INTRINSIC COORDINATES

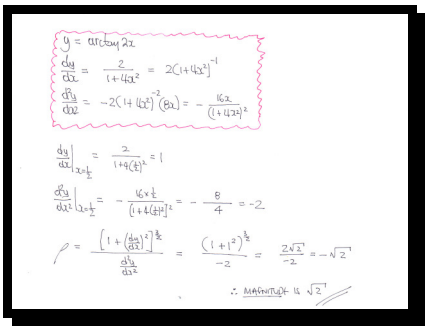
Question 1 (**)

A curve C has Cartesian equation

$$y=\arctan 2x\,,\quad x\in\mathbb{R}\,.$$

Find the magnitude of the radius of curvature at the point on C where $x=\frac{1}{2}$.

$\boxed{\sqrt{2}}$



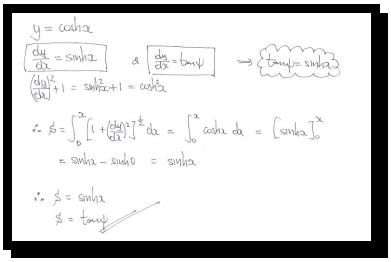
Question 2 (**)

A curve C has Cartesian equation

$$y=\cosh x\,,\quad x\in\mathbb{R}\,.$$

Find an intrinsic equation of C in the form $s=f(\psi)$, where s is measured from the point with coordinates $(0,1)$, and ψ is the angle the tangent to C makes with the positive x axis.

$\boxed{s=\tan\psi}$



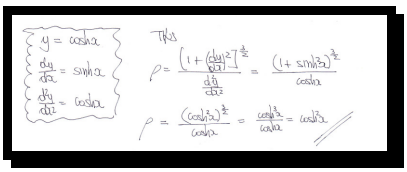
Question 3 (**)

A curve C has Cartesian equation

$$y=\cosh x,\ x\in\mathbb{R}.$$

Find a simplified expression, in terms of x, for the radius of curvature at a general point on C.

$$\boxed{\rho=\cosh^2 x}$$



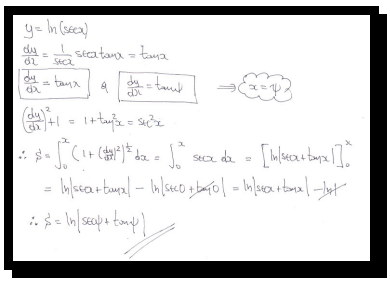
Question 4 (**)

A curve C has Cartesian equation

$$y=\ln\left(\sec x\right),\ 0\le x<\frac{\pi}{2}.$$

Find an intrinsic equation of C in the form $s=f\left(\psi\right)$, where s is measured from the point with coordinates $\left(0,0\right)$, and ψ is the angle the tangent to C makes with the positive x axis.

$$\boxed{s=\ln\left|\tan\psi+\sec\psi\right|}$$



Question 5 (**)

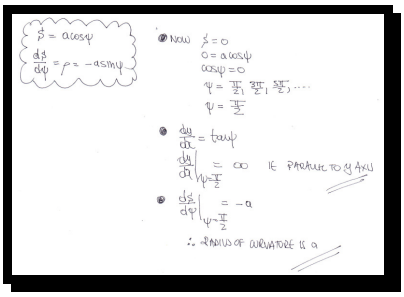
A curve C has intrinsic equation

$$s = a\cos\psi\ ,\ \psi \in [0,\pi],$$

where s denotes the arc length measured from some fixed point and ψ is the angle the tangent to C makes with the positive x axis.

Show that the tangent to C at the point where $s = 0$ is parallel to the y axis and determine the radius of curvature at that point.

$$\boxed{\rho = a}$$



Question 6 (**)

A curve C has intrinsic equation

$$s = 2\sin\psi\,,\ \psi \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right],$$

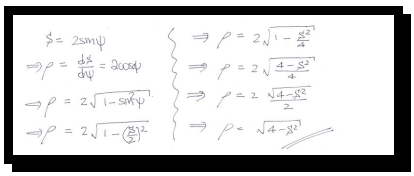
where s denotes the arc length measured from some fixed point and ψ is the angle the tangent to C makes with the positive x axis.

Show clearly that

$$\rho = \sqrt{4-s^2}\,,$$

where ρ is the radius of curvature at a general point on C.

proof



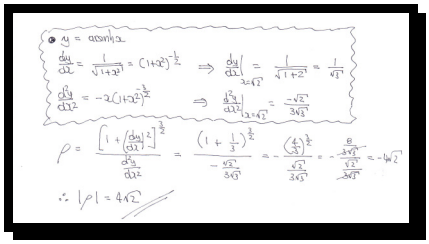
Question 7 (**)

A curve C has Cartesian equation

$$y = \operatorname{arsinh} x\,,\ x \in \mathbb{R}\,.$$

Find the magnitude of the radius of curvature at the point on C where $x = \sqrt{2}$.

$4\sqrt{2}$



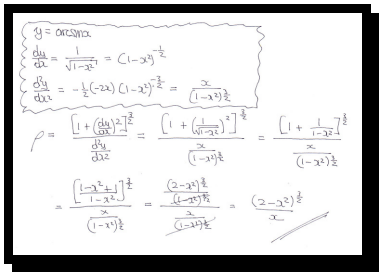
Question 8 (**)

A curve C has Cartesian equation

$$y = \arcsin x\,,\ -1 \le x \le 1\,.$$

Find an expression, in terms of x, for the radius of curvature on C, giving the answer as a single simplified fraction.

$$\boxed{\rho = \frac{\left(2-x^2\right)^{\frac{3}{2}}}{x}}$$



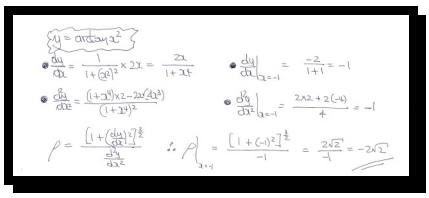
Question 9 (**)

A curve C has Cartesian equation

$$y = \arctan x^2\,,\ 0 \le y < \frac{\pi}{2}\,.$$

Calculate the radius of curvature at the point on C where $x = -1$.

