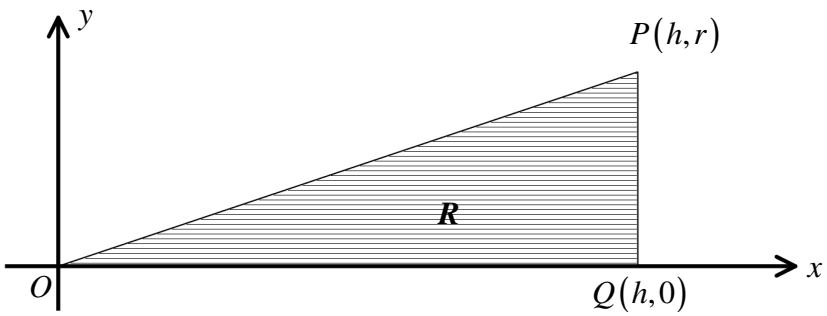
FURTHER INTEGRATION

Question 1



The figure above shows the straight line segment OP, joining the origin to the point $P(h,r)$, where h and r are positive coordinates.

The point $Q(h,0)$ lies on the x axis.

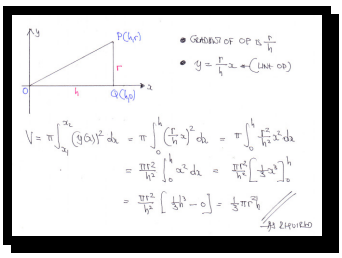
The shaded region R is bounded by the line segments OP, PQ and OQ.

The region R is rotated by 2π radians about the x axis to form a solid cone of height h and radius r.

Show by integration that the volume of the cone V is given by

$$V = \frac{1}{3}\pi r^2 h .$$

proof



Question 2

A curve C is defined parametrically

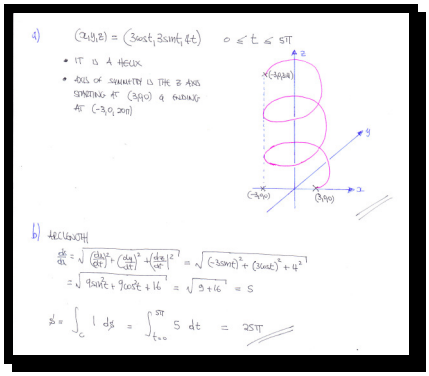
$$(x, y, z) = (3\cos t, 3\sin t, 4t), \quad 0 \le t \le 5\pi .$$

where t is a parameter.

a) Sketch the graph of C.

b) Find the length of C.

$\boxed{25\pi}$



Question 3

A finite region R is defined by the inequalities

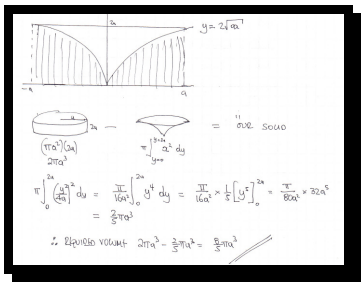
$$y^2 \leq 4ax,\ 0 \leq x \leq a,\ y \geq 0,$$

where a is a positive constant.

The region R is rotated by 2π radians in the y axis forming a solid of revolution.

Determine, in terms of π and a, the exact volume of this solid.

$$\boxed{\frac{8}{5}\pi a^3}$$



Question 4

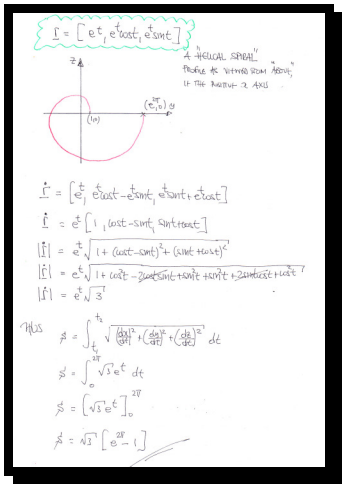
A curve C is defined parametrically

$$(x, y, z) = \left(\mathrm{e}^t, \mathrm{e}^t \cos t, \mathrm{e}^t \sin t\right), \quad 0 \le t \le 2\pi .$$

where t is a parameter.

Describe the graph of C and find its length.

$$\boxed{\text{arclength} = \sqrt{3}\left[\mathrm{e}^{2\pi} - 1\right]}$$



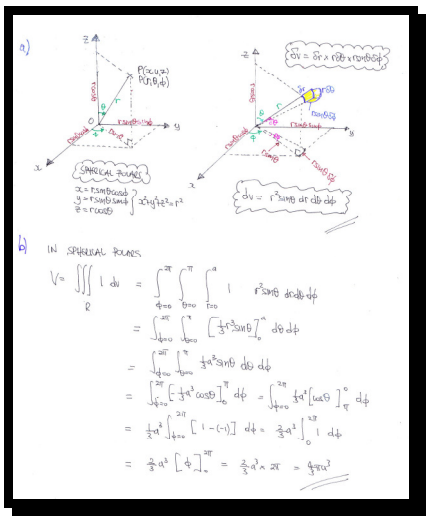
Question 5

a) Determine with the aid of a diagram an expression for the volume element in spherical polar coordinates, (r,θ,φ).

[*You may not use Jacobians in this part*]

b) Use spherical polar coordinates to obtain the standard formula for the volume of a sphere of radius a.

$$\boxed{dv = r^2 \sin\theta \, dr \, d\theta \, d\varphi}$$



Question 6

A family of curves C_n , $n = 1, 2, 3, 4, \ldots$ is defined parametrically by

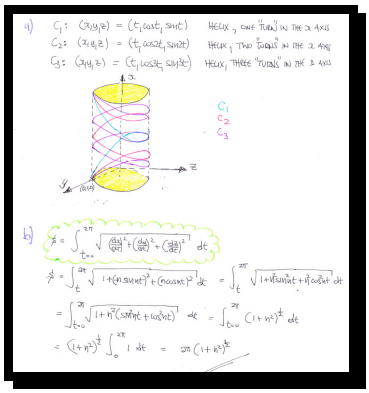
$$C_n : (x, y, z) = (t, \cos nt, \sin nt), \quad 0 \le t \le 2\pi .$$

where t is a parameter.

a) Sketch the graph of C_1, C_2 and C_3.

b) Find an expression for the length of C_n.

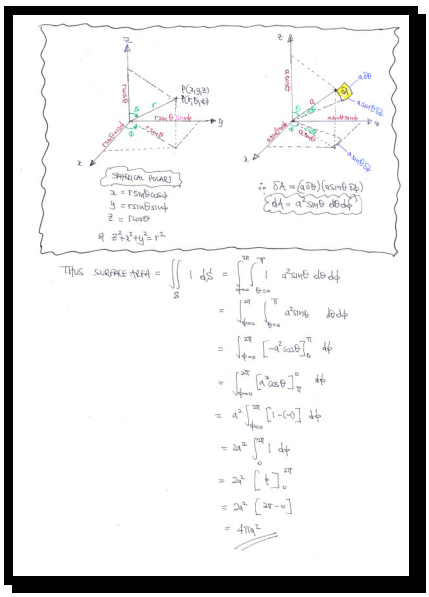
$$\boxed{2\pi\sqrt{1+n^2}}$$



Question 7

Use spherical polar coordinates, (r,θ,φ), to obtain the standard formula for the surface area of a sphere of radius a.

$$\boxed{dA = a^2 \sin\theta \, d\theta \, d\varphi \Rightarrow A = 4\pi a^2}$$



Question 8

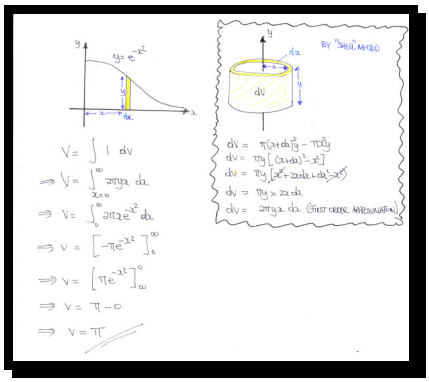
The infinite region R is defined by the inequalities.

$$y \le \mathrm{e}^{-x^2}\,,\ x \ge 0\,,\ y \ge 0\,.$$

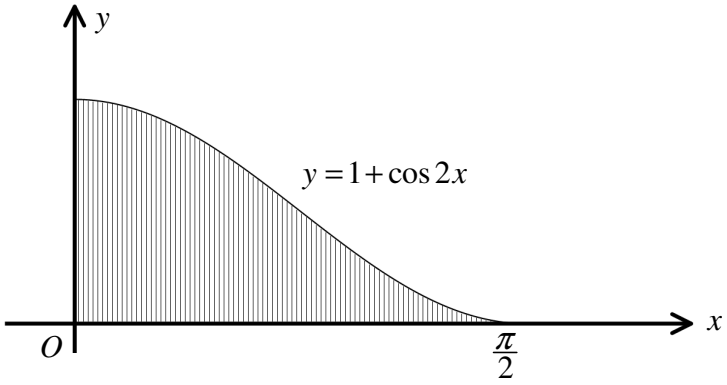
R is rotated by 2π radians in the y axis forming a solid of revolution.

