JACOBIANS

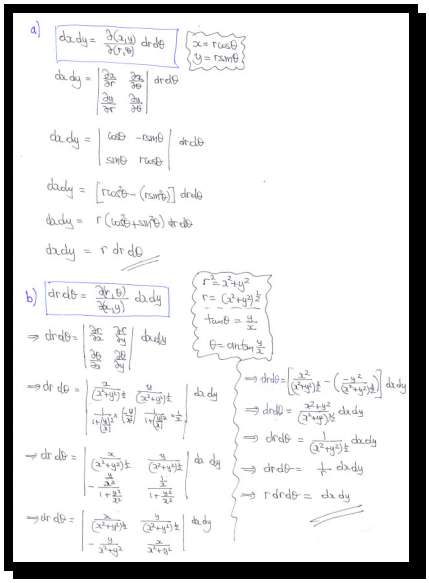
CURVILINEAR COORDINATES

Question 1

a) Determine, by a Jacobian matrix, an expression for the area element in plane polar coordinates, (r,θ).

b) Verify the answer of part **(a)** by performing the same operation in reverse.

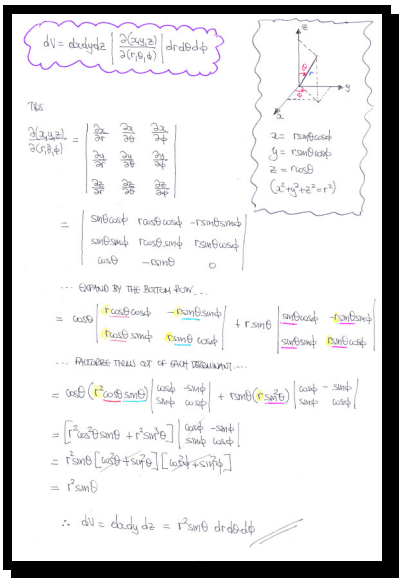
$$\boxed{dA = r\,dr\,d\theta}$$



Question 2

Determine, by a Jacobian matrix, an expression for the volume element in spherical polar coordinates, (r,θ,φ).

$$\boxed{dV = r^2 \sin\theta \, dr \, d\theta \, d\varphi}$$



Question 3

Two sets of variables are related by the equations

$$x = r\cosh\theta \quad \text{and} \quad y = r\sinh\theta,$$

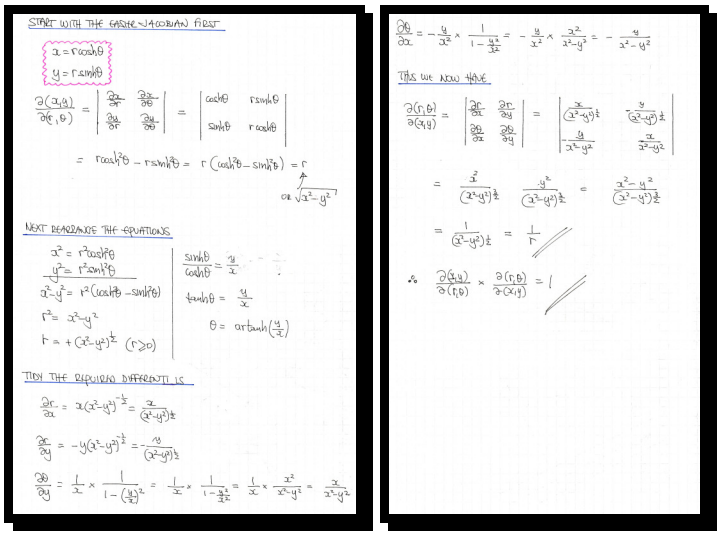
where $r \geq 0$.

Evaluate independently Jacobians

$$I = \frac{\partial(x,y)}{\partial(r,\theta)} \quad \text{and} \quad J = \frac{\partial(r,\theta)}{\partial(x,y)},$$

and hence show that $I = \frac{1}{J}$.

$$\boxed{I = \sqrt{x^2 + y^2} = r}, \quad \boxed{J = \frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2}}}$$



Question 4

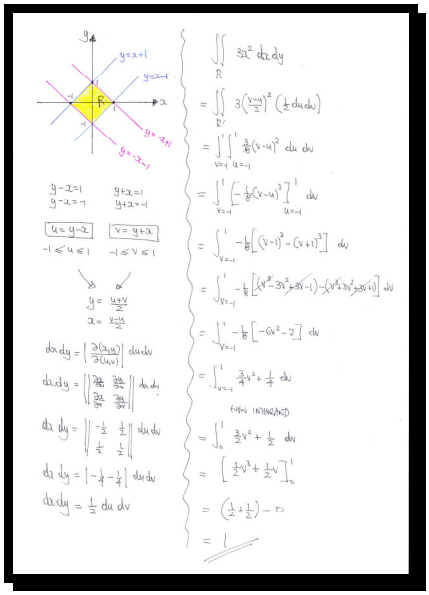
The finite region R is bounded by the straight lines with equations

$$y = x - 1, \quad y = x + 1, \quad y = -x - 1 \quad \text{and} \quad y = -x + 1.$$

Find an exact value for

$$\iint_R 3x^2 \, dx\, dy .$$

1



Question 5

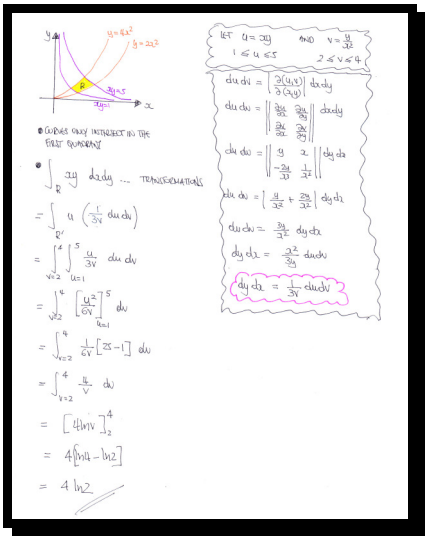
The finite region R is bounded by the curves with equations