$$\boxed{-2\sqrt{2}}$$



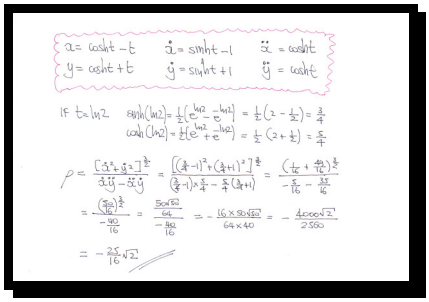
Question 10 (+)**

A curve C has parametric equation s

$$x=\cosh t-t\,,\ y=\cosh t+t\,,\ t\in\mathbb{R}\,.$$

Find the exact value of the radius of curvature at the point on C where $t=\ln 2$.

$$\boxed{\rho=\frac{25}{16}\sqrt{2}}$$



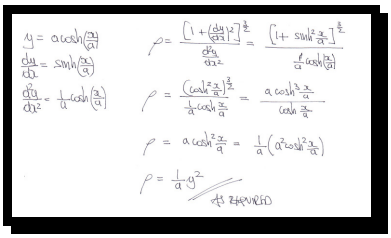
Question 11 (+)**

A curve C has Cartesian equation

$$y=a\cosh\left(\frac{x}{a}\right),\text{ where } a \text{ is a constant.}$$

Show that the radius of curvature on C, is given by $\frac{1}{a}y^2$

$$\boxed{\text{proof}}$$



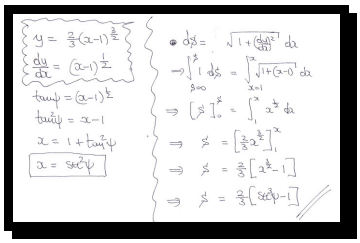
Question 12 (+)**

A curve C has Cartesian equation

$$y=\frac{1}{2}(x-1)^{\frac{3}{2}},\ x\in\mathbb{R},\ x\geq 1.$$

Find an intrinsic equation of C in the form $s=f(\psi)$, where s is measured from the point with coordinates $(1,0)$, and ψ is the angle the tangent to C makes with the positive x axis.

$$\boxed{s=\frac{2}{3}\left(\sec^3\psi-1\right)}$$



Question 13 (+)**

The radius of curvature at a general point on a curve C is given by

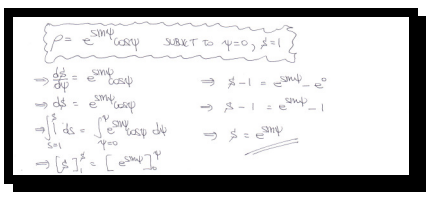
$$e^{\sin\psi}\cos\psi,$$

where ψ is the angle the tangent to C makes with the positive x axis..

It is further given that when $\psi=0$, $s=1$, where s is the arc length measured from some fixed point.

Find an intrinsic equation for C, in the form $s=f(\psi)$.

$$\boxed{s=e^{\sin\psi}}$$



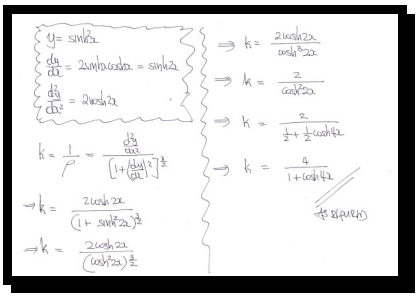
Question 14 (***)

A curve C has Cartesian equation

$$y=\sinh^2 x,\ x\in\mathbb{R}.$$

Express the **curvature** at a general point on C in terms of $\cosh 4x$.

$$\boxed{\kappa=\frac{4}{1+\cosh 4x}}$$



Question 15 (***)

A curve C has intrinsic equation

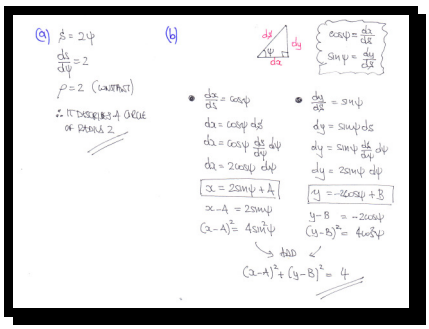
$$s = 2\psi ,$$

where s is measured from some arbitrary point, and ψ is the angle the tangent to C makes with the positive x axis..

a) Describe C geometrically, with reference to $\dfrac{ds}{d\psi}$.

b) Use a calculus method to obtain a Cartesian equation for C, in terms of suitable constants.

$$\boxed{(x-a)^2+(y-b)^2=4}$$



Question 16 (***)

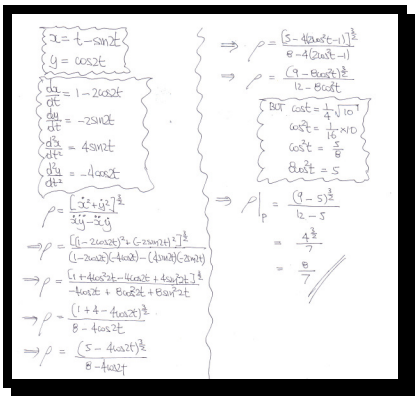
A curve C has parametric equation s

$$x = t - \sin 2t\ ,\ \ y = \cos 2t\ ,\ \ 0 \le t < \frac{\pi}{2}.$$

The point P lies on C where $\cos t = \frac{1}{4}\sqrt{10}$.

Calculate the radius of curvature at P.

$$\boxed{\rho\big|_P = \frac{8}{7}}$$



Question 17 (*+)**

A curve C has intrinsic equation

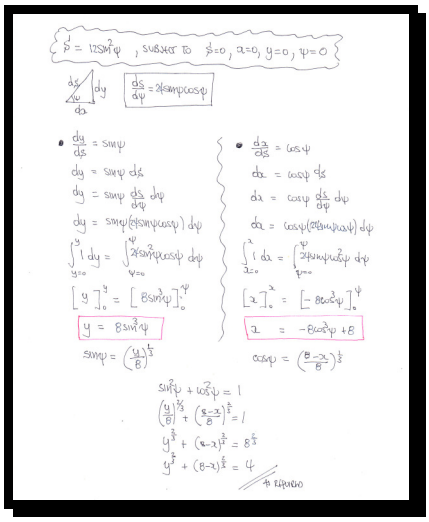
$$s=12\sin^2\psi\,,$$

where s is measured from a Cartesian origin, and ψ is the angle the tangent to C makes with the positive x axis.

