t-1+(t+1)\mathrm{e}^{-2t} & 0\le t\le 2\\ \mathrm{e}^{-2t}\left[t+1+\mathrm{e}^{4}(3t-5)\right] & t>2\end{cases}}$$



EVALUATION OF INTEGRALS

Question 1

$$\int_0^{\infty} t\,\mathrm{e}^{-2t}\cos t \ \ dt\,.$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\boxed{\frac{3}{25}}$$

$\int_0^{\infty} te^{-2t}\cos t\,dt = \frac{3}{25}$

- CONSIDER THE LAPLACE TRANSFORM OF $t\cos t$

$\Rightarrow \mathcal{L}[t\cos t] = -\frac{d}{ds}\left[\mathcal{L}[\cos t]\right]$

$\Rightarrow \int_0^{\infty} (t\cos t)e^{-st}\,dt = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]$

$\Rightarrow \int_0^{\infty} te^{-st}\cos t\,dt = -\frac{(s^2+1)\times 1 - s(2s)}{(s^2+1)^2}$

$\Rightarrow \int_0^{\infty} te^{-st}\cos t\,dt = -\frac{s^2+1-2s^2}{(s^2+1)^2}$

$\Rightarrow \int_0^{\infty} te^{-st}\cos t\,dt = -\frac{1-s^2}{(1+s^2)^2}$

$\Rightarrow \int_0^{\infty} te^{-st}\cos t\,dt = \frac{s^2-1}{(s^2+1)^2}$

- SET $s=2$

$\Rightarrow \int_0^{\infty} te^{-2t}\cos t\,dt = \frac{2^2-1}{(2^2+1)^2}$

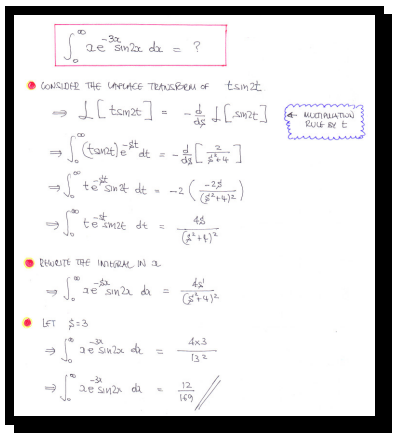
$\Rightarrow \int_0^{\infty} te^{-2t}\cos t\,dt = \frac{3}{25}$

Question 2

$$\int_0^{\infty} x\mathrm{e}^{-3x}\sin 2x \ \ dx .$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\frac{12}{169}$$

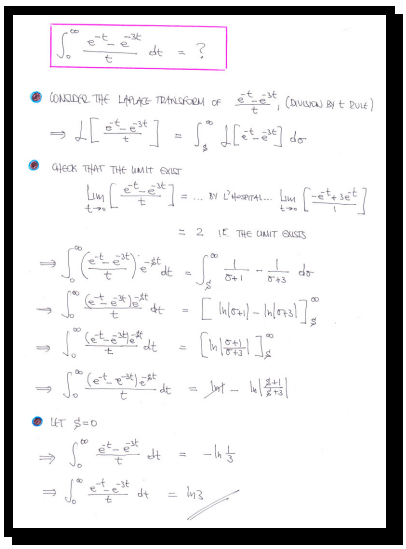


Question 3

$$\int_0^{\infty} \frac{\mathrm{e}^{-t}-\mathrm{e}^{-3t}}{t}\,dt\,.$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$\ln 3$

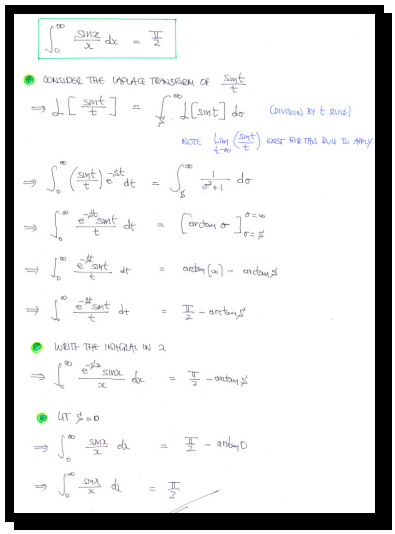


Question 4

$$\int_0^{\infty} \frac{\sin x}{x}\, dx\,.$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\boxed{\frac{\pi}{2}}$$

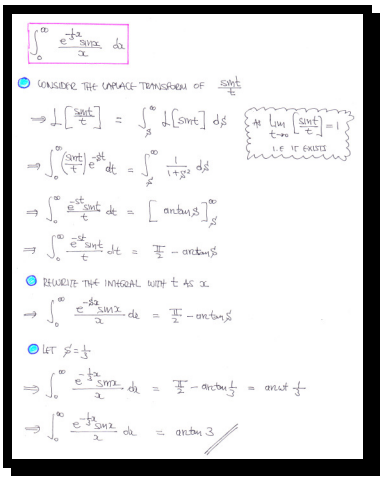


Question 5

$$\int_0^{\infty} \frac{\mathrm{e}^{-\frac{1}{3}x}\sin x}{x}\,dx\,.$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$\arctan 3$

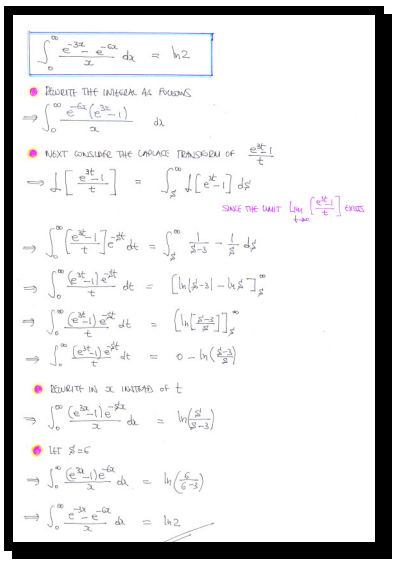


Question 6

$$\int_0^{\infty} \frac{\mathrm{e}^{-3x}-\mathrm{e}^{-6x}}{x}\,dx\,.$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$\ln 2$