$$y = 2x^2, \quad y = 4x^2, \quad xy = 1 \quad \text{and} \quad xy = 5.$$

Find an exact value for

$$\iint_R xy \, dx\, dy.$$

$\boxed{4\ln 2}$



Question 6

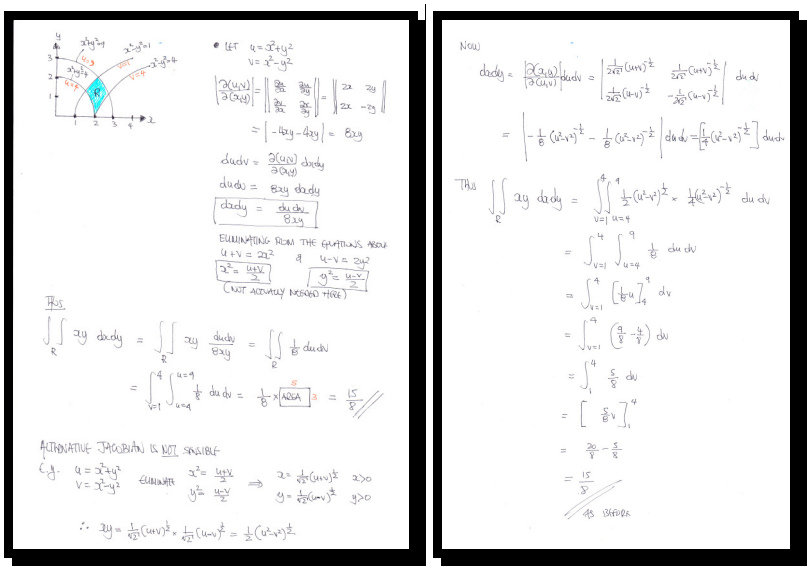
The finite region R in the first quadrant is defined by the inequalities

$$4 \le x^2 + y^2 \le 9 \quad \text{and} \quad 1 \le x^2 - y^2 \le 4 .$$

Evaluate the following integral

$$\iint_R xy \, dx \, dy .$$

$$\frac{15}{8}$$



Question 7

The finite region R is bounded by the straight lines with equations

$$x+y=1, \quad x+y=2, \quad y=x \quad \text{and} \quad y=0.$$

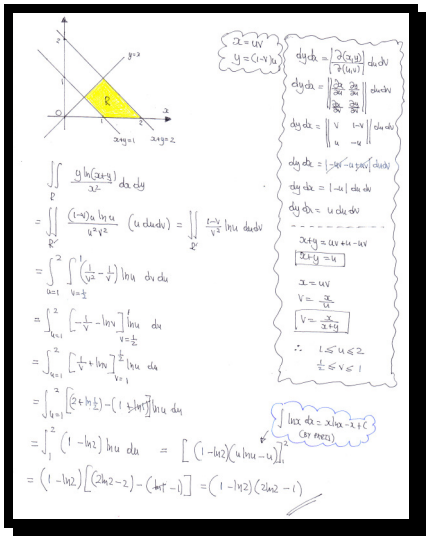
Use the transformation equations

$$x=uv \quad \text{and} \quad y=u(1-v),$$

to find an exact value for

$$\iint_R \frac{y\ln(x+y)}{x^2}\,dx\,dy.$$

$$\boxed{(1-\ln 2)(-1+2\ln 2)}$$



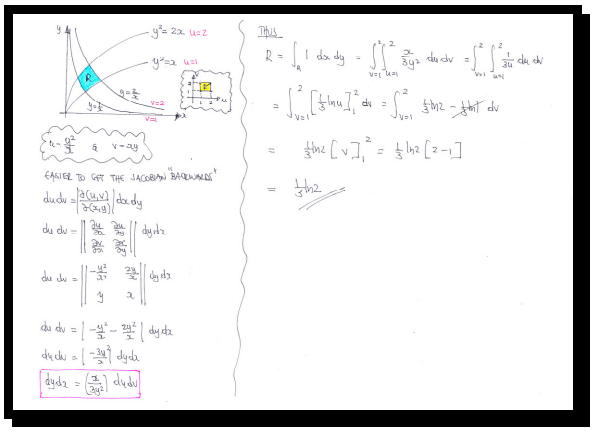
Question 8

The finite region R satisfies the inequalities

$$x \leq y^2 \leq 2x \quad \text{and} \quad \frac{1}{x} \leq y \leq \frac{2}{x}.$$

Find the area of R, giving the answer as an exact simplified logarithm.

$$\frac{1}{3}\ln 2$$



Question 9

An ellipse has Cartesian equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

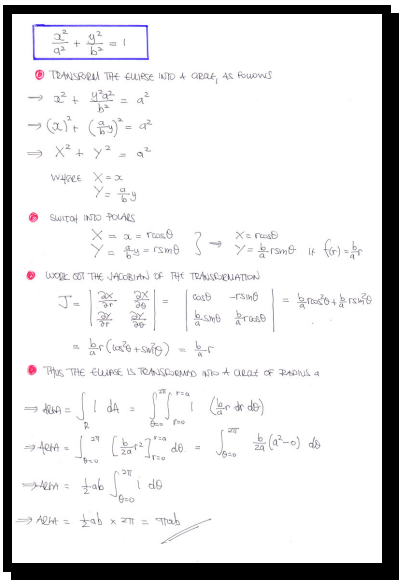
where a and b are positive constants.

Use the transformation equations

$$x=r\cos\theta \quad \text{and} \quad y=f\left(r\right)\sin\theta,$$

where f is a function to be found, to determine the area enclosed by the ellipse.