Show that a Cartesian equation of C is

$$y^{\frac{2}{3}}+(8-x)^{\frac{2}{3}}=4\,.$$

proof



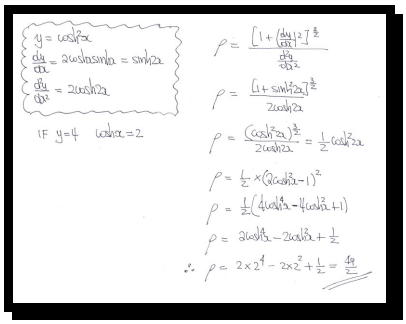
Question 18 (*+)**

A curve C has Cartesian equation

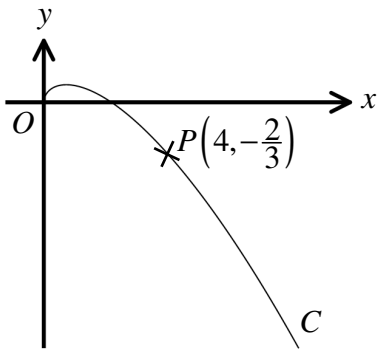
$$y=\cosh^2 x,\ x\in\mathbb{R}.$$

Show that the radius of curvature at the point on C where $y=4$ is $24\frac{1}{2}$.

proof



Question 19 (*+)**

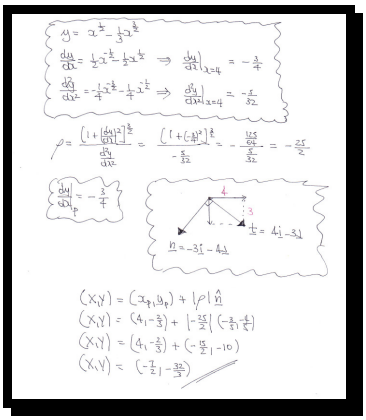


The figure above shows the curve C with Cartesian equation

$$y = x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}, \ x \in \mathbb{R}, \ x \geq 0 .$$

Show that the centre of curvature at the point $P\left(4,-\frac{2}{3}\right)$ on C, is $\left(-\frac{7}{2},-\frac{32}{3}\right)$.

proof



Question 20 (*+)**

A curve C has Cartesian equation

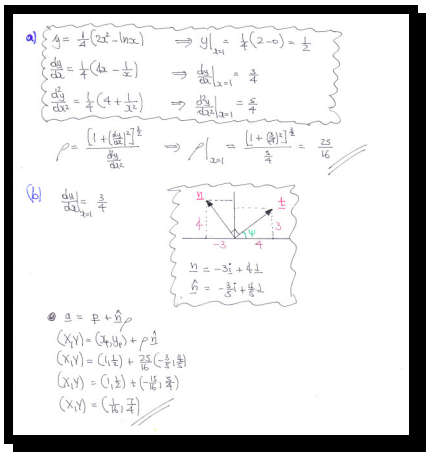
$$y=\frac{1}{2}\left(2x^2-\ln x\right),\ x\in\mathbb{R},\ x>0.$$

The point P lies on C where $x=1$.

a) Determine the radius of curvature at P.

b) Find the exact coordinates of the centre of curvature at P.

$$\boxed{\rho=\frac{25}{16}},\ \boxed{\left(\frac{1}{16},\frac{7}{4}\right)}$$



Question 21 (*+)**

A curve C has Cartesian equation

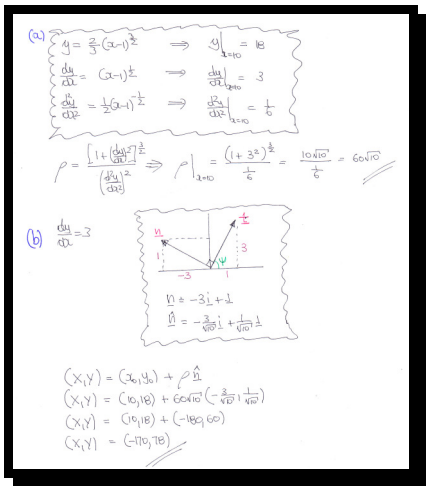
$$y=\frac{2}{3}(x-1)^{\frac{3}{2}},\ x\in\mathbb{R},\ x\geq 1.$$

The point P lies on C where $x=10$.

a) Determine the radius of curvature at P.

b) Find the exact coordinates of the centre of curvature at P.

$$\boxed{\rho=60\sqrt{10}},\ \boxed{(-170,78)}$$



Question 22 (*+)**

A curve C has Cartesian equation

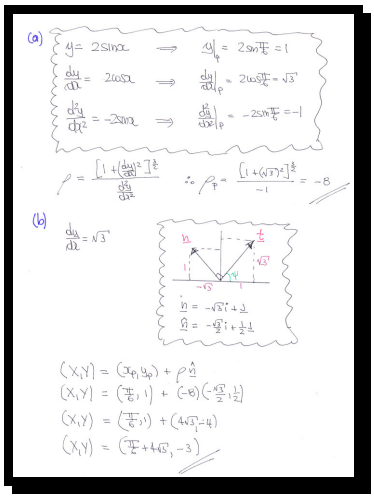
$$y = 2\sin x\,,\ x \in \mathbb{R}\,.$$

The point P lies on C where $x = \dfrac{\pi}{6}$.

a) Determine the radius of curvature at P.

b) Find the exact coordinates of the centre of curvature at P.

$$\boxed{\rho = -8}\,,\ \boxed{\left(\frac{\pi}{6}+4\sqrt{3},-3\right)}$$



Question 23 (*+)**

A cycloid C has parametric equations

$$x = 2t + 2\sin t\,,\ \ y = 2 - 2\cos t\,,\ \ 0 \le t < \frac{\pi}{2}.$$

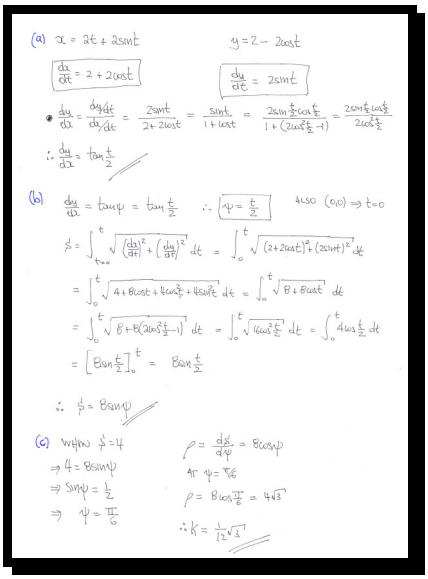
a) Show clearly that

$$\frac{dy}{dx} = \tan\left(\frac{1}{2}t\right).$$

b) Find an intrinsic equation for C, in the form $s = f(\psi)$, where s is measured from a Cartesian origin, and ψ is the angle the tangent to C makes with the positive x axis.

c) Calculate the curvature at the point on C where $s = 4$.