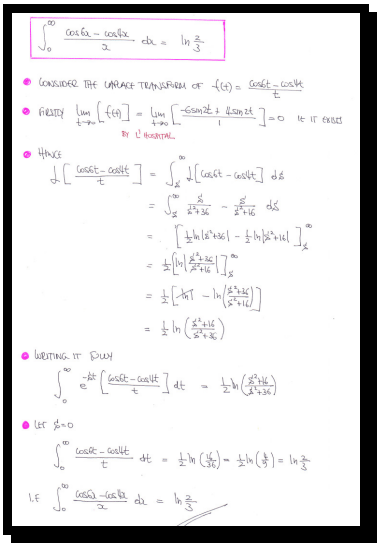


Question 7

$$\int_0^{\infty} \frac{\cos 6x - \cos 4x}{x} \, dx \, .$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\boxed{\ln \frac{2}{3}}$$

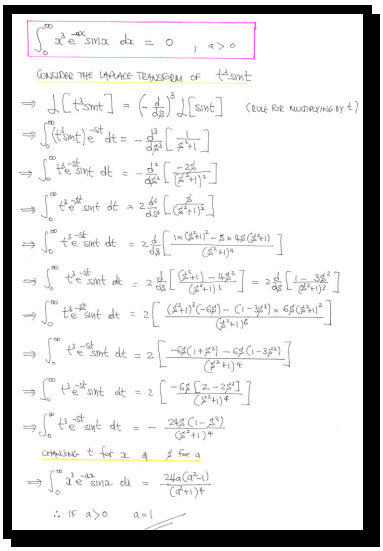


Question 8

$$\int_0^{\infty} x^3 \mathrm{e}^{-ax} \sin x \ dx \,, \ a>0 \,.$$

Given that the value of the above integral is zero, use Laplace transform techniques to find the value of a.

$$\boxed{a=1}$$



Question 9

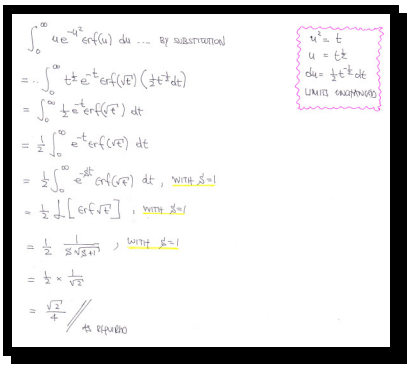
Use Laplace transforms techniques to show that

$$\int_0^{\infty} u\,\mathrm{e}^{-u^2}\,\mathrm{erf}\,(u)\ du\ =\ \frac{1}{4}\sqrt{2}\,.$$

You may assume that

$$\mathcal{L}\left[\,\mathrm{erf}\,\sqrt{t}\,\right]=\frac{1}{s\sqrt{s+1}}\,.$$

proof

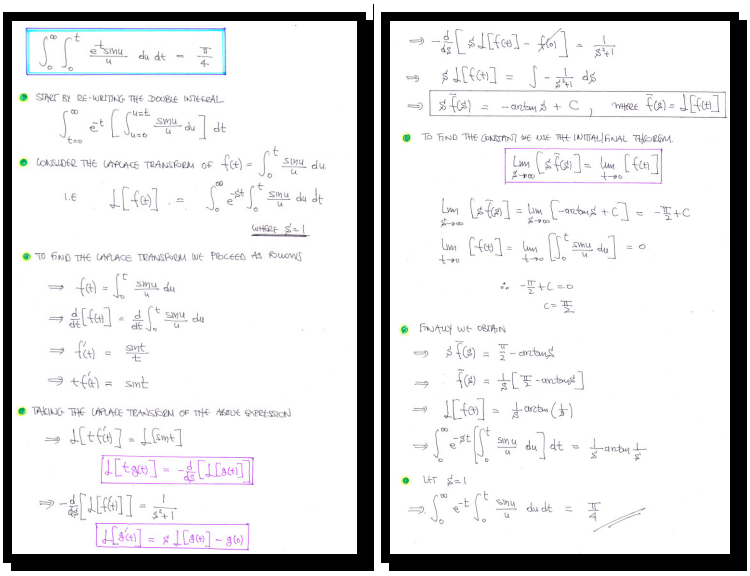


Question 10

$$\int_0^{\infty} \int_0^{t} \frac{\mathrm{e}^{-t} \sin u}{u} \, du \, dt \, .$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\frac{\pi}{4}$$

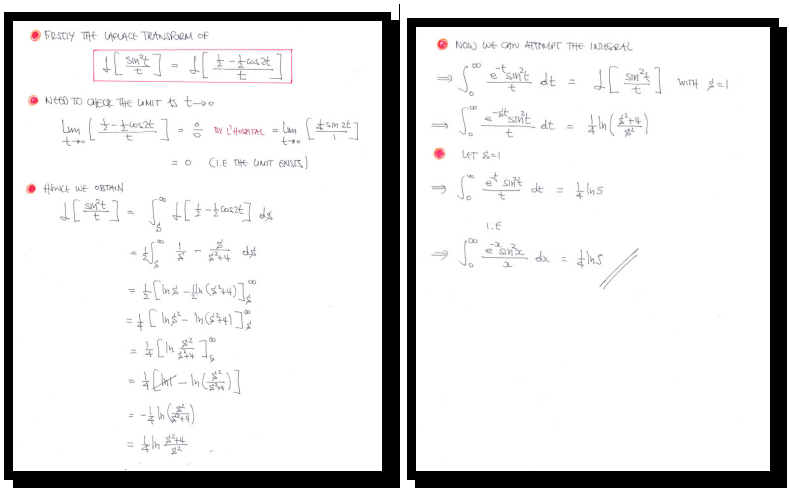


Question 11

$$\int_0^{\infty} \frac{\mathrm{e}^{-x}\sin^2 x}{x}\,dx\,.$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\frac{1}{4}\ln 5$$