πab



Question 10

The finite region R is bounded by the straight lines with equations

$$y = x \quad \text{and} \quad y = 4x ,$$

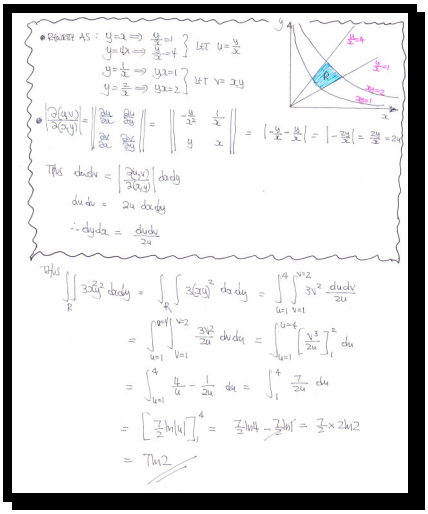
and the hyperbolae with equations

$$y = \frac{1}{x} \quad \text{and} \quad y = \frac{2}{x} , \ x \neq 0 .$$

Show clearly that

$$\iint_R 3x^2 y^2 \, dx \, dy = 7 \ln 2 .$$

proof



Question 11

The **unbounded** region R is defined by the curves with equations

$$y = x^2, \quad y = 2x^2 \quad \text{and} \quad y = \frac{1}{4x^2}.$$

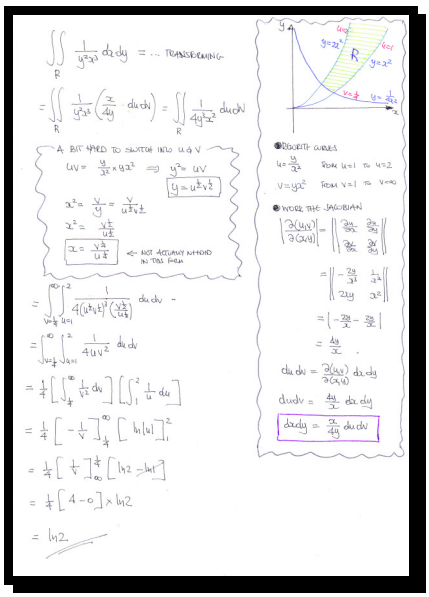
Use the transformation equations

$$u = \frac{y}{x^2} \quad \text{and} \quad v = yx^2,$$

to find an exact value for

$$\iint_R \frac{1}{y^2 x^3}\, dx\, dy.$$

$\ln 2$



Question 12

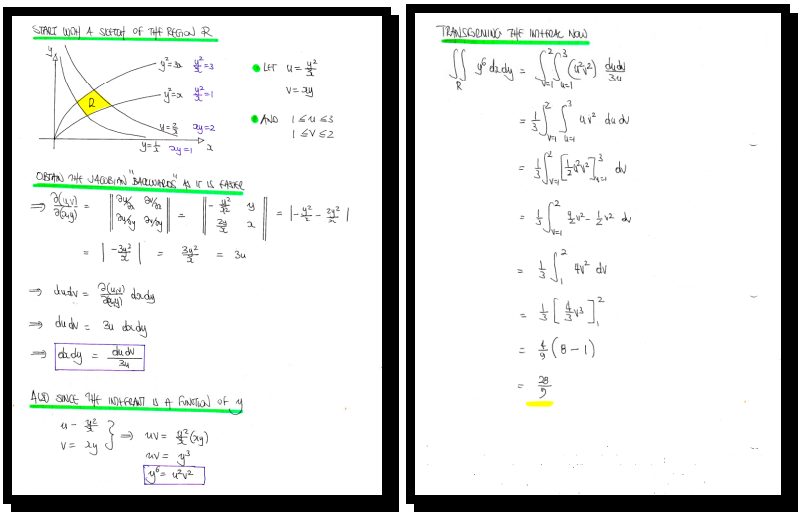
The finite region R, in the first quadrant, satisfies the inequalities

$$x \le y^2 \le 3x \quad \text{and} \quad \frac{1}{x} \le y \le \frac{2}{x}.$$

Find the exact value of

$$\int_R y^6 \, dx\, dy .$$

$\square$, $\boxed{\frac{28}{9}}$



Question 13

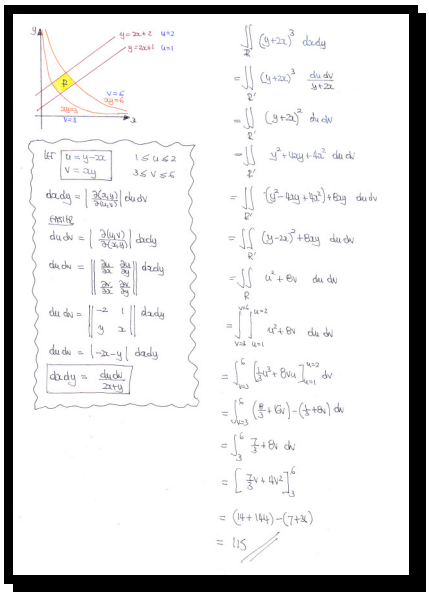
The finite region R, in the first quadrant, is bounded by the curves with equations

$$y = 2x + 1, \quad y = 2x + 2, \quad y = \frac{3}{x} \quad \text{and} \quad y = \frac{6}{x}.$$

Show clearly that

$$\iint_R (y + 2x)^3 \, dx\, dy = 115.$$

proof



Question 14

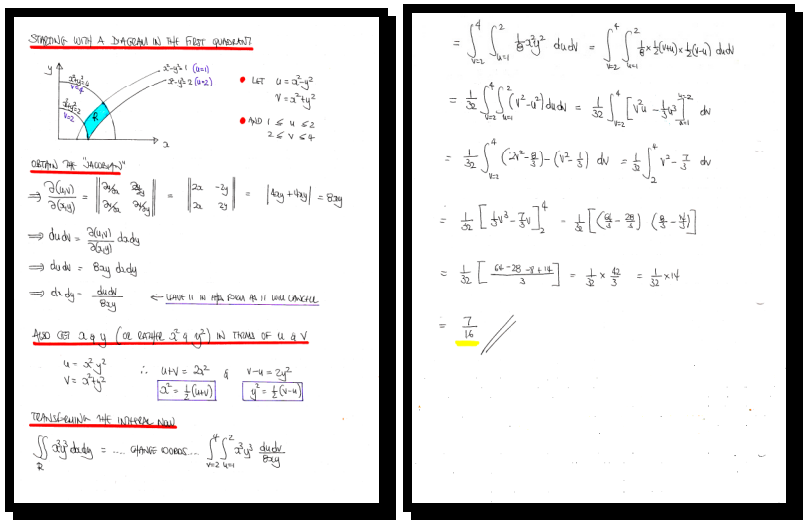
The finite region R is defined by the inequalities

$$2 \le x^2 + y^2 \le 4 \quad \text{and} \quad 1 \le x^2 - y^2 \le 2 .$$

Given further that $x > 0$ and $x > 0$, evaluate the following integral

$$\iint_R x^3 y^3 \, dx \, dy .$$

$\square$, $\boxed{\frac{7}{16}}$



Question 15

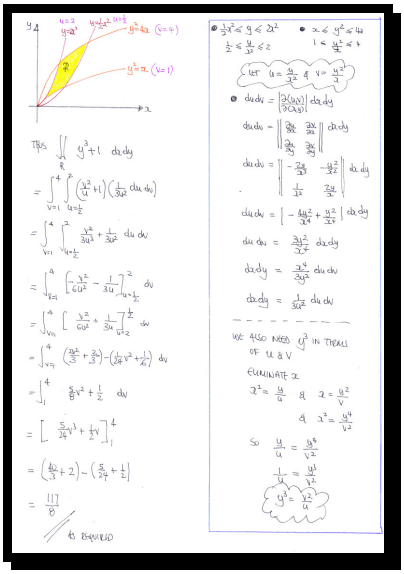
The finite region R is bounded by the parabolas with equations

$$y=\frac{1}{2}x^2, \quad y=2x^2, \quad y^2=x \quad \text{and} \quad y^2=4x.$$

Show clearly that

$$\iint_R y^3+1 \; dx\,dy = \frac{117}{8}.$$

proof



Question 16

The finite region R is bounded by the straight lines with equations

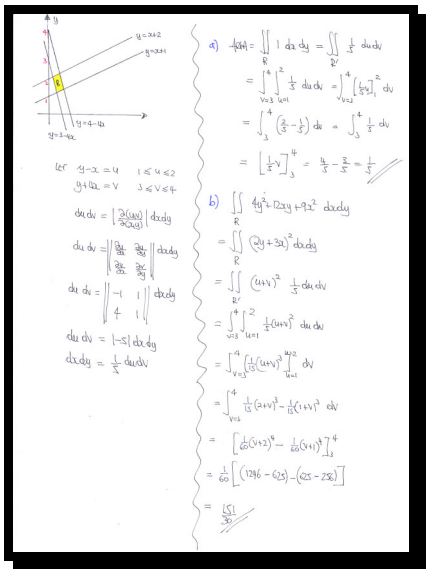
$$y = x+1, \quad y = x+2, \quad y = 3-4x \quad \text{and} \quad y = 4-4x.$$

a) Find the exact area of R.