$$\boxed{s = 8\sin\psi}\,,\ \ \boxed{\kappa = \frac{1}{12}\sqrt{3}}$$



Question 24 (*+)**

A curve C has parametric equations

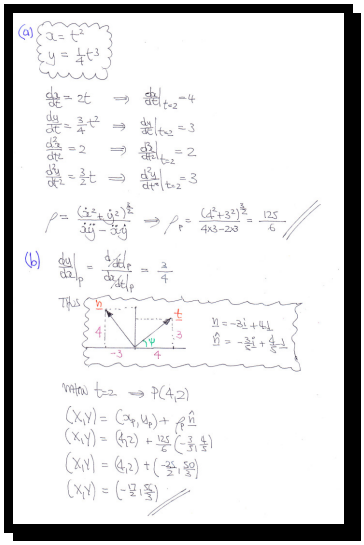
$$x=t^2,\ y=\frac{1}{4}t^3,\ t\in\mathbb{R}.$$

The point P lies on C where $t=2$.

a) Determine the radius of curvature at P.

b) Find the exact coordinates of the centre of curvature at P.

$$\boxed{\rho=\frac{125}{6}},\ \boxed{\left(-\frac{17}{2},\frac{56}{3}\right)}$$



Question 25 (*+)**

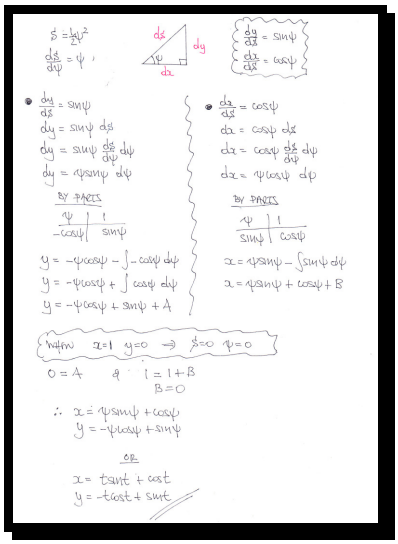
A curve C has intrinsic equation

$$s=\frac{1}{2}\psi^2,$$

where s is measured from the point with Cartesian coordinates $(1,0)$, and ψ is the angle the tangent to C makes with the positive x axis..

Use a calculus method to obtain two parametric equations for C, in terms of a suitable parameter.

$$\boxed{\begin{aligned} x&= t\sin t+\cos t \\ y&=-t\cos t+\sin t \end{aligned}}$$



Question 26 (*+)**

The radius of curvature at a general point on a curve C is given by

$$2\sin\psi,$$

where ψ is the angle the tangent to C makes with the positive x axis..

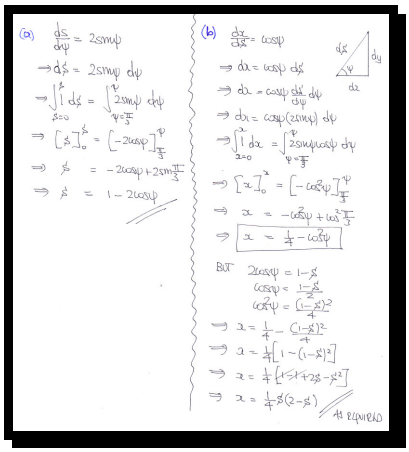
It is further given that the arc length s is measured from the point with Cartesian coordinates $(0,1)$, where the value of ψ at that point is $\frac{\pi}{3}$.

a) Find an intrinsic equation for C, in the form $s=f(\psi)$.

b) Show clearly that

$$x=\frac{1}{4}s(2-s).$$

$$\boxed{s=1-2\cos\psi}$$



Question 27 (*+)**

The radius of curvature at a general point on a curve C is given by

$$2s+1,$$

where s is the arc length measured from the Cartesian origin.

It is further given when $s=0$, $\psi=0$, where ψ is the angle the tangent to C makes with the positive x axis..

a) Find an intrinsic equation for C, in the form $s=f(\psi)$.

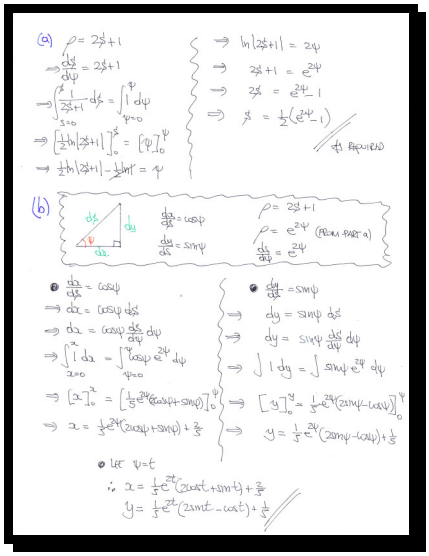
b) Determine a set of parametric equations for C.

You may assume without proof

$$\int \mathrm{e}^{2u}\cos u\ du=\frac{1}{5}\mathrm{e}^{2u}\left(2\cos u+\sin u\right)+\text{constant}$$

$$\int \mathrm{e}^{2u}\sin u\ du=\frac{1}{5}\mathrm{e}^{2u}\left(2\sin u-\cos u\right)+\text{constant}.$$

$$\boxed{s=\frac{1}{2}\left(\mathrm{e}^{2\psi}-1\right)},\quad \boxed{x=\frac{1}{5}\mathrm{e}^{2t}\left(2\cos t+\sin t\right)+\frac{2}{5},\ \ y=\frac{1}{5}\mathrm{e}^{2t}\left(2\sin t-\cos t\right)+\frac{2}{5}}$$



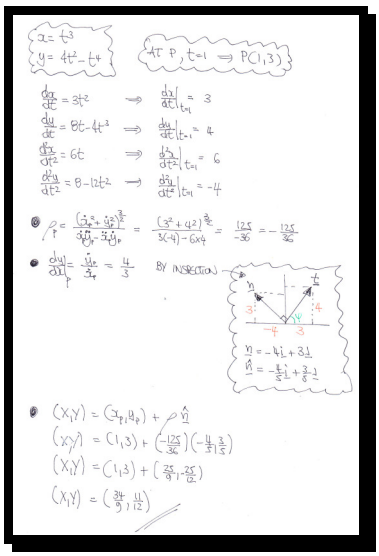
Question 28 (****)

A curve C has parametric equations

$$x = t^3, \ y = 4t^2 - t^4, \ t \in \mathbb{R}.$$

Find the exact coordinates of the centre of curvature at the point P on C where $t = 1$.