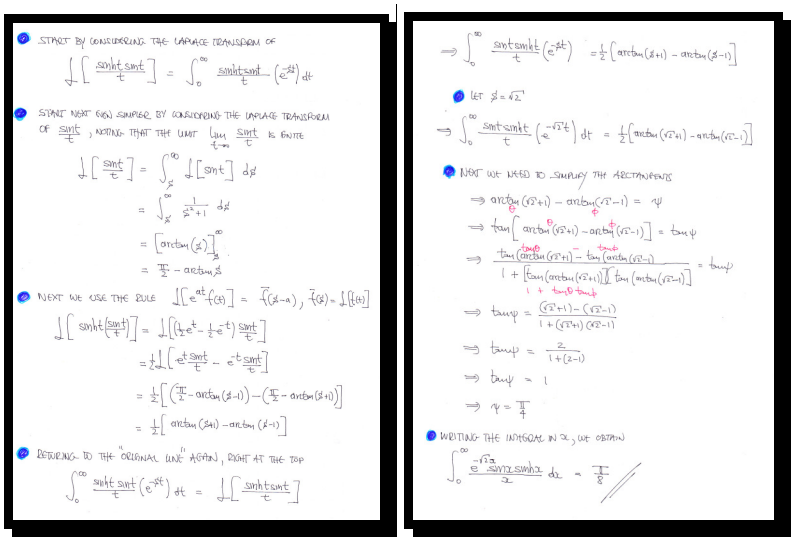


Question 12

$$\int_0^{\infty} \frac{\mathrm{e}^{-\sqrt{2}\,x}\,\sinh x\,\sin x}{x}\,dx\,.$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\boxed{\frac{\pi}{8}}$$