b) Show clearly that

$$\iint_R 4y^2+12xy+9x^2 \ dx\,dy = \frac{151}{30}.$$

$$\boxed{\text{area} = \frac{1}{5}}$$



Question 17

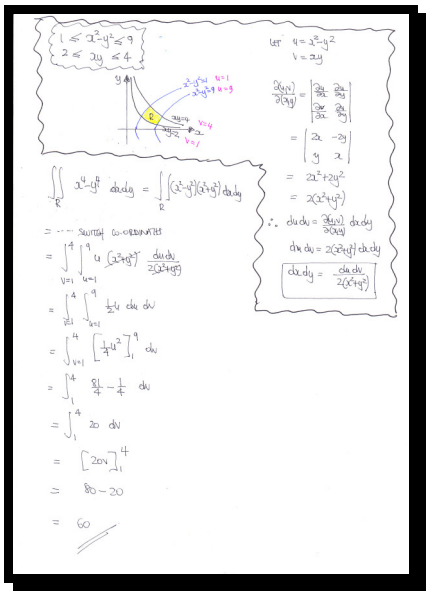
The finite region R is defined by the inequalities

$$1 \le x^2 - y^2 \le 9 \quad \text{and} \quad 2 \le xy \le 4 .$$

Given further that $x > 0$ and $x > 0$, evaluate the following integral

$$\iint_R \left(x^4 - y^4 \right) dx\, dy .$$

60



Question 18

The finite region R is bounded by the straight lines with equations

$$y = x - 1 \quad \text{and} \quad y = x - 3 ,$$

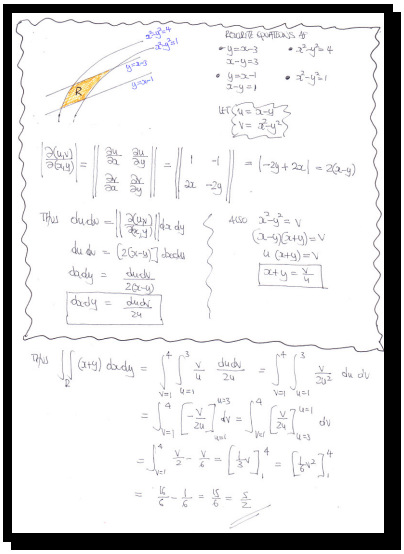
and the hyperbolae with equations

$$x^2 - y^2 = 1 \quad \text{and} \quad x^2 - y^2 = 4 .$$

Evaluate the integral

$$\iint_R (x + y)\, dx\, dy .$$

$\boxed{\frac{5}{2}}$



Question 19

The finite region R is bounded by the curves with equations

$$6xy = \pi \quad \text{and} \quad 2xy = \pi ,$$

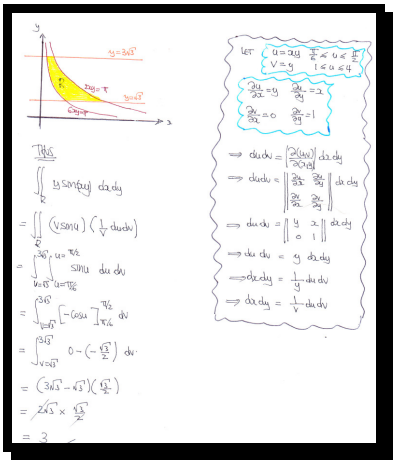
and the straight lines with equations

$$y = \sqrt{3} \quad \text{and} \quad y = 3\sqrt{3} .$$

evaluate the following integral

$$\iint_R y \sin(xy)\, dx\, dy .$$

3



Question 20

The finite region R satisfies the inequalities

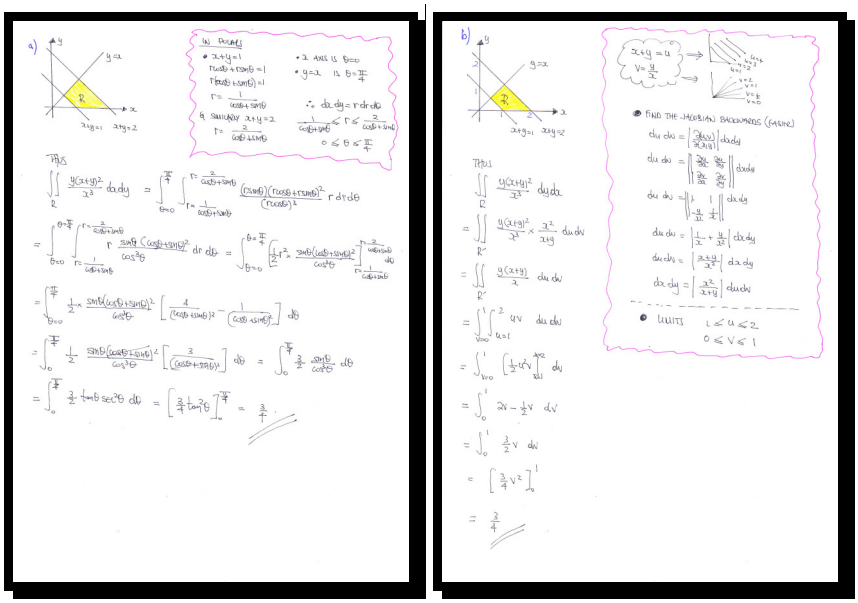
$$1 \le x+y \le 2 \quad \text{and} \quad 0 \le y \le x .$$

a) Use plane polar coordinates (r,θ) to find the value of

$$\iint_R \frac{y(x+y)^2}{x^3}\,dx\,dy .$$

b) Verify the answer obtained in part **(a)** by transforming the integral to different coordinate system.

$$\frac{3}{4}$$



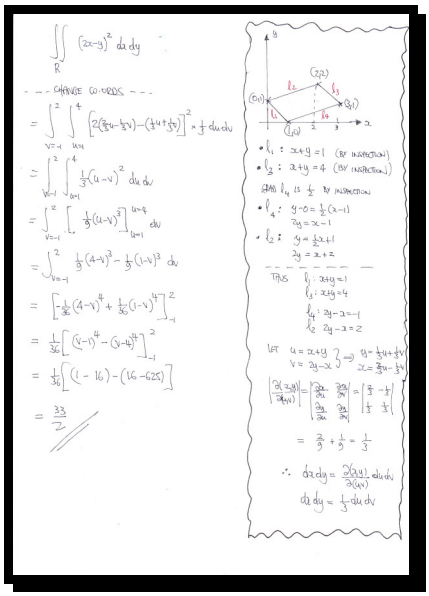
Question 21

The finite region R in the x-y plane is defined as the region enclosed by the straight line segments joining the points with coordinates at $(1,0)$, $(1,0)$, $(1,0)$ and $(1,0)$, in that order.