Determine the exact volume of this solid.

$$\boxed{V = \pi}$$



Question 9



The figure above shows the graph of the curve with equation

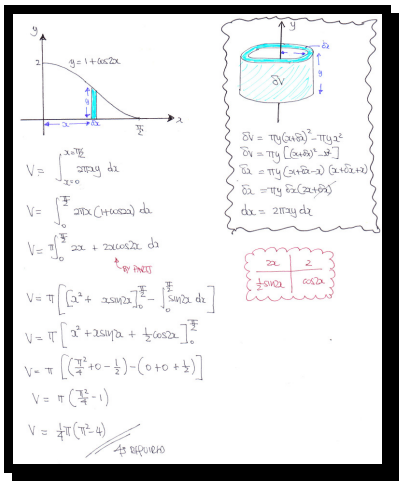
$$y = 1 + \cos 2x \,, \ 0 \leq x \leq \frac{\pi}{2} .$$

The shaded region bounded by the curve and the coordinate axes is rotated by 2π radians about the y axis to form a solid of revolution.

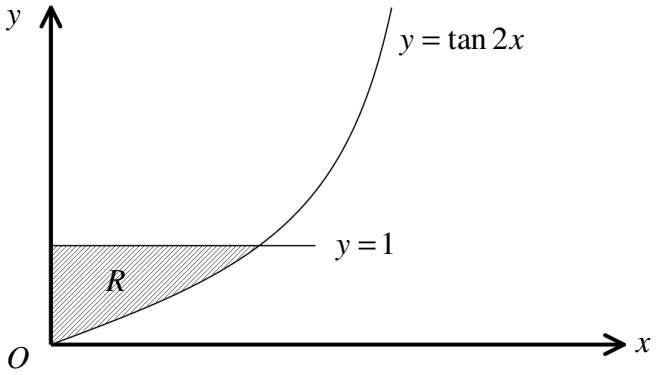
Show that the volume of the solid is

$$\frac{1}{4}\pi\left(\pi^2 - 4\right).$$

$\boxed{}$, proof



Question 10



The figure above shows the graph of the curve with equation

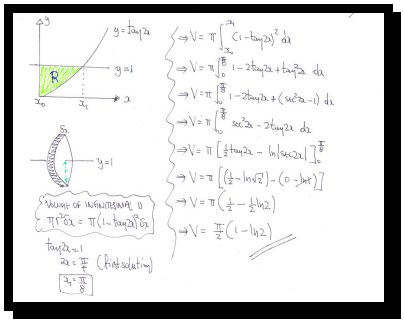
$$y = \tan 2x\,,\ 0 \le x \le \frac{\pi}{4}\,.$$

The finite region R is bounded by the curve, the y axis and the horizontal line with equation $y = 1$.

The region R is rotated by 2π radians about the line with equation $y = 1$ forming a solid of revolution.

Determine an exact volume for this solid.

$$\boxed{\frac{\pi}{2}\left(1 - \ln 2\right)}$$



Question 11

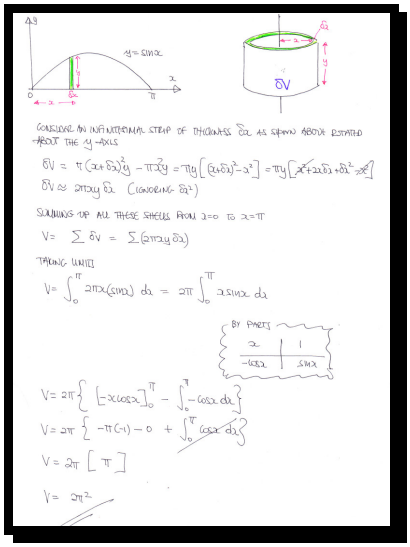
The finite region bounded the curve with equation

$$y = \sin x, \ 0 \leq x \leq \pi$$

and the x axis, is rotated by $360°$ about the y axis to form a solid of revolution.

Find, in exact form, the volume of the solid.

$$\boxed{V = 2\pi^2}$$



Question 12

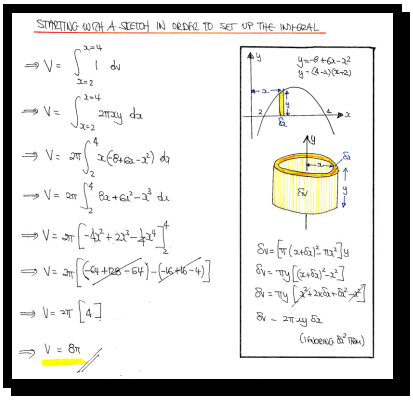
A quadratic curve C has equation

$$y=(4-x)(x-2), \quad x\in\mathbb{R}.$$

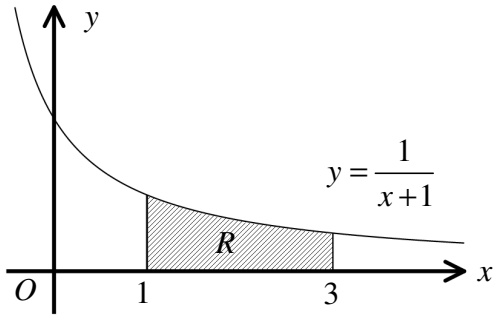
The finite region bounded by C and the x axis is fully revolved about the y axis, forming a solid of revolution S.

Determine in exact form the volume of S.

$\boxed{}$, $\boxed{V=8\pi}$



Question 13



The figure above shows the graph of the curve with equation

$$y=\frac{1}{x+1},\ x\in\mathbb{R},\ x\neq -1.$$

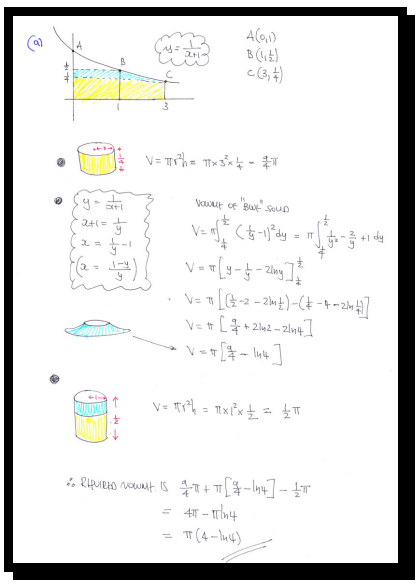
The finite region R is bounded by the curve, the x axis and the lines with equations $x=1$ and $x=3$.

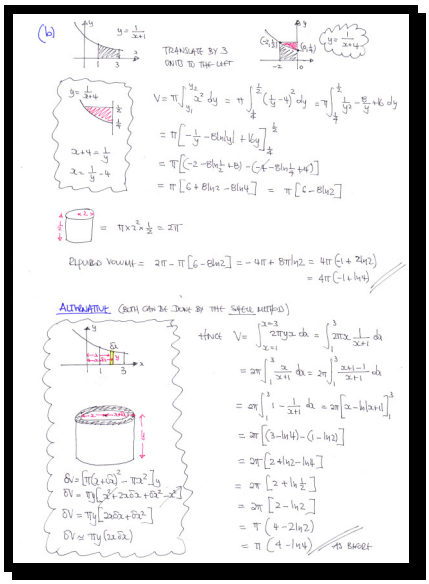
Determine the exact volume of the solid formed when the region R is revolved by 2π radians about …

a) … the y axis.