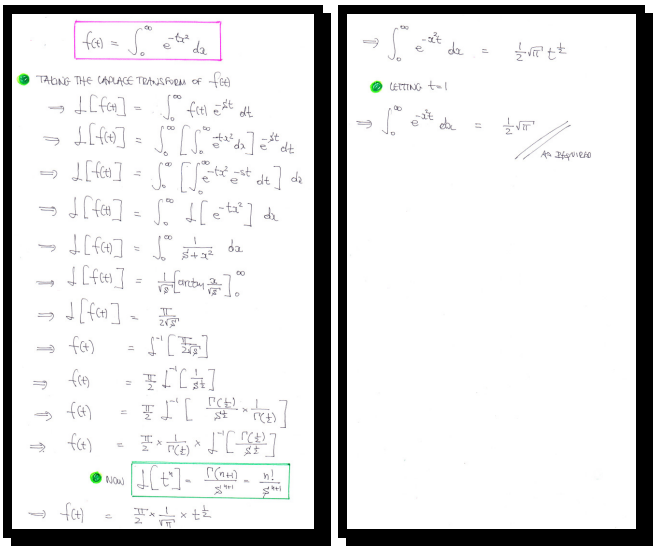
Question 13

$$f(t) \equiv \int_0^{\infty} \mathrm{e}^{-tx^2} \, dx \, .$$

By considering the Laplace transform of $f(t)$, show that

$$\int_0^{\infty} \mathrm{e}^{-x^2} \, dx = \frac{1}{2}\sqrt{\pi} \, .$$

proof



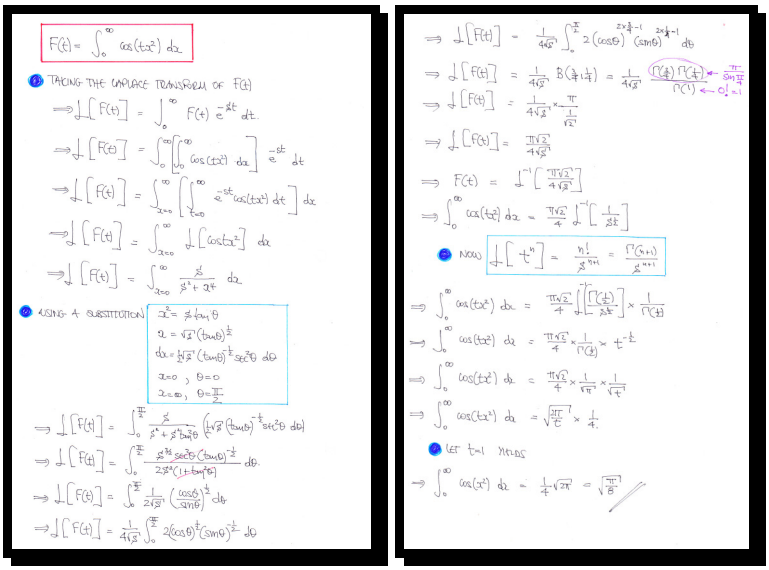
Question 14

$$F(t) \equiv \int_0^{\infty} \cos\left(tx^2\right)\, dx\,.$$

By considering the Laplace transform of $F(t)$, show that

$$\int_0^{\infty} \cos\left(x^2\right)\, dx = \sqrt{\frac{\pi}{8}}\,.$$

proof



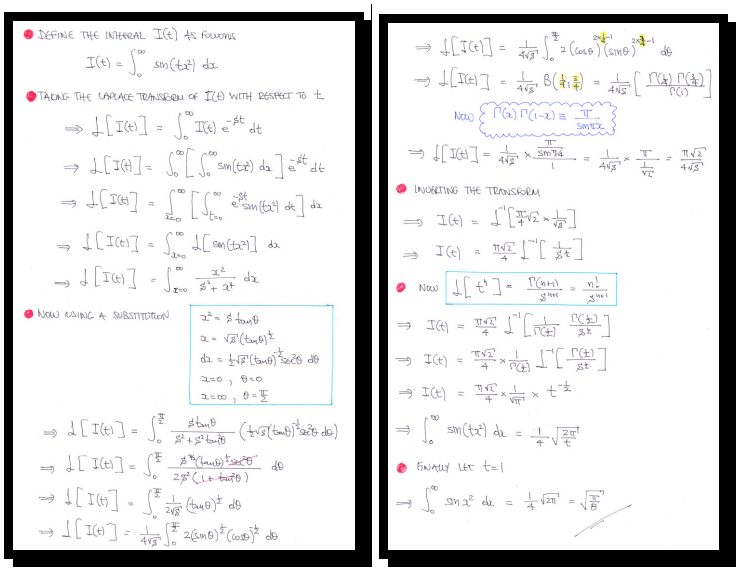
Question 15

$$I(t) \equiv \int_0^{\infty} \sin\left(tx^2\right)\, dx\,.$$

By considering the Laplace transform of $I(t)$, show that

$$\int_0^{\infty} \sin\left(x^2\right)\, dx = \sqrt{\frac{\pi}{8}}\,.$$

proof



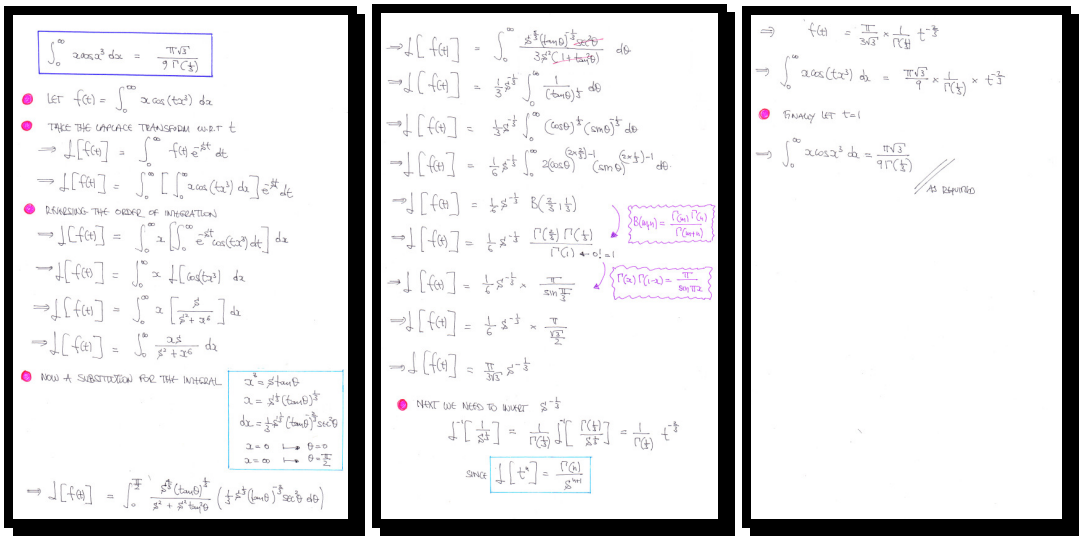
Question 16

$$f(t) \equiv \int_0^{\infty} x\cos\left(tx^3\right)\, dx\,.$$

By considering the Laplace transform of $f(t)$, show that

$$\int_0^{\infty} x\cos\left(x^3\right)\, dx = \frac{\pi\sqrt{3}}{9\Gamma\left(\frac{1}{3}\right)}\,.$$

proof



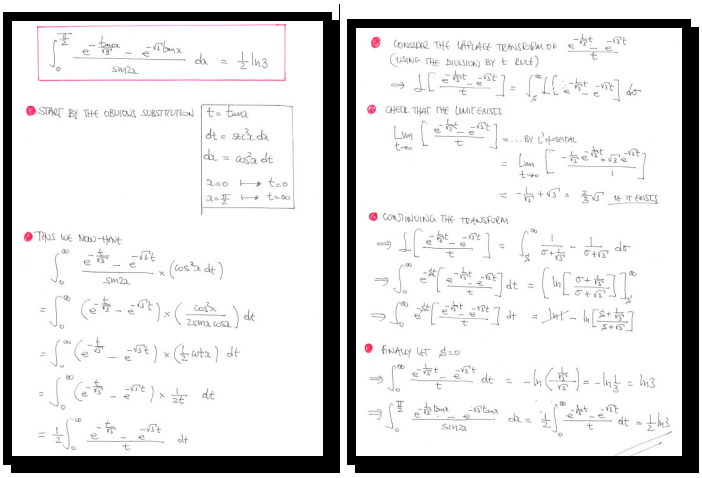
Question 17

$$I=\int_0^{\frac{1}{2}\pi}\frac{\exp\left(-\frac{1}{\sqrt{3}}\tan x\right)-\exp\left(-\sqrt{3}\tan x\right)}{\sin 2x}\,dx\,.$$

Use Laplace transforms to show that

$$I=\frac{1}{2}\ln 3\,.$$

proof

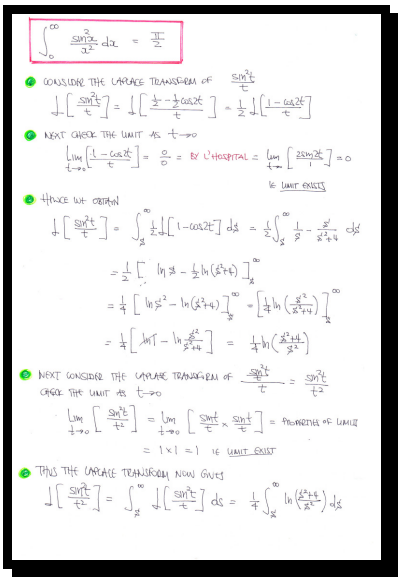


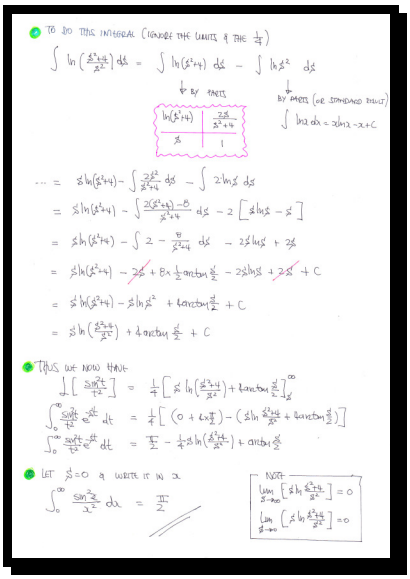
Question 18

$$\int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx\,.$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\boxed{\frac{\pi}{2}}$$





Question 19

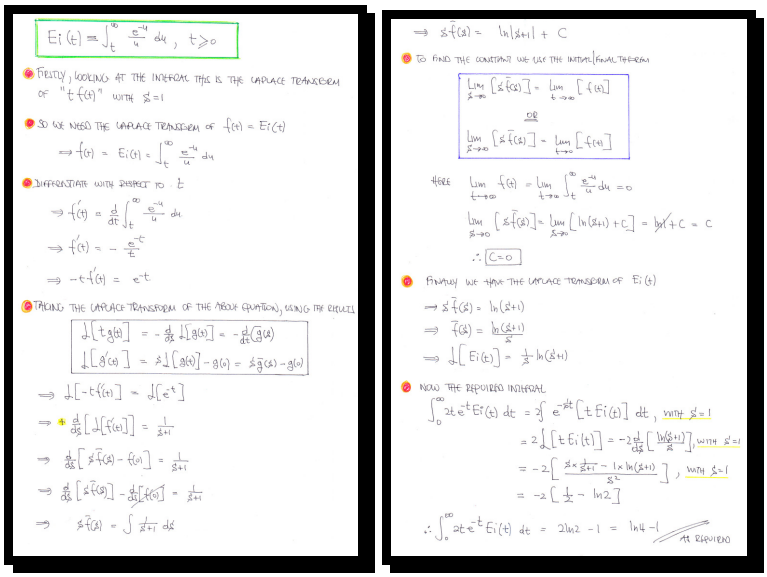
The Exponential integral function $\mathrm{Ei}(t)$ is defined as

$$\mathrm{Ei}(t) \equiv \int_t^{\infty} \frac{\mathrm{e}^{-u}}{u}\, du\,,\ t \geq 0\,.$$