Evaluate the following integral

$$\iint_R (2x-y)^2 \, dx\,dy .$$

$$\frac{33}{2}$$



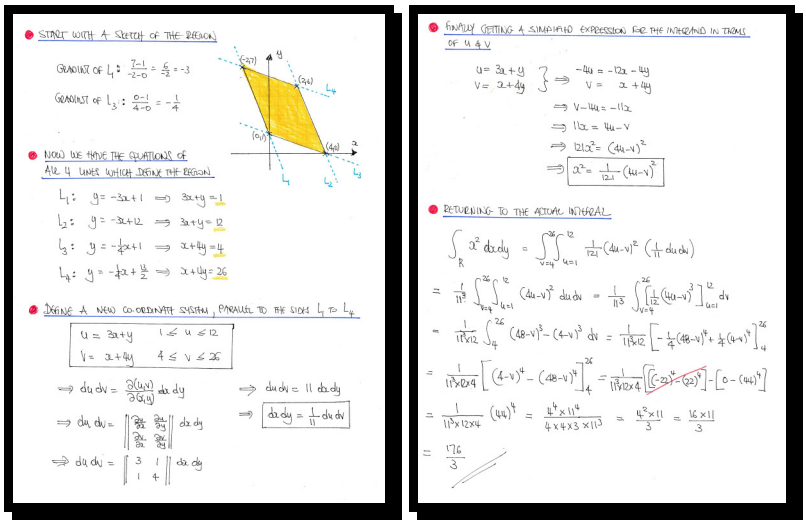
Question 22

The finite region R in the x-y plane, is defined as the interior of a parallelogram with vertices at $(4,0)$, $(0,1)$, $(-2,7)$ and $(2,6)$.

Evaluate the integral

$$\int_R x^2 \, dxdy \, .$$

$$\boxed{}, \quad \boxed{\frac{176}{3}}$$



Question 23

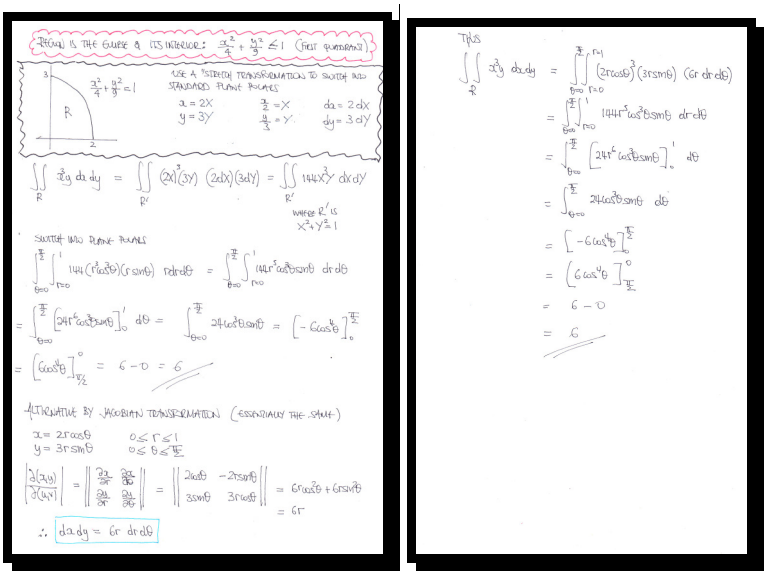
Given that R is the finite region in the x-y plane, defined as

$$\frac{x^2}{4}+\frac{y^2}{9}\leq 1,\ x\geq 0,\ y\geq 0,$$

evaluate the integral

$$\int_R yx^3\ dxdy.$$

6



Question 24

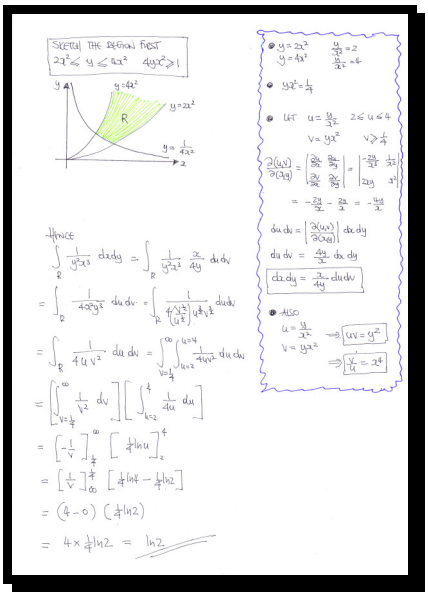
Given that R is the region of the x-y plane, defined as