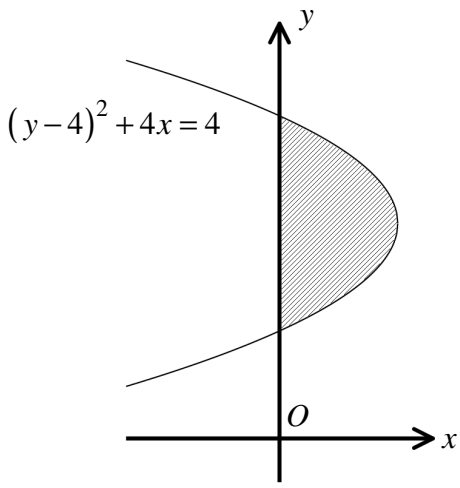
b) … the straight line with equation $x=3$.

$\boxed{\pi(4-\ln 4)}$, $\boxed{4\pi(-1+\ln 4)}$





Question 14



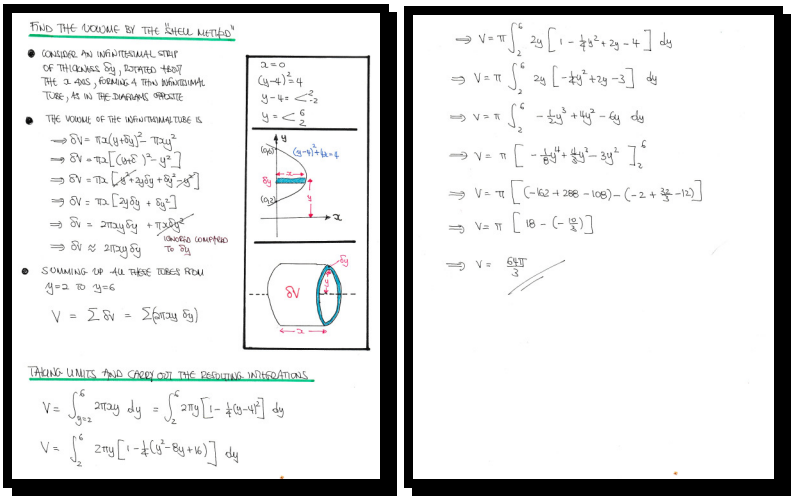
The figure above shows the curve with equation

$$(y-4)^2+4x=4\,.$$

The finite region bounded the curve and the y axis, shown shaded in the figure, is rotated by a full turn about the x axis to form a solid of revolution.

Find, in exact form, the volume of the solid.

$$V=\frac{64\pi}{3}$$



Question 15

A tube in the shape of a right circular cylinder of radius 4 m and height 0.5 m, emits heat from its curved surface only.

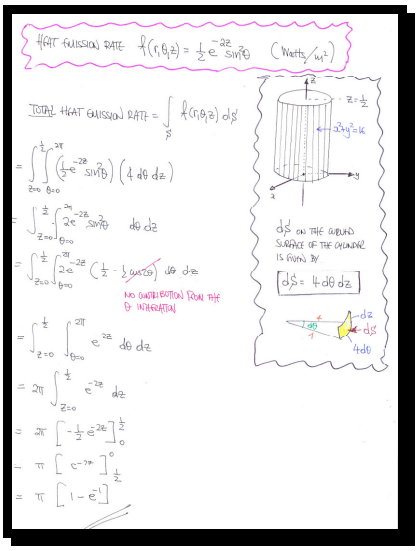
The heat emission rate, in Wm^{-2}, is given by

$$\frac{1}{2}\text{e}^{-2z}\sin^2\theta,$$

where θ and z are standard cylindrical polar coordinates, whose origin is at the centre of one of the flat faces of the cylinder.

Given that the cylinder is contained in the part of space for which $z \geq 0$, determine the total heat emission rate from the tube.

$$\boxed{\pi\left(1-\text{e}^{-1}\right)}$$



Question 16

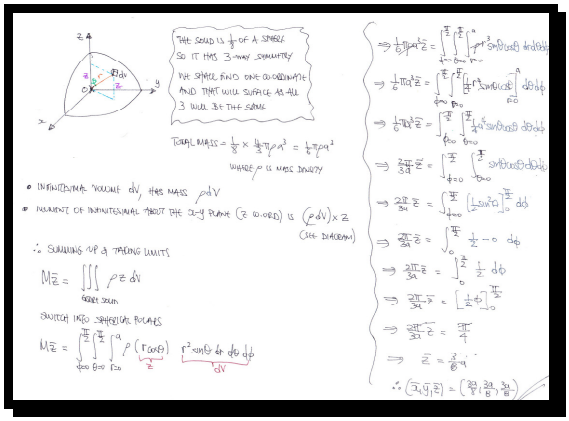
A uniform solid has equation

$$x^2 + y^2 + z^2 = a^2 ,$$

with $x > 0$, $y > 0$, $z > 0$, $a > 0$.

Use integration in spherical polar coordinates, (r, θ, φ), to find in Cartesian form the coordinates of the centre of mass of the solid.

$$\left(\frac{3}{8}a, \frac{3}{8}a, \frac{3}{8}a \right)$$



Question 17

A hemispherical surface, of radius a m, is electrically charged.

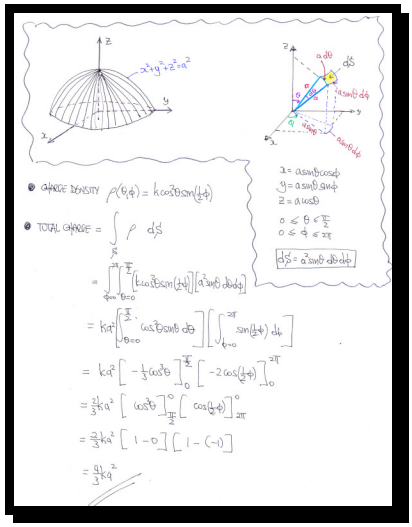
The electric charge density $\rho(\theta,\varphi)$, in C m^{-2}, is given by

$$\rho(\theta,\varphi) = k\cos^2(\theta)\sin\left(\frac{1}{2}\varphi\right),$$

where k is a positive constant, and θ and φ are standard spherical polar coordinates, whose origin is at the centre of the flat open face of the hemisphere.

Given that the hemisphere is contained in the part of space for which $z \geq 0$, determine the total charge on its surface.

$$\boxed{\frac{3}{4}ka^2}$$



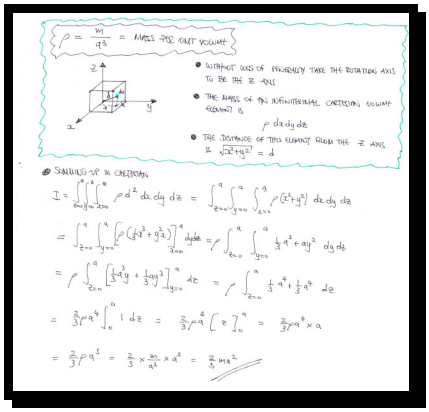
Question 18

A uniform solid cube, of mass m and side length a, is free to rotate about one of its edges, L.

Use multiple integration in Cartesian coordinates, to find the moment of inertia of this cube about L, giving the answer in terms of m and a.

*You may **not** use any standard rules or standard results about moments of inertia in this question apart from the definition of moment of inertia.*

$$\boxed{\frac{2}{3}ma^2}$$



Question 19

A hemispherical solid piece of glass, of radius a m, has small air bubbles within its volume.

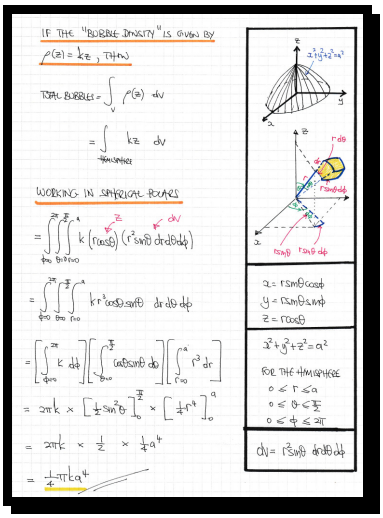
The air bubble density $\rho(z)$, in m^{-3}, is given by

$$\rho(z) = k z ,$$

where k is a positive constant, and z is a standard cartesian coordinate, whose origin is at the centre of the flat face of the solid.

Given that the solid is contained in the part of space for which $z \geq 0$, determine the total number of air bubbles in the solid.