By considering the Laplace transform of $\mathrm{Ei}(t)$, show that

$$\int_0^{\infty} 2t\,\mathrm{e}^{-t}\,\mathrm{Ei}(t)\ dt \ = \ \ln 4 - 1\,.$$

proof

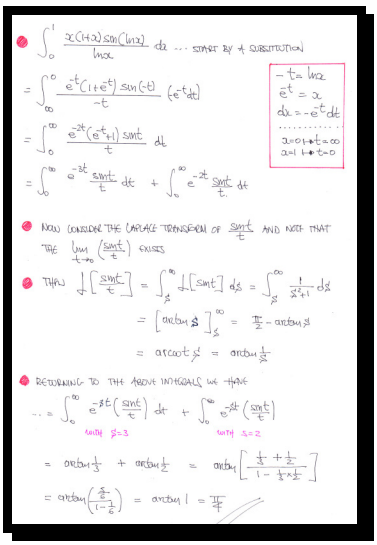


Question 20

$$\int_0^{\infty} \frac{x\left(1+x\right)\sin\left(\ln x\right)}{\ln x}\, dx\,.$$

Given that the above integral is finite, use Laplace transform techniques to find its exact value.

$$\frac{\pi}{4}$$



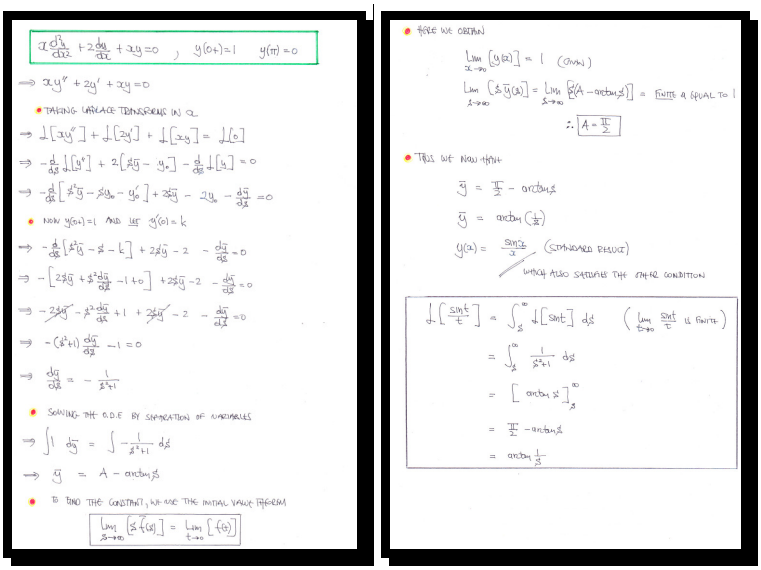
SOLVING O.D.E.s, P.D.E.s & INTEGRAL EQUATIONS

Question 1

Use Laplace transforms to solve the following differential equation

$$x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+xy=0, \quad y(0+)=1, \quad y(\pi)=0.$$

$$\boxed{y(x)=\frac{\sin x}{x}}$$

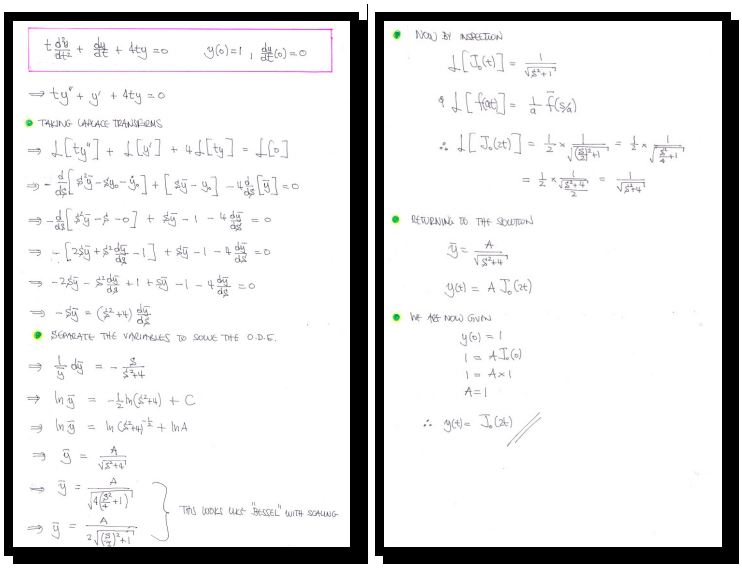


Question 2

Use Laplace transforms to solve the following differential equation

$$t\frac{d^2y}{dt^2}+\frac{dy}{dt}+4ty=0\,,\quad y(0)=1\,,\quad \frac{dy}{dt}(0)=0\,.$$