$$2x^2 \le y \le 4x^2 \quad \text{and} \quad 4yx^2 \ge 1,$$

evaluate the integral

$$\int_R \frac{1}{y^2x^3}\,dxdy\,.$$

$\ln 2$

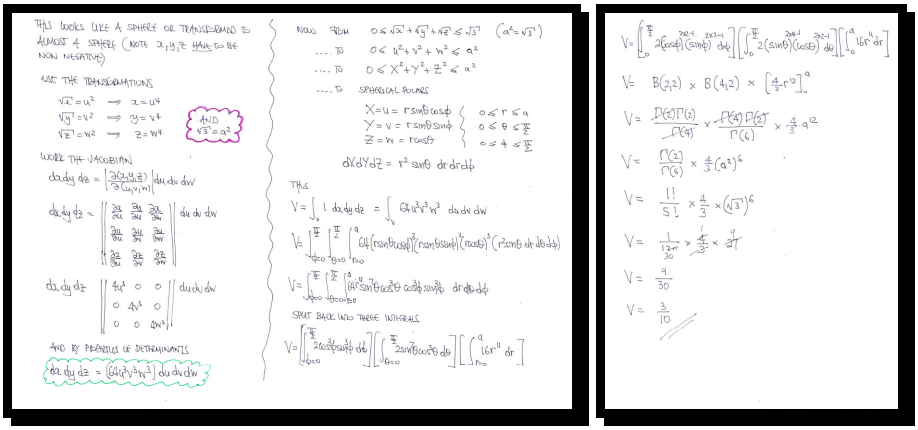


Question 25

By suitably changing coordinates, find the volume of the solid defined as

$$0 \le \sqrt{x} + \sqrt{y} + \sqrt{z} \le \sqrt{3} .$$

$\frac{3}{10}$



Question 26

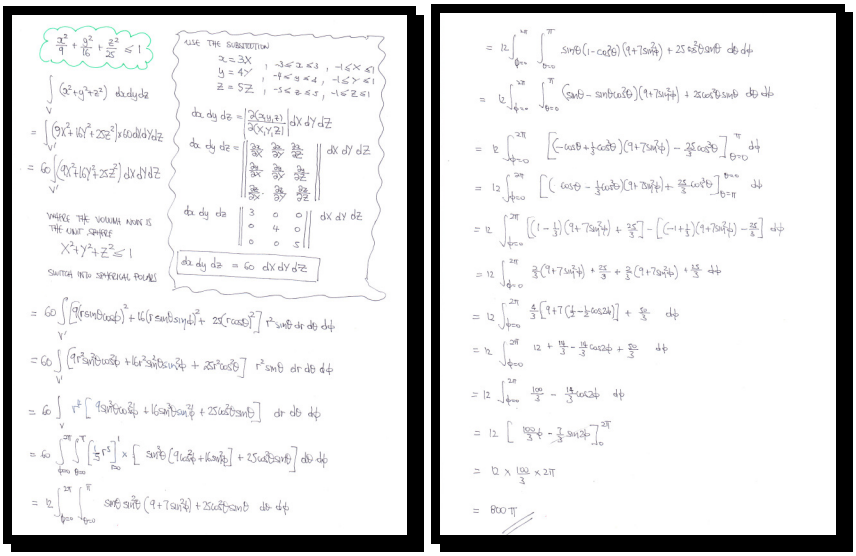
The finite region R is defined as the region enclosed by the ellipsoid with Cartesian equation

$$\frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{25}=1.$$

By first transforming the Cartesian coordinates into a new Cartesian coordinate system, use spherical polar coordinates, (r,θ,φ), find the value of

$$\iiint_R \left(x^2+y^2+z^2\right)\,dx\,dy\,dz.$$

$\boxed{800\pi}$



Question 27

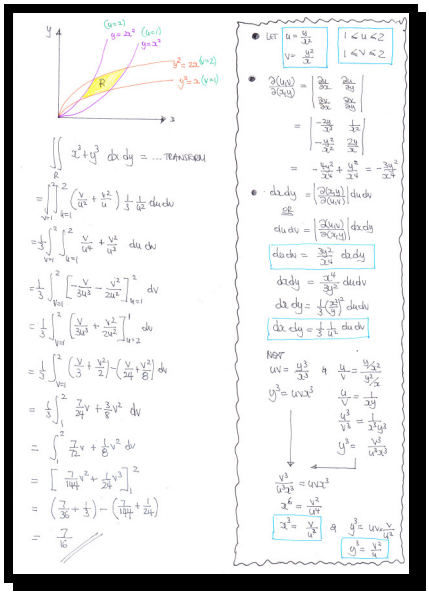
The finite region R in the x-y plane, is defined

$$x^2 \le y \le 2x^2 \quad \text{and} \quad x \le y^2 \le 2x \, .$$

Evaluate the integral

$$\int_R x^3 + y^3 \, dxdy \, .$$

$$\frac{7}{16}$$



Question 28

The finite region R is bounded by the coordinate axes and the straight line with Cartesian equation

$$x+y=1$$

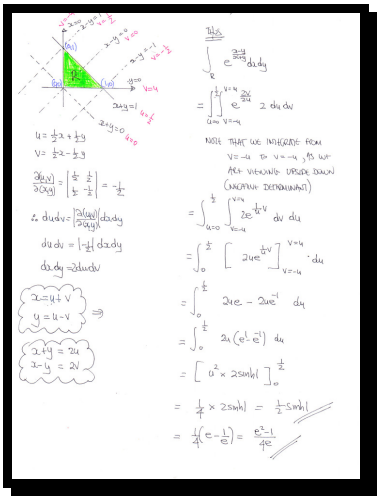
Use the coordinate transformation equations

$$u=\frac{1}{2}x+\frac{1}{2}y \quad \text{and} \quad v=\frac{1}{2}x-\frac{1}{2}y$$

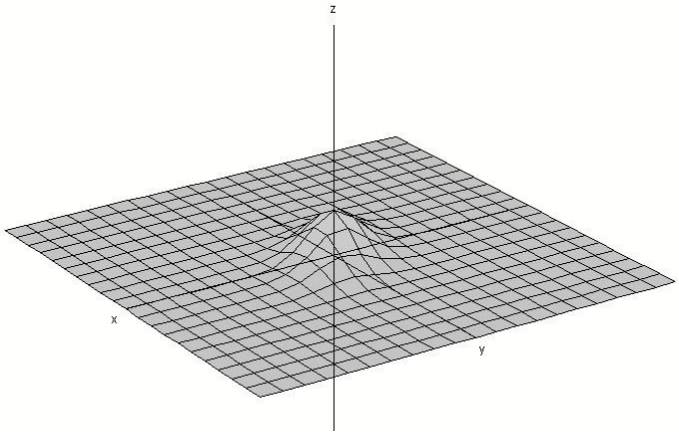
to find an exact value for

$$\int_R \mathrm{e}^{\frac{x-y}{x+y}}\,dxdy\,.$$

$$\boxed{\frac{\mathrm{e}^2-1}{4\mathrm{e}}=\frac{1}{2}\sinh 1}$$



Question 29



The figure above shows the graph of a "hill", modelled by the function $z = f(x, y)$, defined in the entire x-y plane by

$$z = \mathrm{e}^{-\left(\frac{5}{4}x^2 - xy + 2y^2\right)}.$$

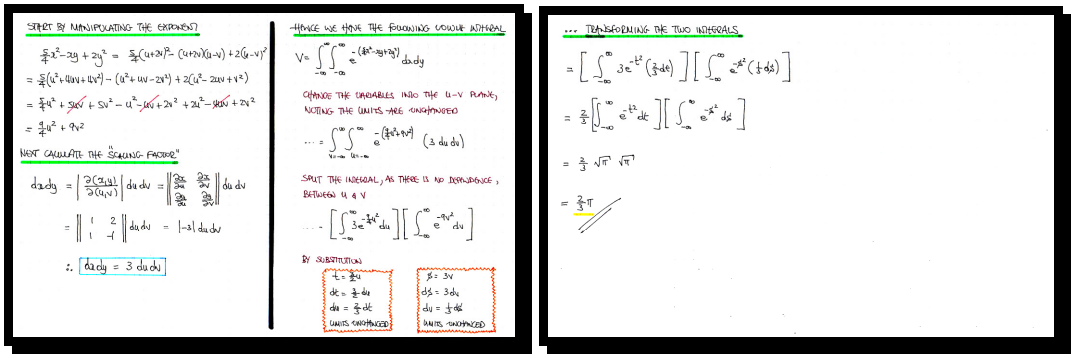
Use the transformation equations

$$x = u + 2v \text{ and } y = u - v$$

to show that the volume of the "hill" is $\dfrac{2\pi}{3}$.