$$\boxed{\left(\frac{34}{9}, \frac{11}{12}\right)}$$



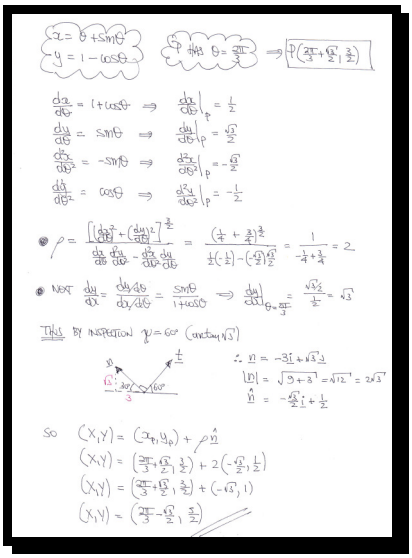
Question 29 (****)

A cycloid C has parametric equations

$$x=\theta+\sin\theta,\ y=1-\cos\theta,\ 0\le\theta\le 2\pi .$$

Find the exact coordinates of the centre of curvature at the point on C where $\theta=\dfrac{2\pi}{3}$.

$$\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2},\frac{5}{2}\right)$$



Question 30 (****)

The gradient at every point on a curve C is given by

$$\frac{dy}{dx}=\frac{1}{2}s,$$

where s is the arc length along C measured from the point P whose Cartesian coordinates are $(0,2)$. It is further given that $\psi=0$ at P, where ψ is the angle the tangent to C makes with the positive x axis.

a) Show clearly that

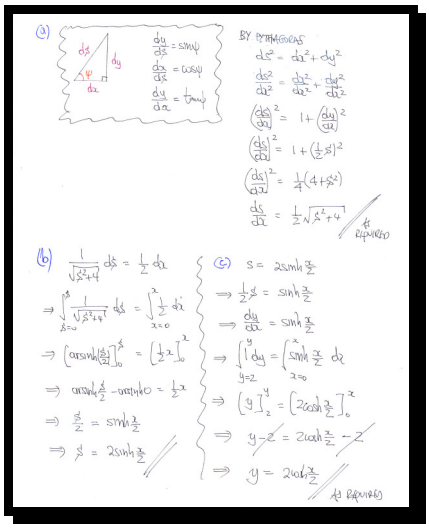
$$\frac{ds}{dx}=\frac{1}{2}\sqrt{s^2+4}.$$

b) Express s as a function of x.

c) Deduce that

$$y=2\cosh\left(\frac{1}{2}x\right).$$

$$\boxed{s=2\sinh\left(\frac{1}{2}x\right)}$$



Question 31 (****)

An ellipse has equation

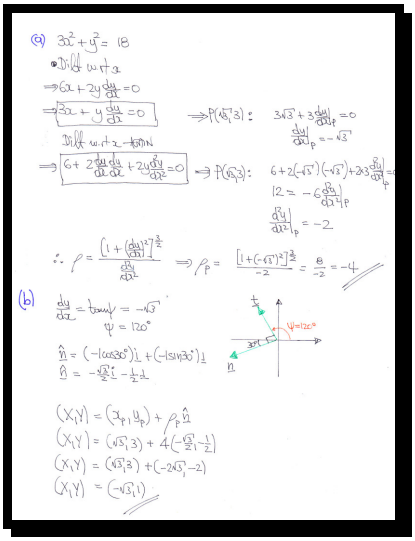
$$3x^2 + y^2 = 18 .$$

The point $P\left(\sqrt{3},3\right)$ lies on C.

a) Determine the radius of curvature at P.

b) Find the exact coordinates of the centre of curvature at P.

$\boxed{\rho = -4}$, $\boxed{\left(-\sqrt{3},1\right)}$



Question 32 (****)

A curve C has intrinsic equation

$$s=\ln\left(\tan\frac{\psi}{2}\right),\ 0<\psi<\pi,$$

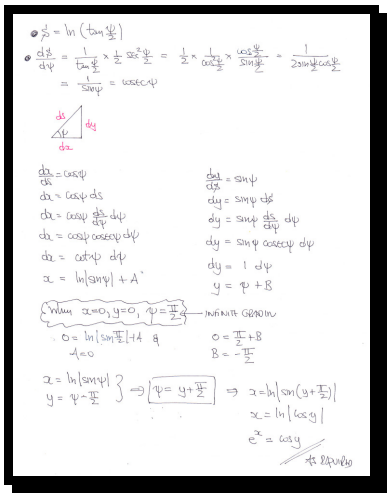
where s is measured from a fixed point, and ψ is the angle the tangent to C makes with the positive x axis.

It is given that the tangent to C at a Cartesian origin has infinite gradient.

Show that a Cartesian equation of C is

$$\mathrm{e}^{x}=\cos y\,.$$

proof



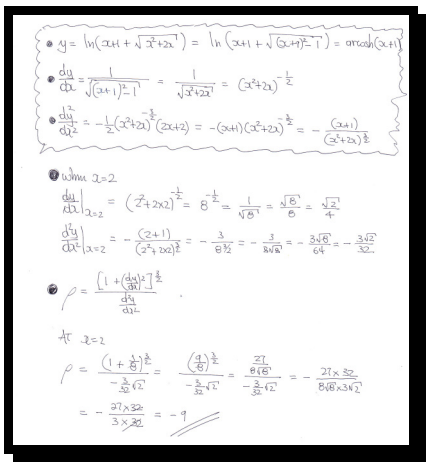
Question 33 (****)

A curve C has Cartesian equation

$$y=\ln\left(x+1+\sqrt{x^2+2x}\right),\ x\in\mathbb{R}.$$

Determine the radius of curvature at the point on C where $x=2$.

$$\boxed{\rho=-9}$$



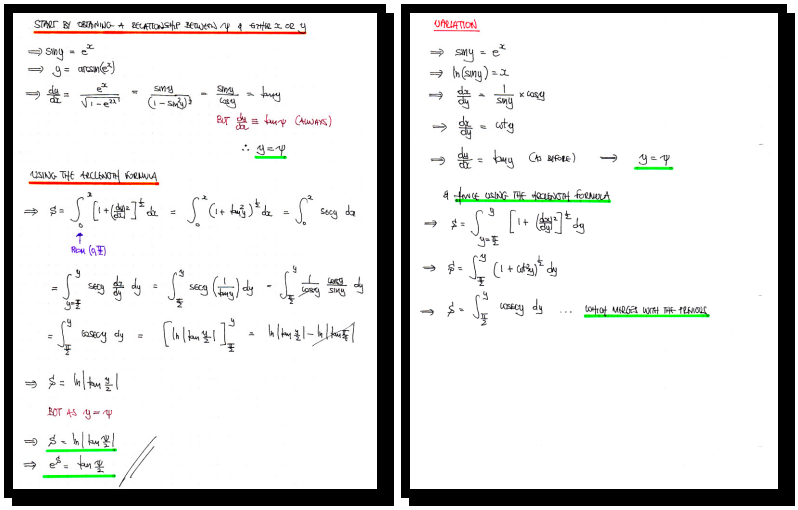
Question 34 (****)

A curve C has Cartesian equation

$$\sin y = \mathrm{e}^x, \ x \leq 0 .$$

Find an intrinsic equation for C, in the form $s = f(\psi)$, where s is measured from the point with Cartesian coordinates $\left(0, \frac{\pi}{2}\right)$, and ψ is the angle the tangent to C makes with the positive x axis.