$$\boxed{y(t)=J_0(2t)}$$

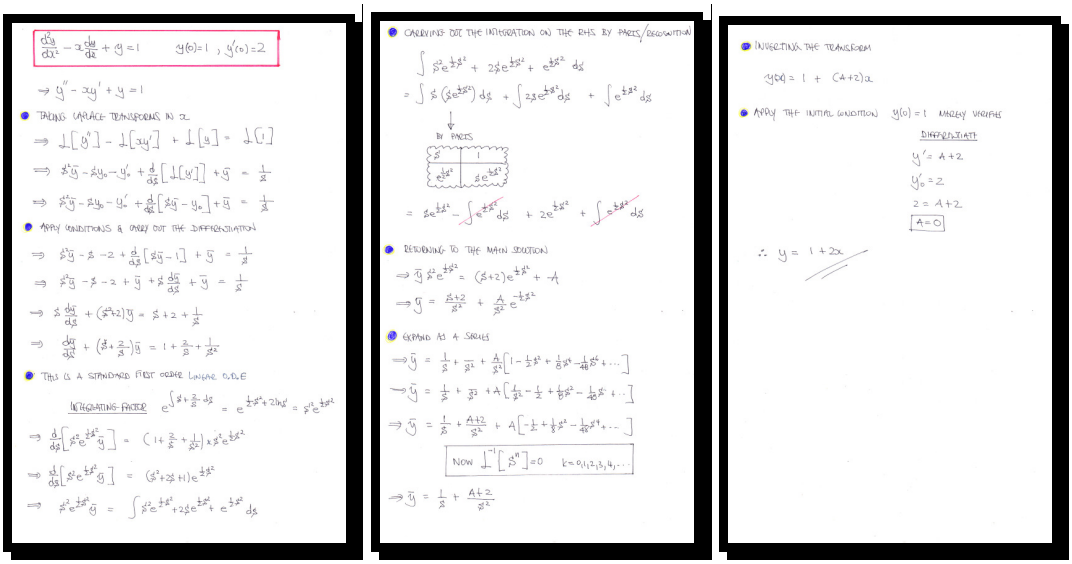


Question 3

Use Laplace transforms to solve the following differential equation

$$\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=1, \quad y(0)=1, \quad \frac{dy}{dx}(0)=2.$$

$$\boxed{y(x)=2x+1}$$



Question 4

The function $u = u(t, y)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} + y\frac{\partial u}{\partial y} = y\,, \quad t \geq 0\,, \quad y > 0\,,$$

subject to the following conditions

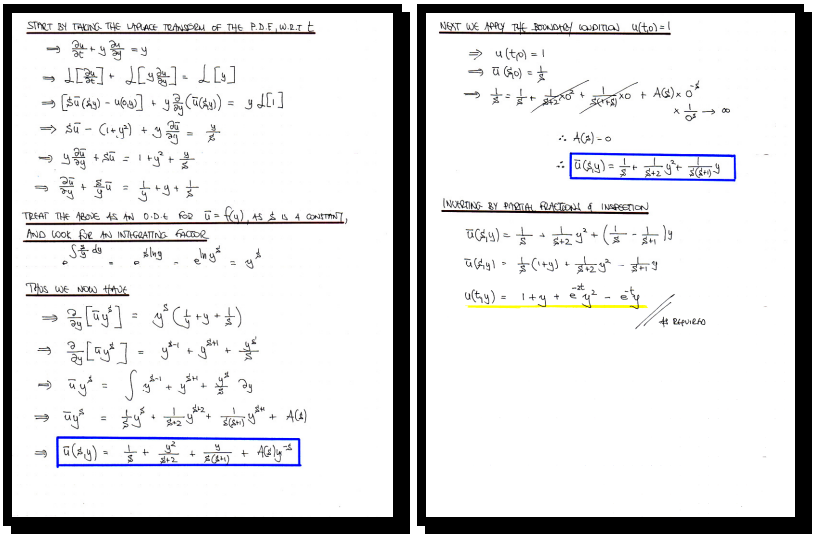
i. $u(0, y) = 1 + y^2\,,\ y > 0\,.$

ii. $u(t, 0) = 1\,,\ t \geq 0\,.$

Use Laplace transforms in t to show that

$$u(t, y) = 1 + y - y\,\mathrm{e}^{-t} + y^2\,\mathrm{e}^{-2t}\,.$$

[ ], [proof]



Question 5

The function $z = z(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial x} = 2\frac{\partial z}{\partial t} + z, \quad x \geq 0, \quad t \geq 0,$$

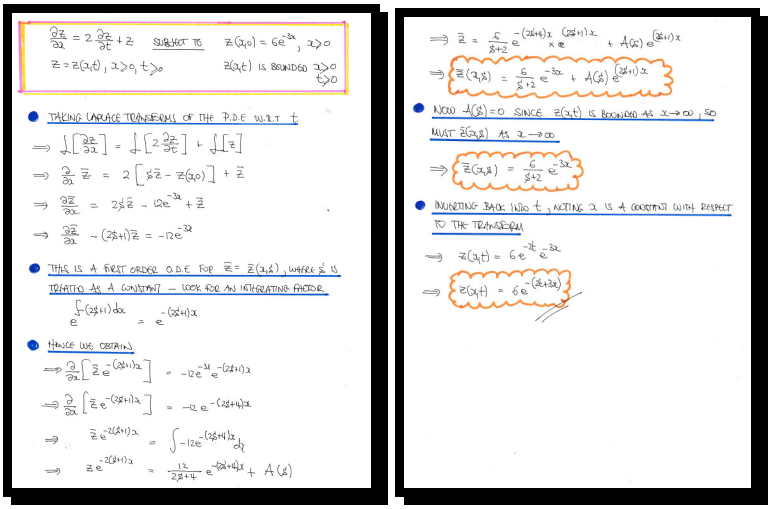
subject to the following conditions

i. $z(x,0) = 6\mathrm{e}^{-3x}$, $x > 0$.

ii. $z(x,t)$, is bounded for all $x \geq 0$ and $t \geq 0$.

Find the solution of partial differential equation by using Laplace transforms.

$\boxed{}$, $\boxed{z(x,t) = 6\mathrm{e}^{-(3x+2t)}}$



Question 6

$$\theta(x)=8\sin(2\pi x),\ 0\le x\le 1$$

The above equation represents the temperature distribution $\theta\,^{\circ}\mathrm{C}$, maintained along the 1 m length of a thin rod.

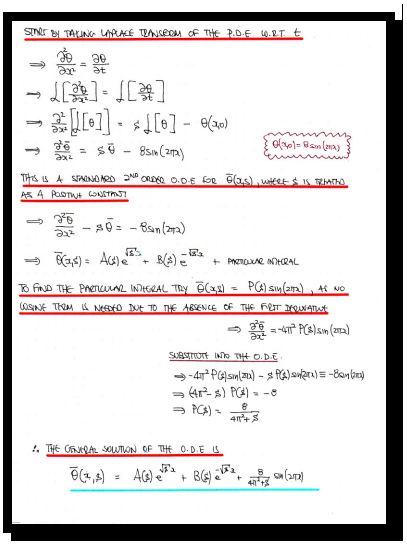
At time $t=0$, the temperature θ is suddenly dropped to $\theta=0\,^{\circ}\mathrm{C}$ at both the ends of the rod at $x=0$, and at $x=1$, and the source which was previously maintaining the temperature distribution is removed.