You may assume without proof that $\displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{-s^2}\,ds = \sqrt{\pi}$.

[ ], [proof]



Question 30

The finite region R is bounded by the coordinate axes and the straight line with Cartesian equation

$$x+y=1$$

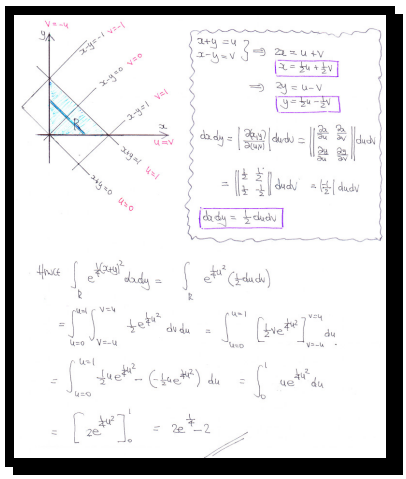
Use the transformation equations

$$u=x+y \qquad \text{and} \qquad v=x-y$$

to find an exact value for

$$\int_R \mathrm{e}^{\frac{1}{4}(x+y)^2}\,dxdy\,.$$

$$\boxed{2\left(\mathrm{e}^{\frac{1}{4}}-1\right)}$$



Question 31

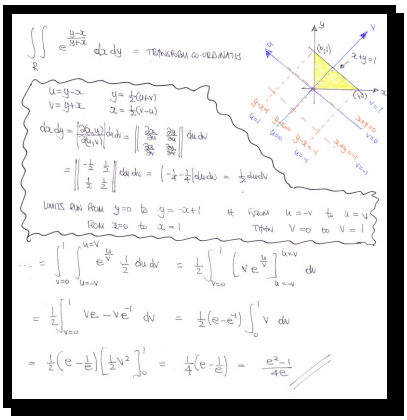
The finite region R is bounded by the coordinate axes and the straight line with Cartesian equation

$$x+y=1$$

Use a suitable coordinate transformation to find an exact value for

$$\int_R \mathrm{e}^{\frac{y-x}{y+x}}\,dxdy\,.$$

$$\boxed{\frac{\mathrm{e}^2-1}{4\mathrm{e}}=\frac{1}{2}\sinh 1}$$



Question 32

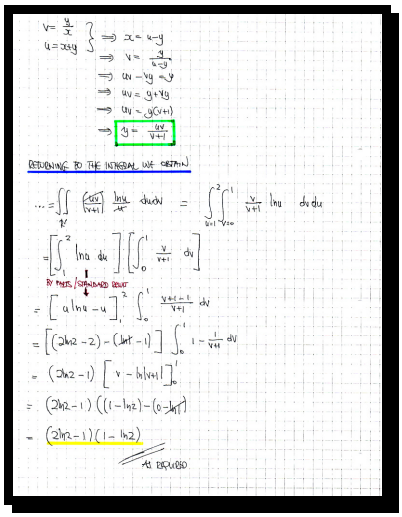
The finite region R satisfies the inequalities

$$1 \le x + y \le 2 \quad \text{and} \quad 0 \le y \le x .$$

Show clearly that

$$\iint_R \frac{y \ln(x+y)}{x^2}\, dx\, dy = (1-\ln 2)(-1+2\ln 2) .$$

, proof



Question 33

The finite region R is defined by the inequalities

$$y \le x, \quad y \le 1-x \quad \text{and} \quad y \ge 0$$

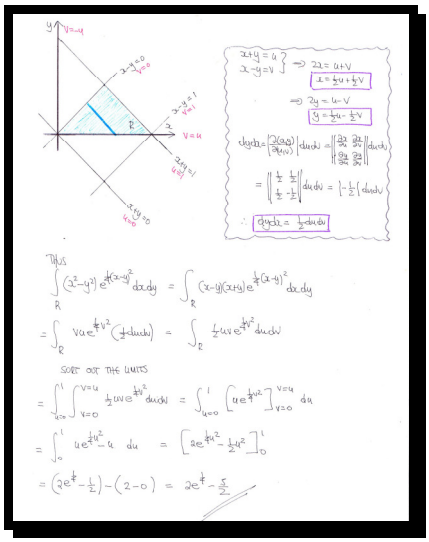
Use the transformation equations

$$u = x + y \qquad \text{and} \qquad v = x - y$$

to find an exact value for

$$\int_R \left(x^2 - y^2\right) \mathrm{e}^{\frac{1}{4}(x-y)^2} \, dxdy .$$

$$\boxed{\frac{1}{2}\left(4\mathrm{e}^{\frac{1}{4}} - 5\right)}$$



Question 34

The finite region R is bounded by the straight lines with equations

$$y = x, \quad x = 1 \quad \text{and} \quad y = 0.$$

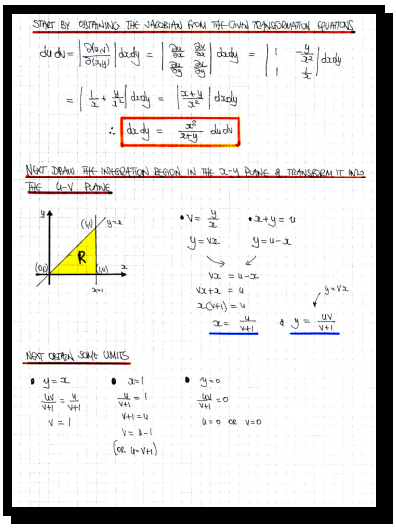
Use the transformation equations

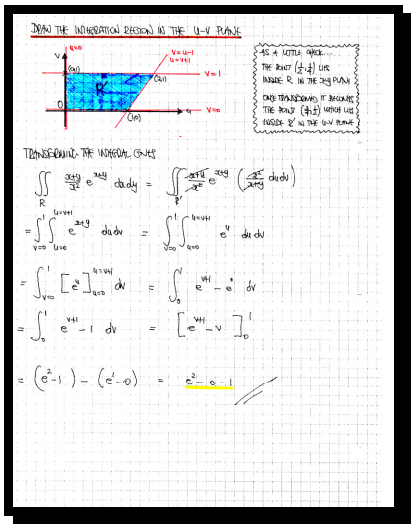
$$u = x + y \quad \text{and} \quad v = \frac{y}{x},$$