$\square$, $\boxed{s = \ln\left|\tan\left(\frac{\psi}{2}\right)\right| \quad \text{or} \quad \mathrm{e}^s = \tan\left(\frac{\psi}{2}\right)}$



Question 35 (****)

A curve C has intrinsic equation

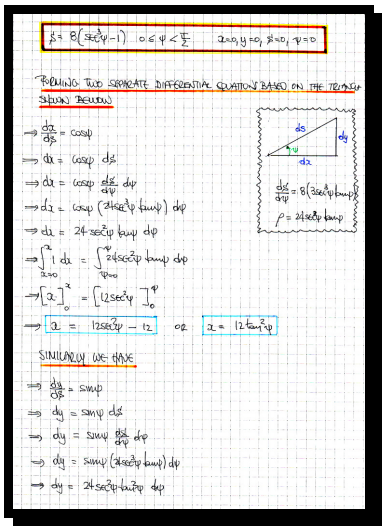
$$s=8\left(\sec^3\psi-1\right),\ 0\le\psi<\frac{\pi}{2},$$

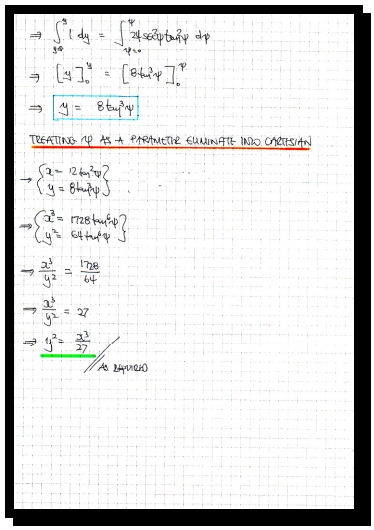
where s is the arc length is measured from a Cartesian origin O, and ψ is the angle the tangent to C makes with the positive x axis. It is further given that $\psi=0$ at the origin O.

Show that a Cartesian equation of C is

$$y^2=\frac{x^3}{27}.$$

[ ], [proof]





Question 36 (****)

A curve C has intrinsic equation

$$s = 4\sin\psi, \quad 0 \le \psi \le \pi,$$

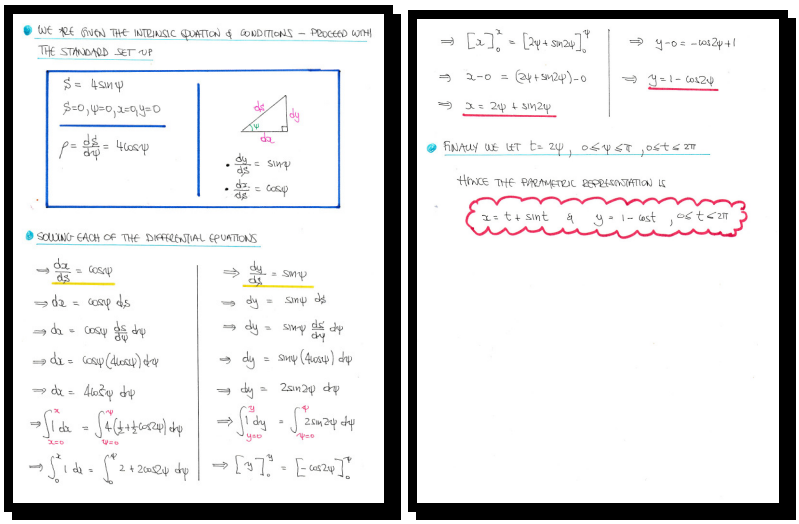
where s is the arc length is measured from the Cartesian origin O, and ψ is the angle the tangent to C makes with the positive x axis.

It is given that the tangent to C at O has zero gradient.

Show that the parametric equations of C are

$$x = t + \sin t, \qquad y = 1 - \cos t, \qquad 0 \le t \le 2\pi.$$

, proof



Question 37 (****)

A curve has Cartesian equation

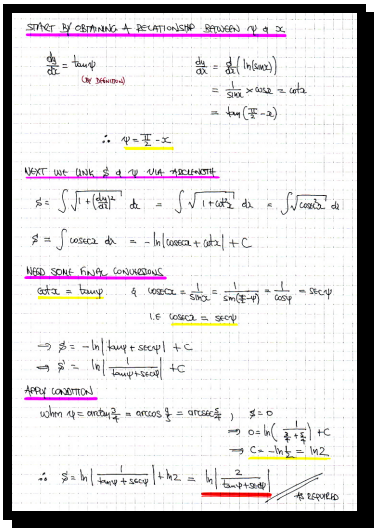
$$y=\ln\left(\sin x\right),\ 0<x<\pi .$$

Show that an intrinsic equation of the curve is

$$s=\ln\left|\frac{2}{\tan\psi+\sec\psi}\right|,$$

where s is the arc length measured from the point where $\psi=\arctan\frac{3}{4}$, where ψ is the angle the tangent to the curve makes with the positive x axis.

$\boxed{}$, $\boxed{\text{proof}}$



Question 38 (****)

A curve C has parametric equations

$$x = 6\tan^2 t\,,\ y = 4\tan^3 t\,,\ 0 \le t < \frac{\pi}{2}.$$

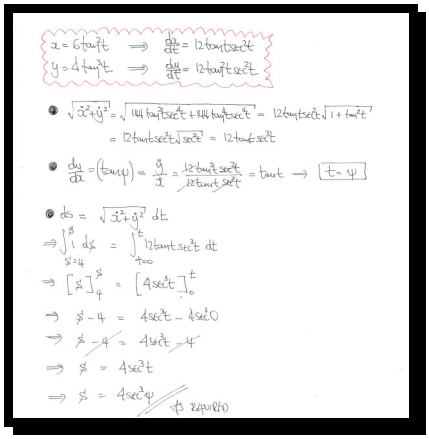
It is further given that when $t = 0$, $s = 4$, where s is the arc length measured from some fixed point.