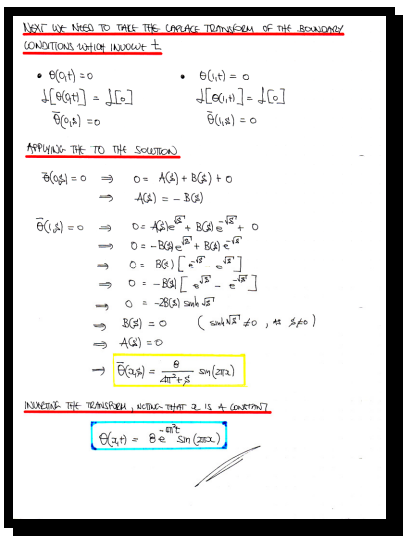
The new temperature distribution along the rod $\theta(x,t)$, satisfies the heat equation

$$\frac{\partial^2\theta}{\partial x^2}=\frac{\partial\theta}{\partial t},\quad 0\le x\le 1,\quad t\ge 0.$$

Use Laplace transforms to determine an expression for $\theta(x,t)$.

$$\boxed{\phantom{\theta(x,t)=8e^{-4\pi^2t}\sin(2\pi x)}},\quad \boxed{\theta(x,t)=8e^{-4\pi^2t}\sin(2\pi x)}$$





Question 7

The temperature $\theta(x,t)$ in a semi-infinite thin rod satisfies the heat equation

$$\frac{\partial^2\theta}{\partial x^2}=\frac{\partial\theta}{\partial t},\quad x\geq 0,\quad t\geq 0.$$

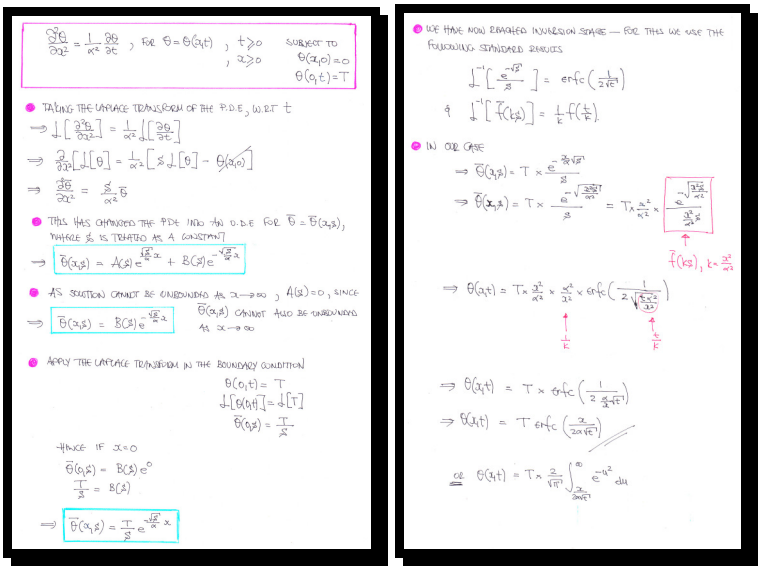
The initial temperature of the rod is $0\ ^\circ\text{C}$, and for $t>0$ the endpoint at $x=0$ is maintained at $T\ ^\circ\text{C}$.

Assuming the rod is insulated along its length, use Laplace transforms to find an expression for $\theta(x,t)$.

You may assume that

- $\mathcal{L}^{-1}\left[\dfrac{\mathrm{e}^{-\sqrt{s}}}{s}\right]=\operatorname{erfc}\left(\dfrac{1}{2\sqrt{t}}\right)$
- $\mathcal{L}^{-1}\left[\bar{f}\left(k\,s\right)\right]=\dfrac{1}{k}f\left(\dfrac{t}{k}\right)$, where k is a constant.

$$\theta(x,t)=\frac{2T}{\sqrt{\pi}}\int_{\frac{x}{2\alpha\sqrt{t}}}^{\infty}\mathrm{e}^{-u^2}\,du\quad=\quad T\operatorname{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right)$$



Question 8

The function $x = f(t)$ satisfies the differential equation

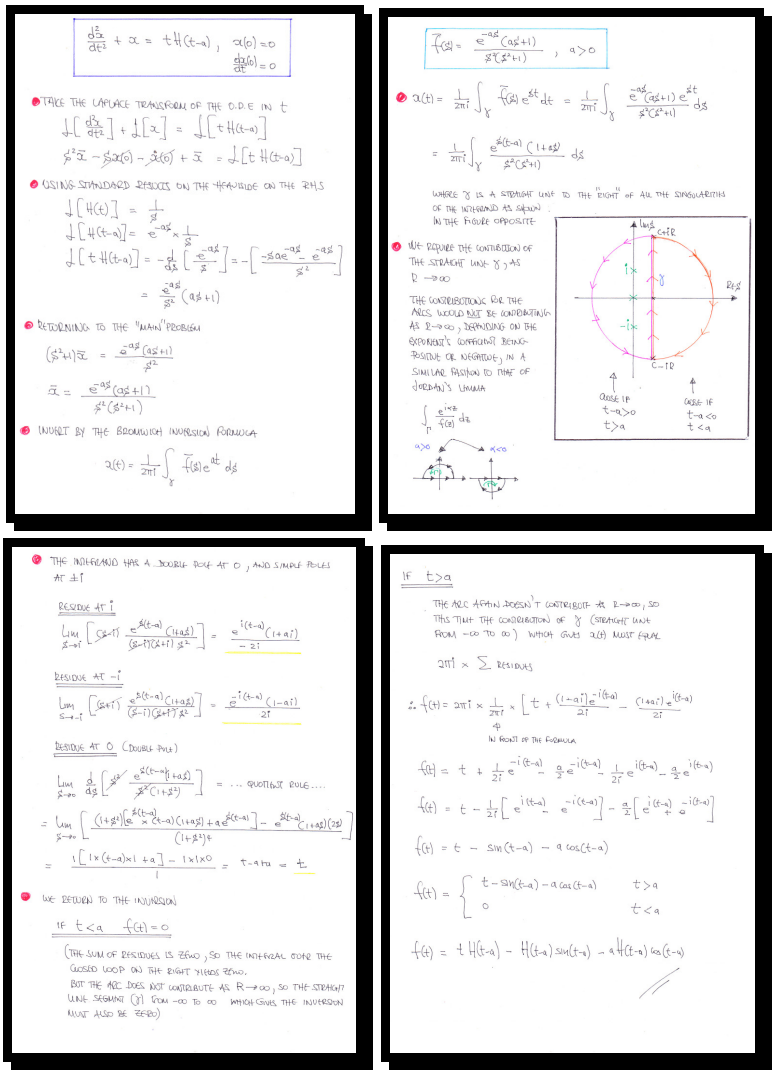
$$\frac{d^2x}{dt^2} + x = t\ \mathrm{H}(t-a),\ t \geq 0,$$

where $\mathrm{H}(t)$ is the Heaviside function and a is a positive constant.