$\boxed{}$, $\boxed{\frac{1}{4}\pi k a^4}$



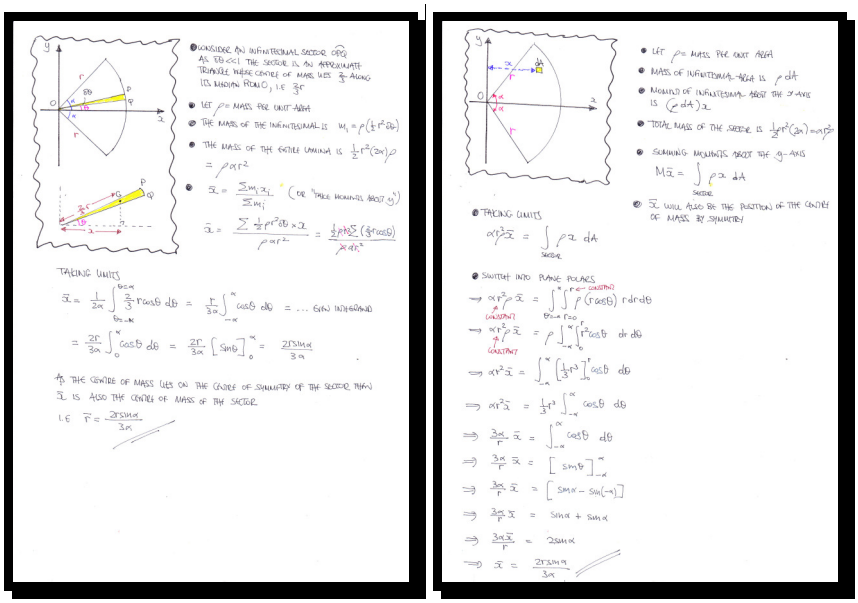
Question 20

A circular sector of radius r subtends an angle of 2α at its centre O. The position of the centre of mass of this sector lies at the point G, along its axis of symmetry.

Use calculus to show that

$$|OG| = \frac{2r\sin\alpha}{3\alpha}.$$

proof



Question 21

A hemispherical solid piece of glass, of radius a m, has small air bubbles within its volume.

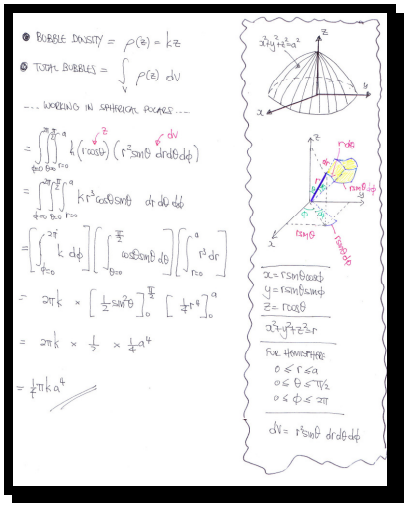
The air bubble density $\rho(z)$, in m^{-3}, is given by

$$\rho(z) = k\,z\,,$$

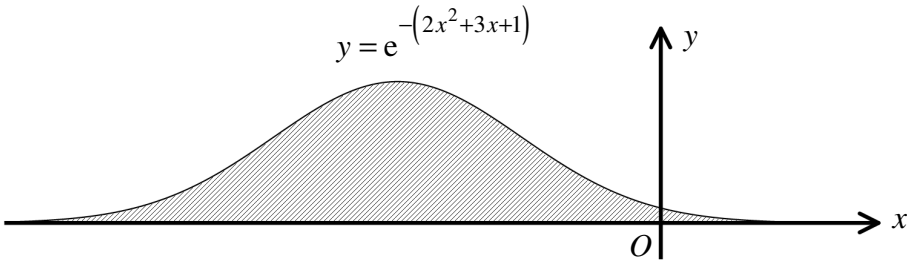
where k is a positive constant, and z is a standard cartesian coordinate, whose origin is at the centre of the flat face of the solid.

Given that the solid is contained in the part of space for which $z \geqslant 0$, determine the total number of air bubbles in the solid.

$\boxed{}$, $\boxed{\frac{1}{4}\pi k a^4}$



Question 22

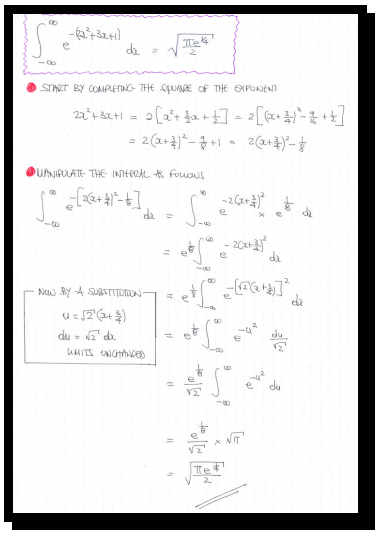


The figure above shows the curve with equation

$$y = \mathrm{e}^{-\left(2x^2+3x+1\right)}, \quad x \in \mathbb{R}.$$

Show that the area between the curve and the x axis is $\sqrt{\frac{1}{2}\pi \mathrm{e}^{\frac{1}{4}}}$

proof



Question 23

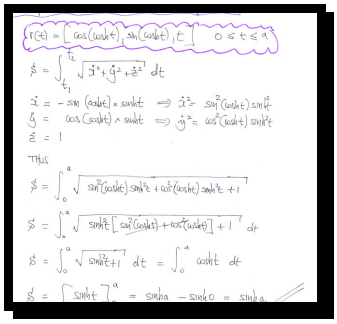
The position vector of a curve C is given by

$$\mathbf{r}(t)=\cos(\cosh t)\mathbf{i}+\sin(\cosh t)\mathbf{j}+t\mathbf{k}\,,$$

where t is a scalar parameter with $0\le t\le a$, $a\in\mathbb{R}$.

Determine the length of C.

$$\boxed{\text{arclength}=\sinh a}$$



Question 24

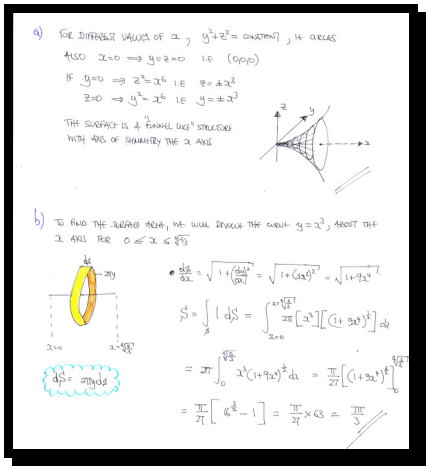
A surface S is has Cartesian equation

$$y^2 + z^2 = x^6, \quad 0 \le x \le \sqrt[4]{\frac{5}{3}}.$$

a) Sketch the graph of S.

b) Find the area of S.

$$\boxed{\frac{7}{3}\pi}$$



Question 25

A solid sphere has equation

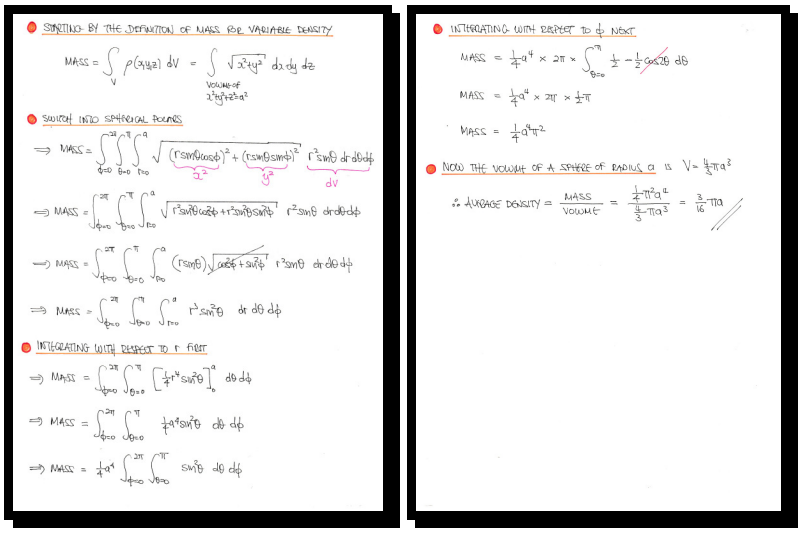
$$x^2 + y^2 + z^2 = a^2 .$$

The density, ρ, at the point of the sphere with coordinates (x_1, y_1, z_1) is given by

$$\rho = \sqrt{x_1^2 + y_1^2} .$$

Determine the **average** density of the sphere.