Show clearly that

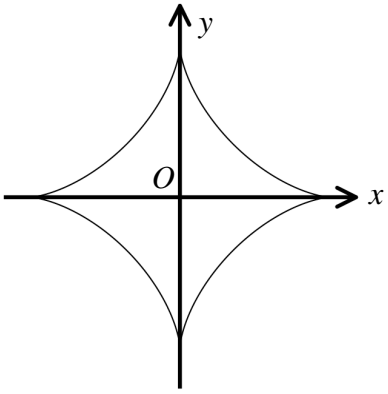
$$s = 4\sec^3 \psi\,,$$

where ψ is the angle the tangent to C makes with the positive x axis.

proof



Question 39 (**+)**



An astroid is given parametrically by

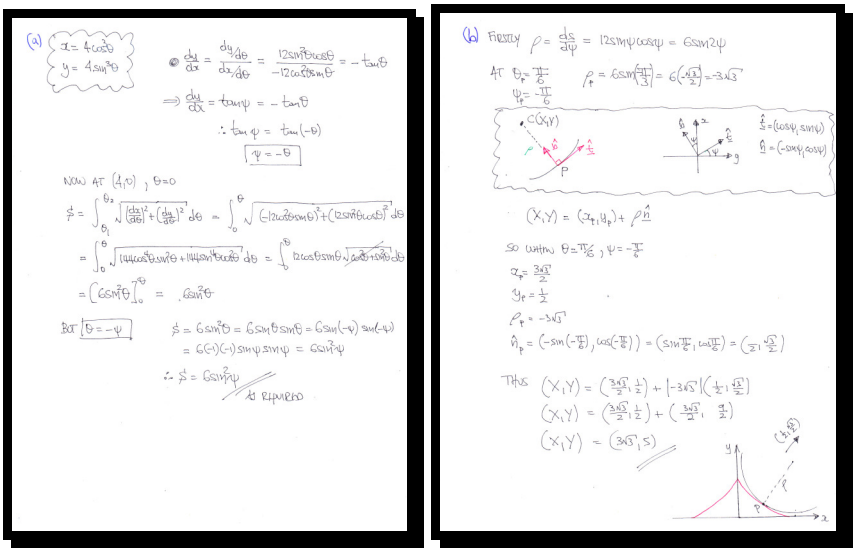
$$x=4\cos^3\theta,\ x=4\sin^3\theta,\ 0\le\theta<2\pi.$$

a) Show that if the arc length s is measured from the point $(4,0)$, and ψ is the angle the tangent to the astroid makes with the positive x axis, then

$$s=6\sin^2\psi.$$

b) Determine the coordinates of the centre of curvature at the point P on the astroid where $\theta=\dfrac{\pi}{6}$

$$\boxed{C\left(3\sqrt{3},5\right)}$$



Question 40 (**+)**

A curve C has parametric equations

$$x = 2\sinh t\,,\ y = \cosh^2 t\,,\ t \in \mathbb{R}\,.$$

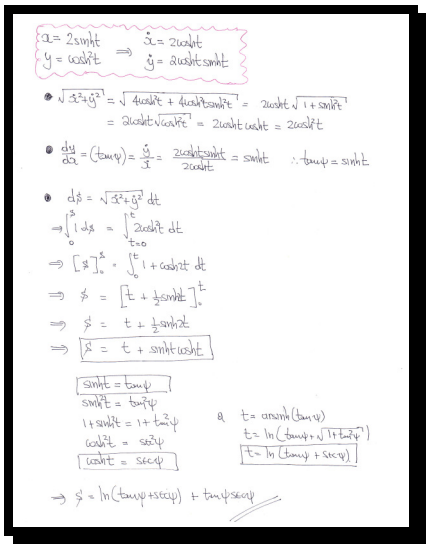
It is further given that the arc length s is measured from the point where $t = 0$.

Show clearly that

$$s = \ln\left(\tan\psi + \sec\psi\right) + \tan\psi\sec\psi\,,$$

where ψ is the angle the tangent to C makes with the positive x axis.

proof



Question 41 (**+)**

The position vector of a curve C is given by

$$\mathbf{r}(t)=\left(\frac{2}{1+t^2}-1\right)\mathbf{i}+\left(\frac{2t}{1+t^2}\right)\mathbf{j},$$

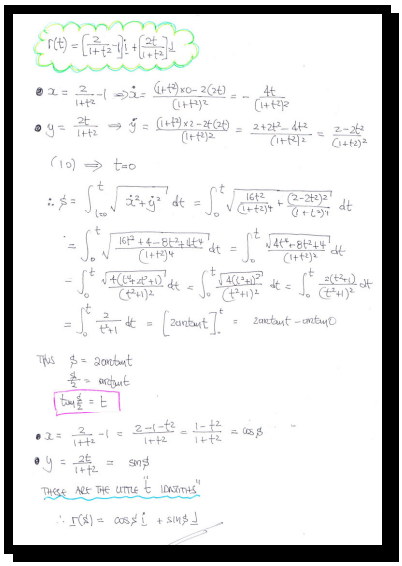
where t is a scalar parameter with $t\in\mathbb{R}$.

Find an expression for the position vector of C, giving the answer in the form

$$\mathbf{r}(s)=f(s)\mathbf{i}+g(s)\mathbf{j},$$

where s is the arc length of a general point on C, measure from the point $(1,0)$.

$$\boxed{\mathbf{r}(s)=(\cos s)\mathbf{i}+(\sin s)\mathbf{j}}$$



Question 42 (**+)**

A curve C has intrinsic equation

$$s=\ln\left(\tan\psi+\sec\psi\right)+\tan\psi\sec\psi\,,\ 0\le\psi<\frac{\pi}{2},$$