Use Laplace transforms followed by inversion using complex variable to show that

$$x = t\,\mathrm{H}(t-a) - \mathrm{H}(t-a)\sin(t-a) + a\ \mathrm{H}(t-a)\cos(t-a).$$

proof



Question 9

The function $x_n = f(t,n)$ satisfies the differential equation

$$t\frac{d^2x}{dt^2}+(1-t)\frac{dx}{dt}+nx=0,\ t\geq 0,\ n\in\mathbb{N}.$$

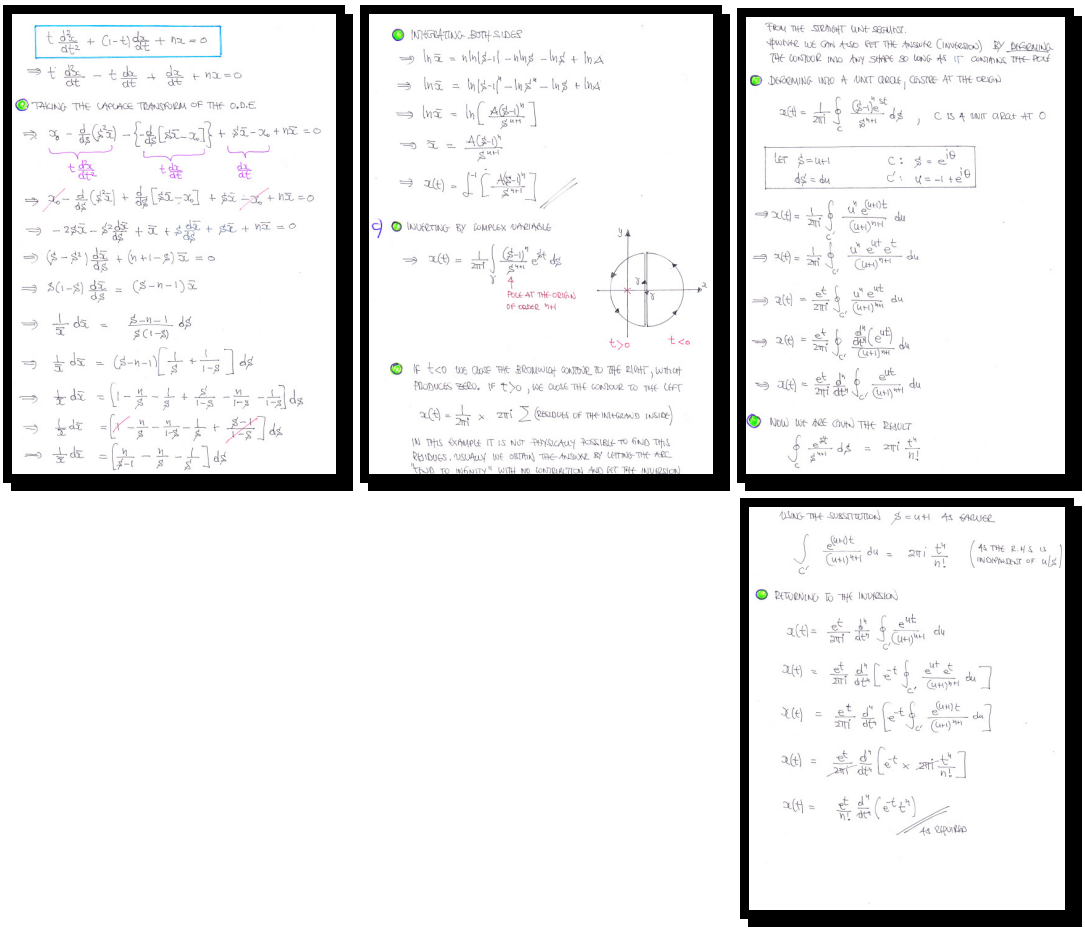
Use Laplace transforms in t, followed by inversion using a unit circle contour, to show that

$$x_n=\frac{\mathrm{e}^t}{n!}\frac{d^n}{dt^n}\left(t^n\,\mathrm{e}^{-t}\right).$$

You may assume that

$$\oint_C \frac{\mathrm{e}^{st}}{s^{n+1}}\,ds=\frac{t^n}{n!}(2\pi\mathrm{i}),\quad \text{where}\quad C:s=\mathrm{e}^{\mathrm{i}\theta},-\pi<\infty\leq\pi$$

proof



Question 10

The function $y = y(t)$, $t \geq 0$ satisfies the following equation.

$$\frac{d^2y}{dt^2} - y + 2\int_0^t \sin(t-u)\, y(u)\, du = \cos t\,.$$

Use Laplace transforms to show that

$$y(t) = \sin\left(\frac{t}{\sqrt{2}}\right)\sinh\left(\frac{t}{\sqrt{2}}\right).$$

proof



Question 11

The one dimensional heat equation for the temperature, $T(x,t)$, satisfies

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\sigma}\frac{\partial T}{\partial t}, \ t \geq 0,$$

where t is the time, x is a spatial dimension and σ is a positive constant.

The temperature $T(x,t)$ is subject to the following conditions.

i. $\lim_{x\to\infty}\left[T(x,t)\right] = 0$

ii. $T(0,t) = 1$

iii. $T(x,0) = 0$

a) Use Laplace transforms to show that

$$\mathcal{L}\left[T(x,t)\right] = \overline{T}(x,s) = \frac{1}{s}\exp\left[-\sqrt{\frac{s}{\sigma}}\, x\right].$$

b) Use contour integration to show further that

$$T(x,t) = 1 - \operatorname{erf}\left[\frac{x}{4\sigma t}\right].$$

You may assume without proof that

- $\displaystyle\int_0^\infty \mathrm{e}^{-ax^2}\cos kx \ dx = \sqrt{\frac{\pi}{4a}}\exp\left[-\frac{k^2}{4a}\right]$
- $\displaystyle\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-\xi^2} \ d\xi$

proof

[solution overleaf]