$\boxed{}$, $\boxed{\bar{\rho} = \frac{3}{16}\pi a}$



Question 26

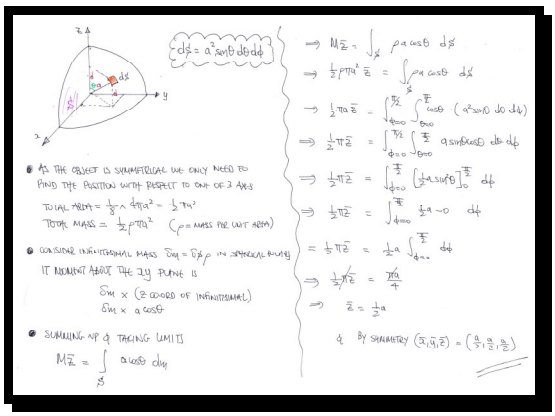
A thin uniform spherical shell with equation

$$x^2 + y^2 + z^2 = a^2 \, , \; a > 0 \, ,$$

occupies the region in the first octant.

Use integration in spherical polar coordinates, (r, θ, φ), to find in Cartesian form the coordinates of the centre of mass of the shell.

$$\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2} \right)$$



Question 27

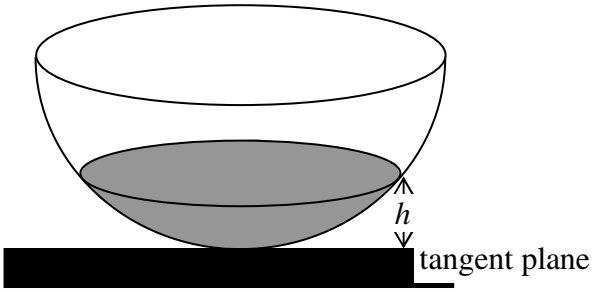


figure 1

Figure 1 shows a hemispherical bowl of radius r cm containing water up to a certain level h cm. The shape of the water in the bowl is called a spherical segment.

It is required to find a formula for the volume of a spherical segment as a function of the radius r cm and the distance of its plane face from the tangent plane, h cm.

The circle with equation

$$x^2 + y^2 = r^2,\ x \geq 0$$

is to be used to find a formula for the volume of a spherical segment.

The part of the circle in the first quadrant between $x = r - h$ and $x = r$ is shown shaded in figure 2, and is labelled as the region R.

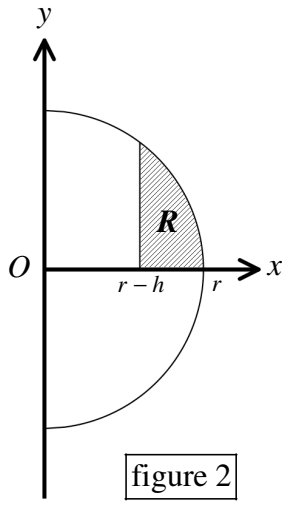


figure 2

[continues overleaf]

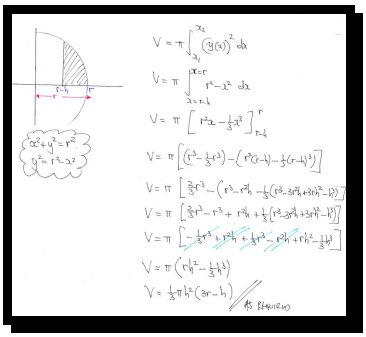
[continued from overleaf]

Show by integration that the volume of the spherical segment V is given by

$$V=\frac{1}{3}\pi h^2(3r-h),$$

where r is the radius of the sphere or hemisphere and h is the distance of its plane face from the tangent plane.

proof



Question 28

A thin plate occupies the region in the x-y plane with equation

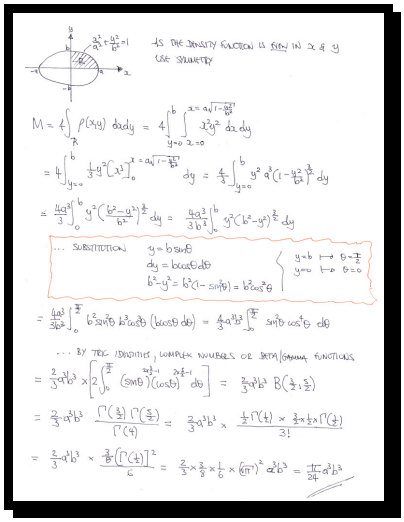
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,.$$

The mass per unit area of the plate ρ, is given by

$$\rho(x,y)=x^2y^2\,.$$

Find a simplified expression for the mass of the plate.

$$\frac{\pi a^3 b^3}{24}$$

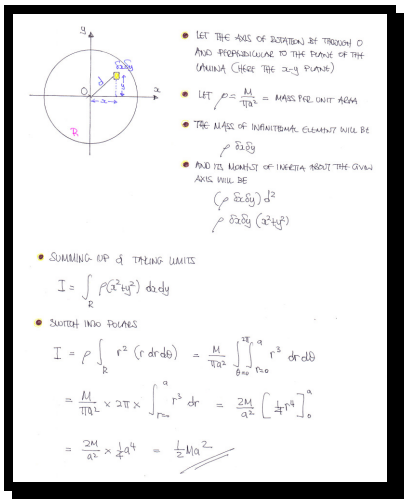


Question 29

A uniform circular lamina has mass M and radius a.

Use double integration in plane polar coordinates to find the moment of inertia of the lamina, when the axis of rotation is perpendicular to the plane of the lamina and passes through its centre.

$$\boxed{\frac{1}{2}Ma^2}$$

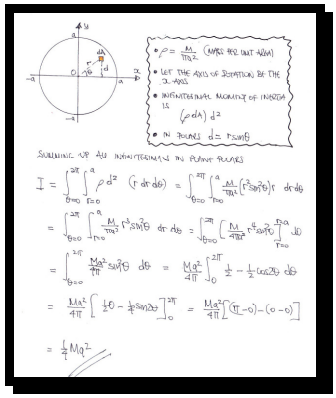


Question 30

A uniform circular lamina has mass M and radius a.

Use double integration to find the moment of inertia of the lamina, when the axis of rotation is a diameter.

$$\boxed{\frac{1}{4}Ma^2}$$

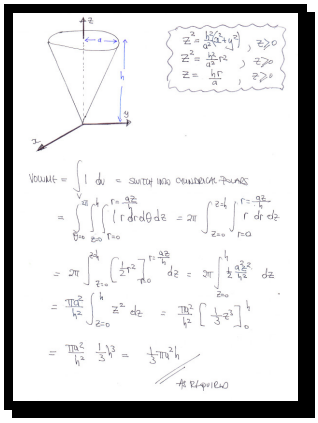


Question 31

Use cylindrical polar coordinates (r,θ,z) to show that the volume of a right circular cone of height h and base radius a is

$$\frac{1}{3}\pi a^2 h\,.$$

proof



Question 32

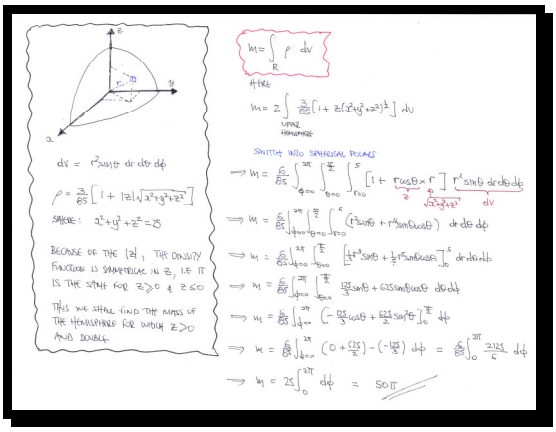
A solid sphere has radius 5 and is centred at the Cartesian origin O.

The density ρ at point $P(x_1, y_1, z_1)$ of the sphere satisfies

$$\rho = \frac{3}{85}\left[1+|z_1|\sqrt{x_1^2+y_1^2+z_1^2}\right].$$

Use spherical polar coordinates, (r,θ,φ), to find the mass of the sphere.

$$\boxed{m = 50\pi}$$

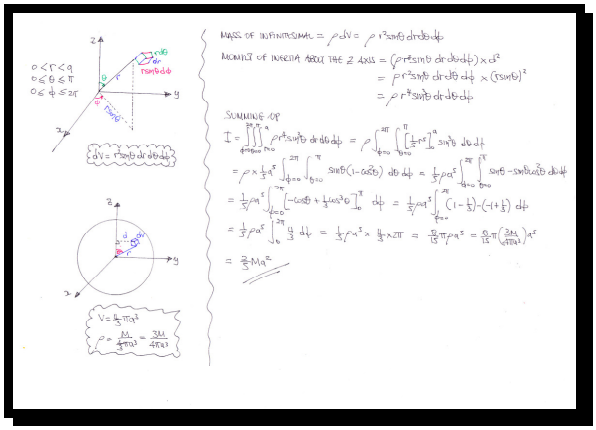


Question 33

A solid uniform sphere has mass M and radius a.