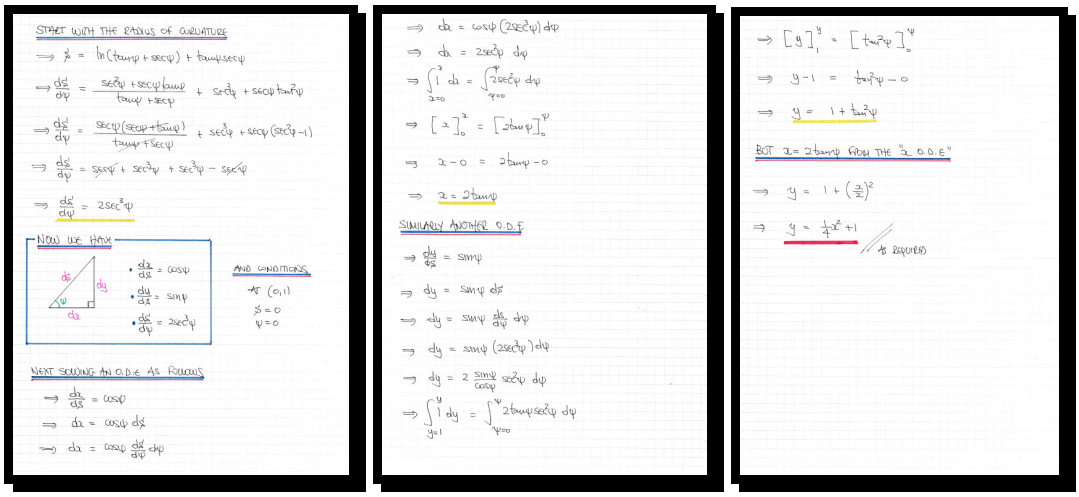
where s is the arc length is measured from the point with Cartesian coordinates $(0,1)$, and ψ is the angle the tangent to C makes with the positive x axis.

It is further given that the gradient at $(0,1)$ is zero.

Show that the Cartesian equation of C is

$$y=\frac{1}{4}x^2+1.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 43 (*****)

The gradient at every point on a curve C is given by

$$\frac{dy}{dx}=\frac{1}{2}s\,,$$

where s is the arc length along C measured from the point P whose Cartesian coordinates are $(0,2)$.

It is further given that $\psi=0$ at P, where ψ is the angle the tangent to C makes with the positive x axis.

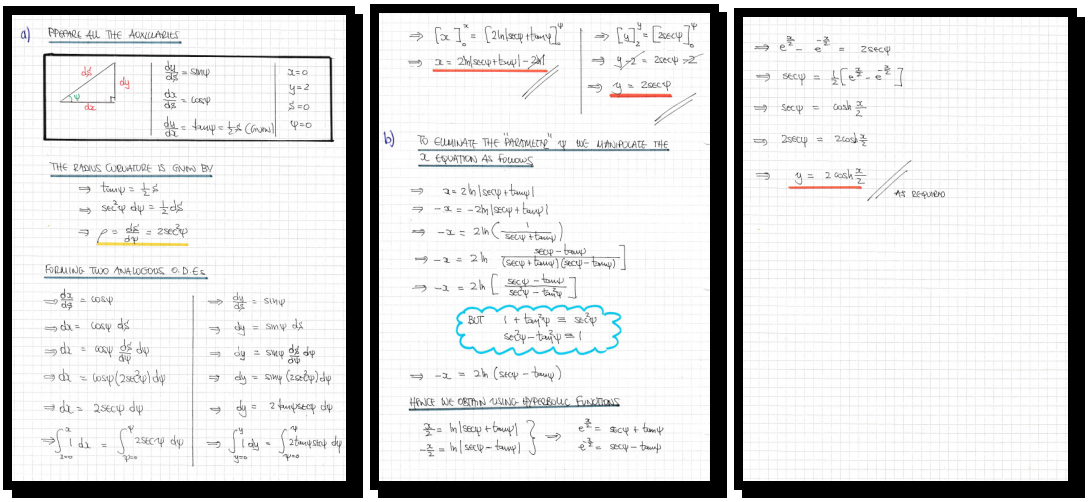
a) Show clearly that

$$x=2\ln\left|\sec\psi+\tan\psi\right|,\qquad y=2\sec\psi\,.$$

b) Eliminate ψ to show further that

$$y=2\cosh\left(\frac{1}{2}x\right).$$

, proof



Question 44 (*****)

The radius of curvature ρ at any point on a curve with Cartesian equation $y=f(x)$ is given by

$$\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

a) Given that the curve can be parameterized as $x=g(t)$, $y=h(t)$, for some parameter t, show that

$$\rho=\frac{\left(\dot{x}^2+\dot{y}^2\right)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}},$$

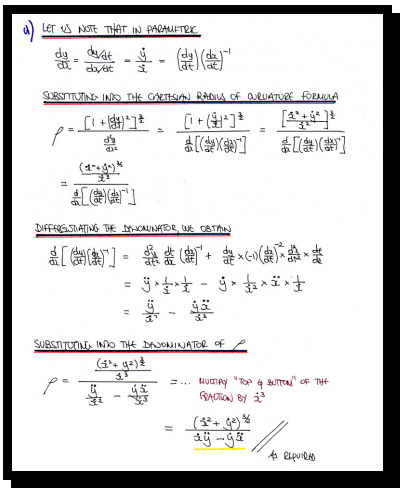
where a dot above a variable denoted differentiation with respect to t.

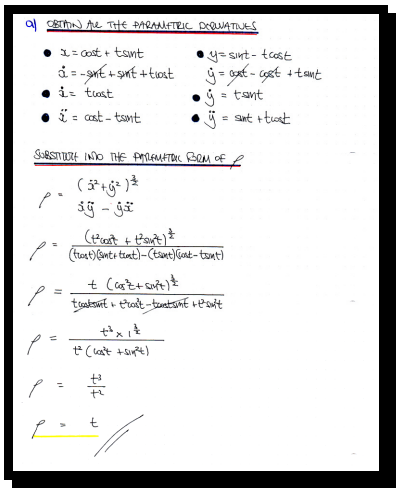
A curve C is given parametrically by

$$x=\cos t+t\sin t,\ y=\sin t-t\cos t,\ 0\le t<2\pi.$$

b) Find an expression for ρ on C, giving the answer in terms of t.

$\square$, $\boxed{\rho=t}$





Question 45 (*****)

A curve C has Cartesian equation $y = f(x)$.

The same curve has intrinsic equation $s = g(\psi)$, where s is measured from an arbitrary point and ψ is the angle the tangent to C makes with the positive x axis.

The radius of curvature ρ at any point on C is defined as $\dfrac{ds}{d\psi}$.

a) Show clearly that $\rho = \dfrac{\left(1+\left(\dfrac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$

b) Given that C can be suitably parameterized as $x = h_1(t)$, $y = h_2(t)$, for some parameter t, show further that

$$\rho = \frac{\left(\dot{x}^2+\dot{y}^2\right)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}},$$

where a dot above a variable denotes differentiation with respect to t.

[ ], [proof]