to find an exact value for

$$\iint_R \left(\frac{x+y}{x^2} \right) \mathrm{e}^{x+y} \, dx \, dy.$$

$\square$, $\mathrm{e}^2 - \mathrm{e} - 1$





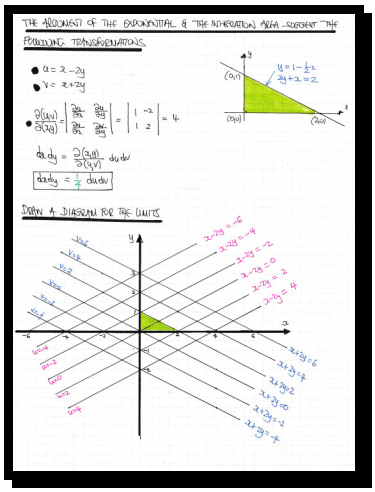
Question 35

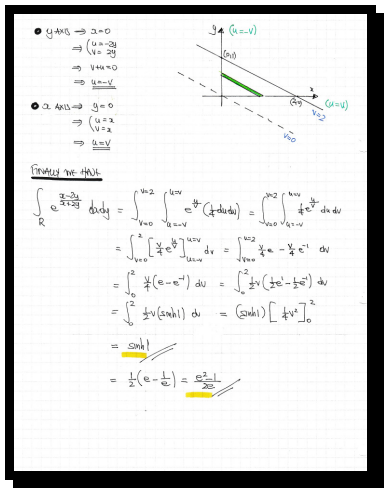
The finite region R in the x-y plane is enclosed by the rectilinear triangle with vertices at $(0,0)$, $(0,1)$ and $(2,0)$.

Use a suitable coordinate transformation to find an exact value for

$$\int_R e^{\frac{x-2y}{x+2y}}\,dxdy\,.$$

$\square$, $\boxed{\dfrac{e^2-1}{2e}=\sinh 1}$





Question 36

$$I = \int_0^{\infty} \int_0^{\infty} \mathrm{e}^{-(x+y)^2} \, dx \, dy \, .$$

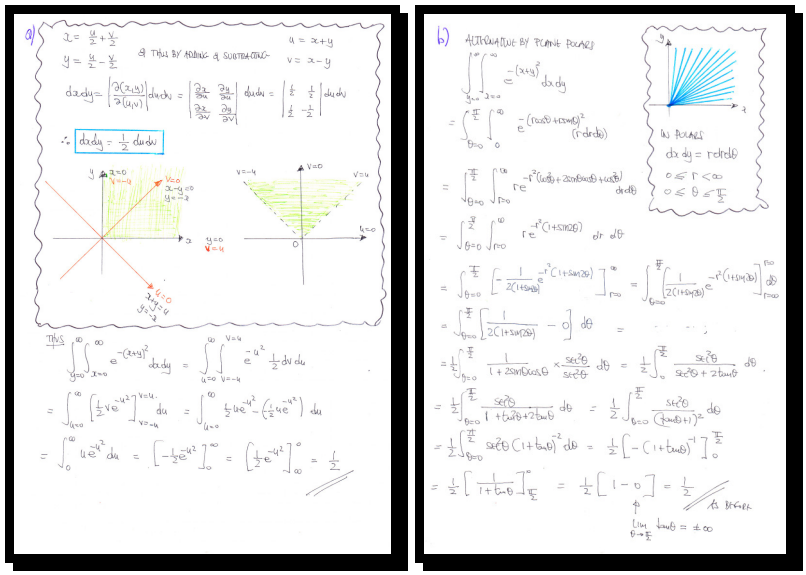
a) Use the coordinate transformation equations

$$x = \frac{1}{2}u + \frac{1}{2}v \text{ and } y = \frac{1}{2}u - \frac{1}{2}v \, ,$$

to find the value of I .

b) Evaluate I in plane polar coordinates, (r, θ), and hence verify the answer of part **(a)**.

$$\boxed{\frac{1}{2}}$$



Question 37

The finite region R is bounded by the curve with Cartesian equation

$$x^4+y^4=1,\ x\geq 0,\ y\geq 0.$$

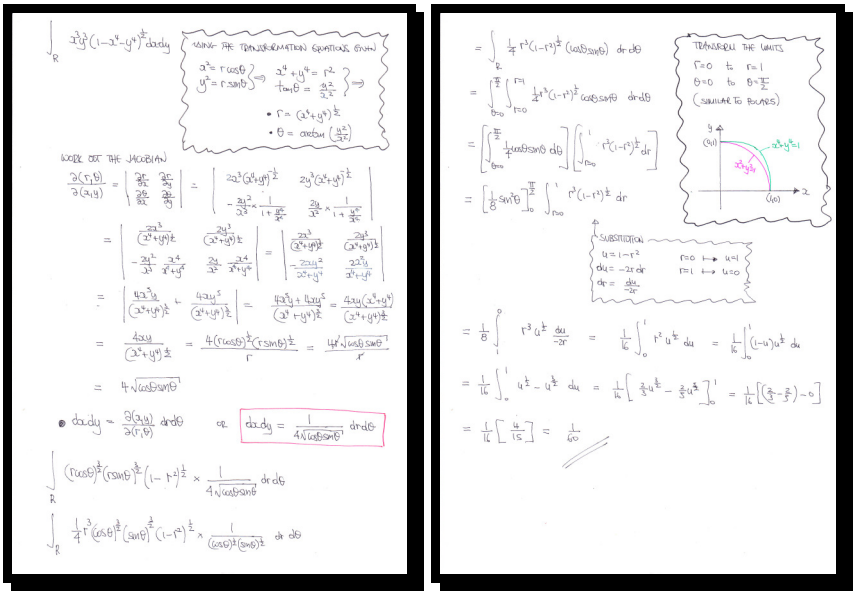
Use the transformation equations

$$x^2=r\cos\theta \quad \text{and} \quad x^2=r\cos\theta,$$

to find the value of

$$\iint_R x^3y^3\sqrt{1-x^4-y^4}\ dxdy.$$

$$\frac{1}{60}$$



Question 38

The finite region V is enclosed by the surface with Cartesian equation

$$x^4 + y^4 + z^4 = 64 .$$

By first transforming the Cartesian coordinates into a new Cartesian coordinate system, use spherical polar coordinates, (r,θ,φ), to show that the volume of V is

$$\frac{8}{3\pi}\left[\Gamma\left(\frac{1}{4}\right)\right]^4 .$$

proof