Use spherical polar coordinates, (r,θ,φ), to show that the moment of inertial of this sphere about one of its diameters is $\frac{2}{5}Ma^2$.

proof

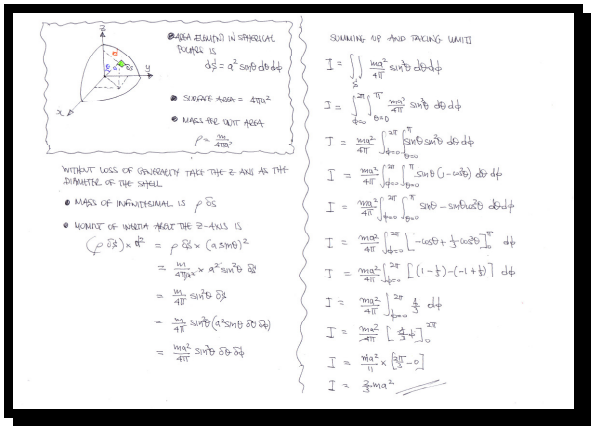


Question 34

A thin uniform spherical shell has mass m and radius a.

Use spherical polar coordinates, (r,θ,φ), to show that the moment of inertial of this spherical shell about one of its diameters is $\frac{2}{3}ma^2$.

proof



Question 35

A building whose plan measures 10 m long by 10 m wide has vertical walls and a suspended fabric roof. The height, z m, of the roof above the ground is modelled in three dimensional Cartesian space by the equation

$$z = \frac{y\left(x^2 + y\right)}{50} + 2\,, \quad -5 \le x \le 5\,, \quad 0 \le y \le 10\,.$$

a) Sketch the graph of the surface which models the roof of the building.

Give a brief description of its shape including its key features with relevant coordinates such as the maximum height and minimum height of the roof.

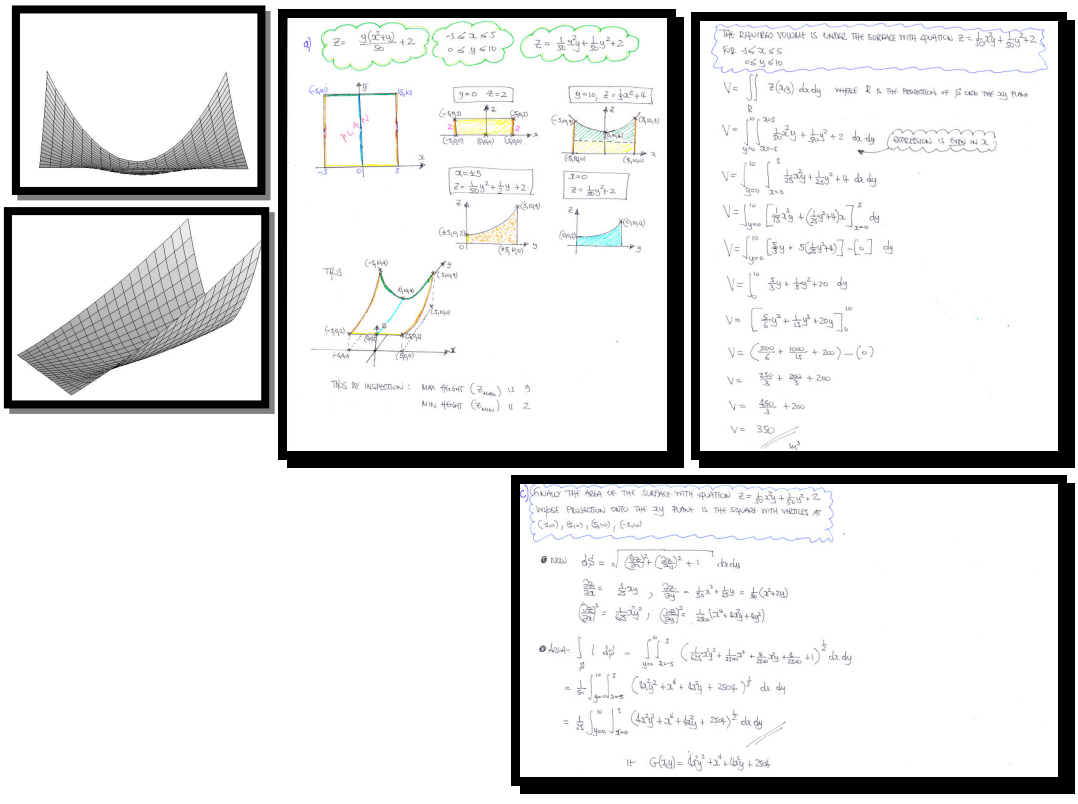
b) Determine the volume of the building enclosed by vertical walls and the suspended fabric roof.

c) Show that the area of the fabric roof is given by

$$\frac{1}{25}\int_{y=0}^{10}\int_{x=0}^{5}\sqrt{G\left(x,y\right)}\,dxdy\,,$$

where $G\left(x,y\right)$ is a function to be found.

$\boxed{\text{Volume} = 350}$, $\boxed{G\left(x,y\right) = 4x^2y^2 + x^4 + 4x^2y + 2504}$



Question 36

A thin plate occupies the region in the x-y plane defined by the inequalities

$$0 \le x \le 2 \quad \text{and} \quad 0 \le y \le 2x .$$

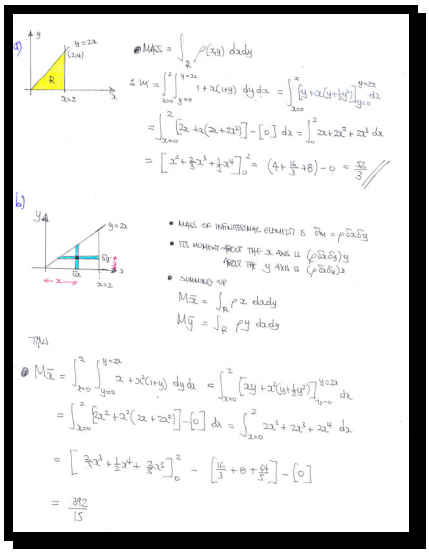
The mass per unit area of the plate ρ, is given by

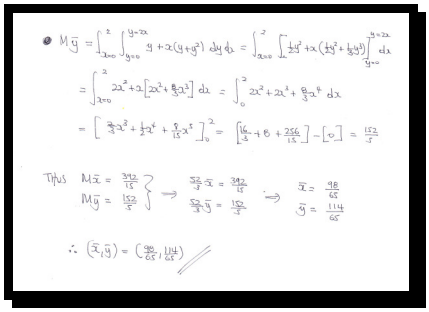
$$\rho(x, y) = 1 + x(1 + y) .$$

a) Find the mass of the plate.

b) Determine the coordinates of the centre of mass of the plate.

$$\boxed{m = \frac{52}{3}}, \quad \boxed{(\bar{x}, \bar{y}) = \left(\frac{98}{65}, \frac{114}{65}\right)}$$





Question 37

The position vector of a curve C is given by

$$\mathbf{r}(t)=\left(\frac{2}{1+t^2}-1\right)\mathbf{i}+\left(\frac{2t}{1+t^2}\right)\mathbf{j},$$

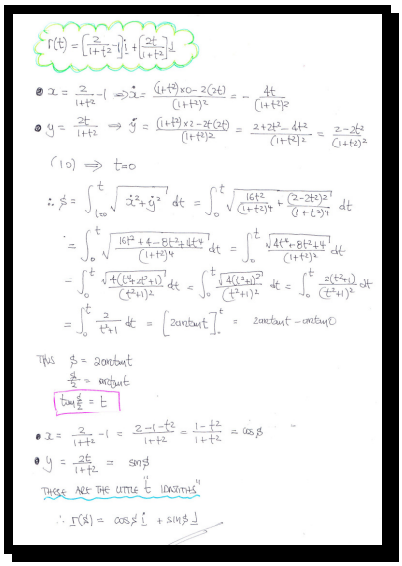
where t is a scalar parameter with $t\in\mathbb{R}$.

Find an expression for the position vector of C, giving the answer in the form

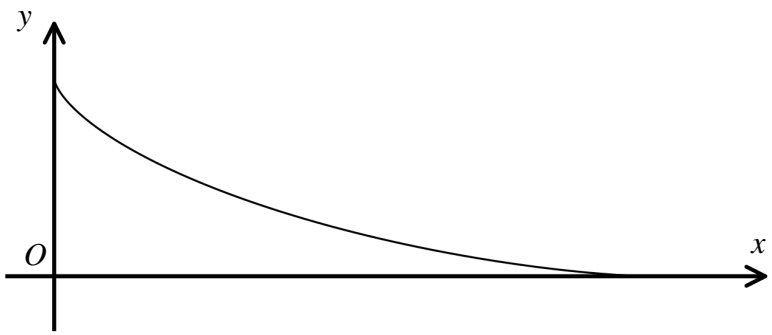
$$\mathbf{r}(s)=f(s)\mathbf{i}+g(s)\mathbf{j},$$

where s is the arc length of a general point on C, measured from the point $(1,0)$.

$$\boxed{\mathbf{r}(s)=(\cos s)\mathbf{i}+(\sin s)\mathbf{j}}$$



Question 38



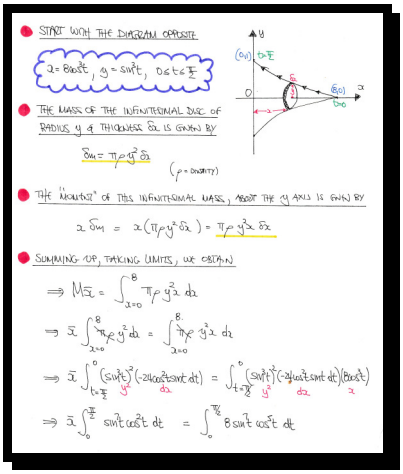
The figure above shows the curve with parametric equations

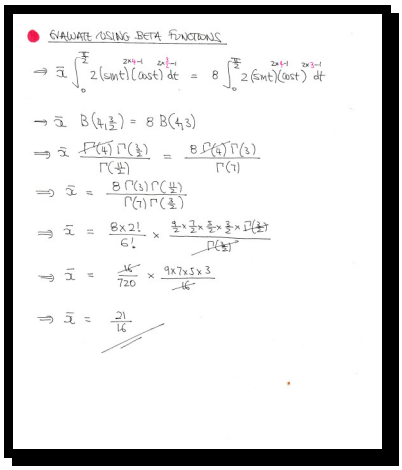
$$x = 8\cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le \frac{1}{2}\pi .$$

The finite region bounded by the curve and the coordinate axes is revolved fully about the x axis, forming a solid of revolution S.

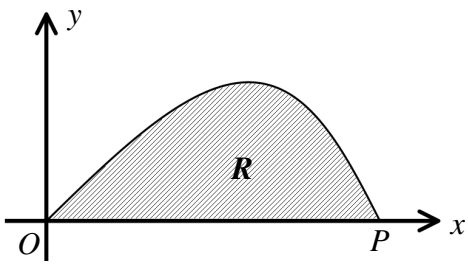
Determine the x coordinate of the centre of mass of S.

$\square$, $\boxed{\bar{x} = \frac{21}{16}}$





Question 39



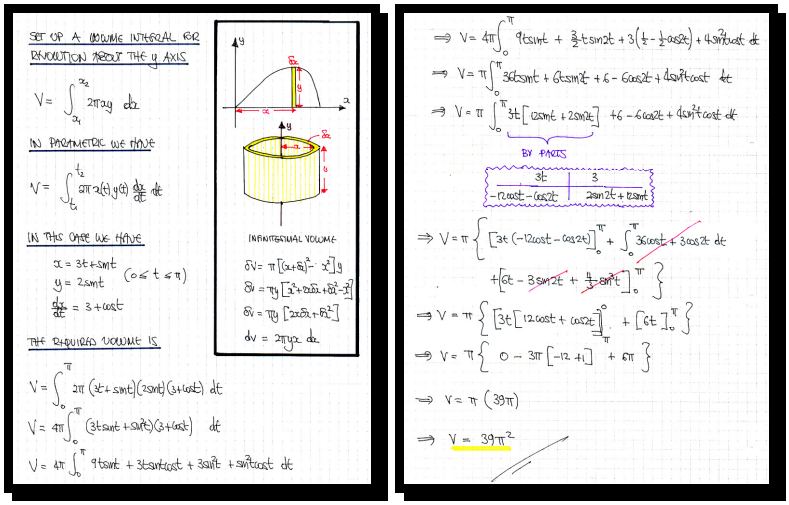
The figure above shows the finite region R, bounded by the coordinate axes and the curve with parametric equations

$$x = 3t + \sin t, \quad y = 2\sin t \ , \quad 0 \le t \le \pi .$$

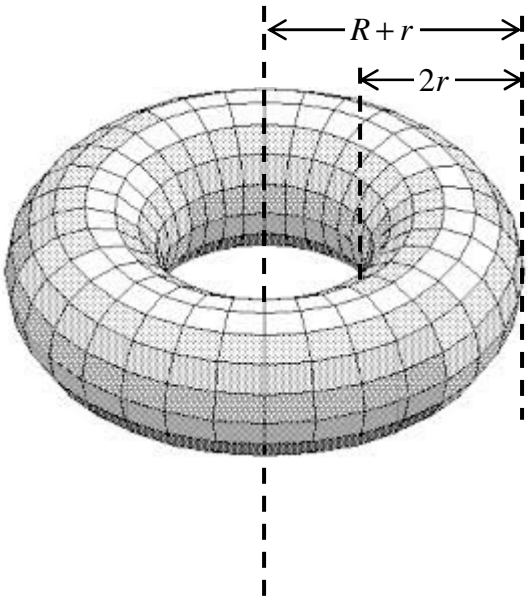
R is fully revolved about the y axis forming a solid of revolution.

Show that the volume of this solid is $39\pi^2$.

[ ], proof



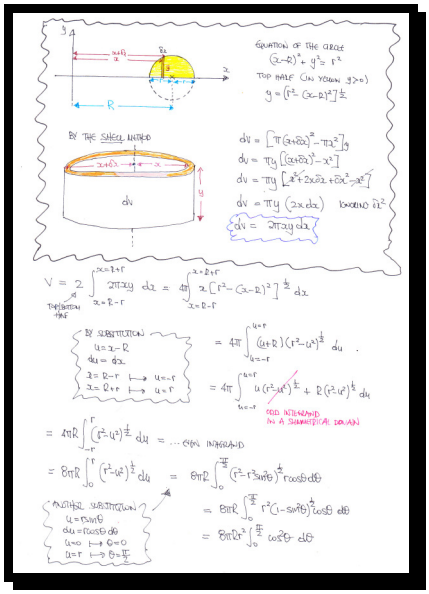
Question 40

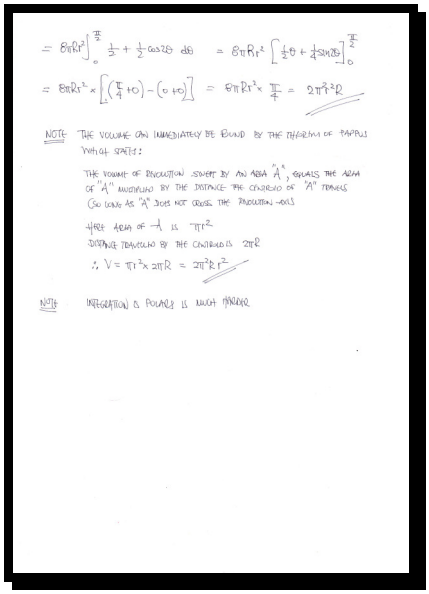


Use direct integration in Cartesian coordinates to show the volume V of the circular ring torus, shown in the figure above, is given by

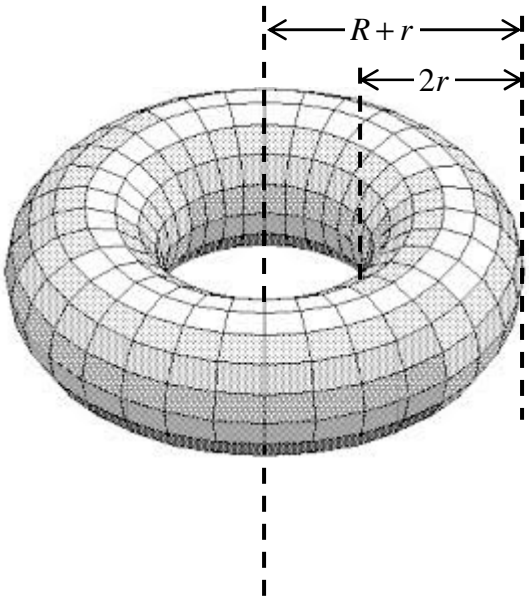
$$V = 2\pi^2 r^2 R\,,\ 0 < r < R\,.$$

proof





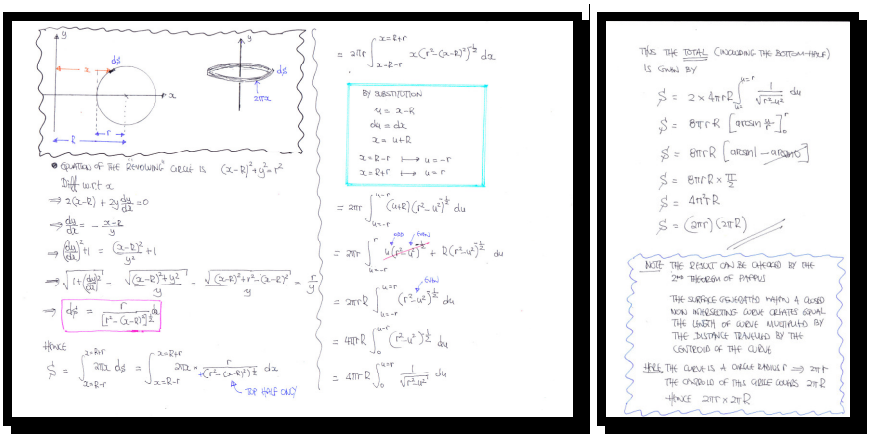
Question 41



Use direct integration in Cartesian coordinates to show the surface area S of the circular ring torus, shown in the figure above, is given by

$$S=(2\pi r)(2\pi R),\ 0<r<R\,.$$

proof



Question 42

The circle with equation

$$x^2 + y^2 = 4 ,$$

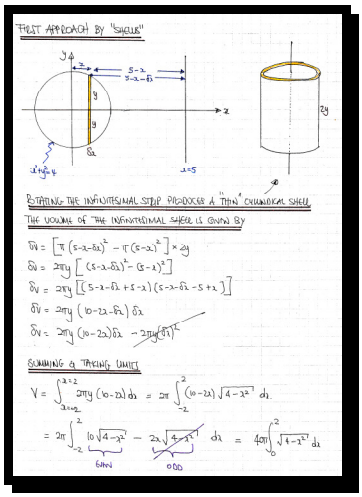
is rotated by 2π radians about the straight line with equation $x = 5$ axis to form a solid of revolution, known as a torus.

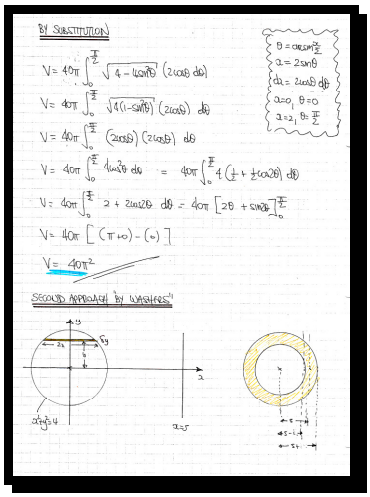
Use integration to show that the volume of the solid is

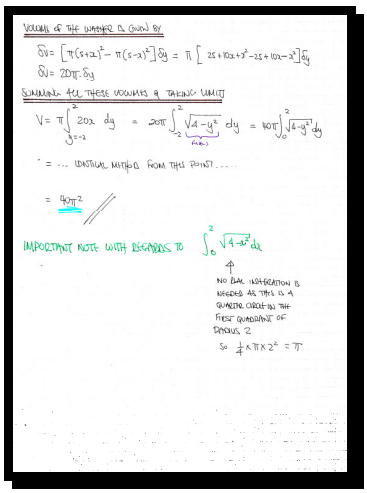
$$40\pi^2 .$$

You may not use the formula for the volume of a torus or the theorem of Pappus.

[ ], [proof]







Question 43

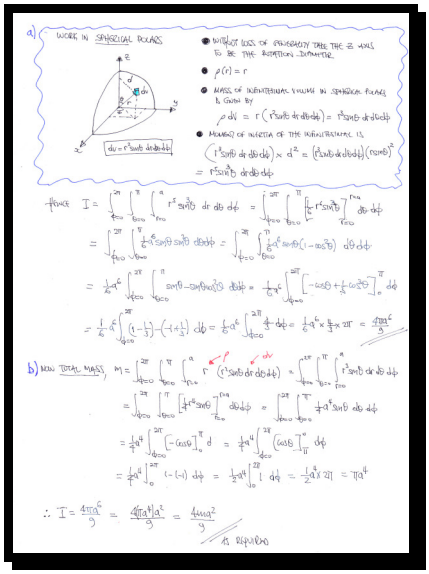
A solid uniform sphere of radius a, has variable density $\rho(r)=r$, where r is the radial distance of a given point from the centre of the sphere.

a) Use spherical polar coordinates, (r,θ,φ), to find the moment of inertia of this sphere I, about one of its diameters.

b) Given that the total mass of the sphere is m, show that

$$I=\frac{4}{9}ma^2\,.$$

$$\boxed{I=\frac{4}{9}\pi a^6}$$



Question 44

A solid sphere has equation

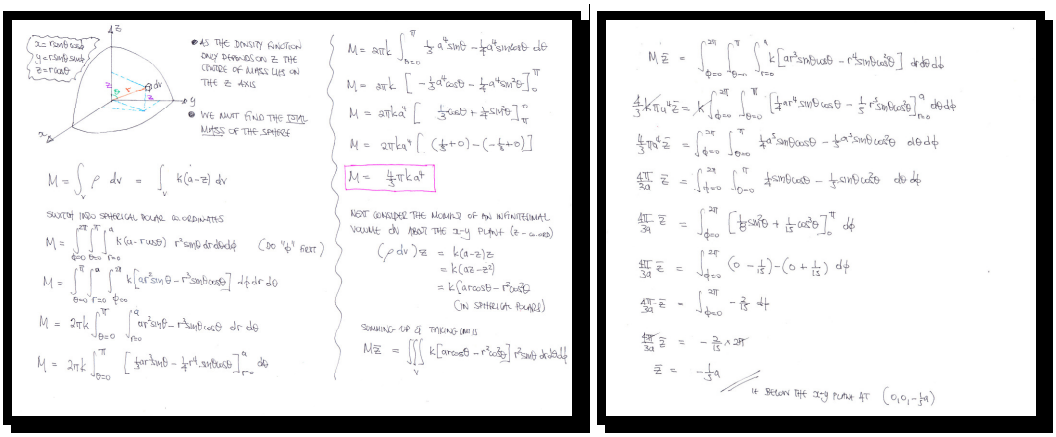
$$x^2+y^2+z^2=a^2,\ a>0.$$

The sphere has variable density ρ, given by

$$\rho=k(a-z),\ k>0.$$

Use integration in spherical polar coordinates, (r,θ,φ), to find in Cartesian form the coordinates of the centre of mass of the sphere.

$$\boxed{\left(0,0,-\frac{1}{5}a\right)}$$



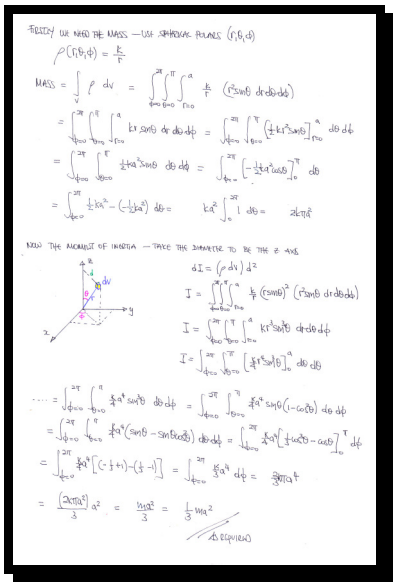
Question 45

A solid sphere has radius a and mass m.

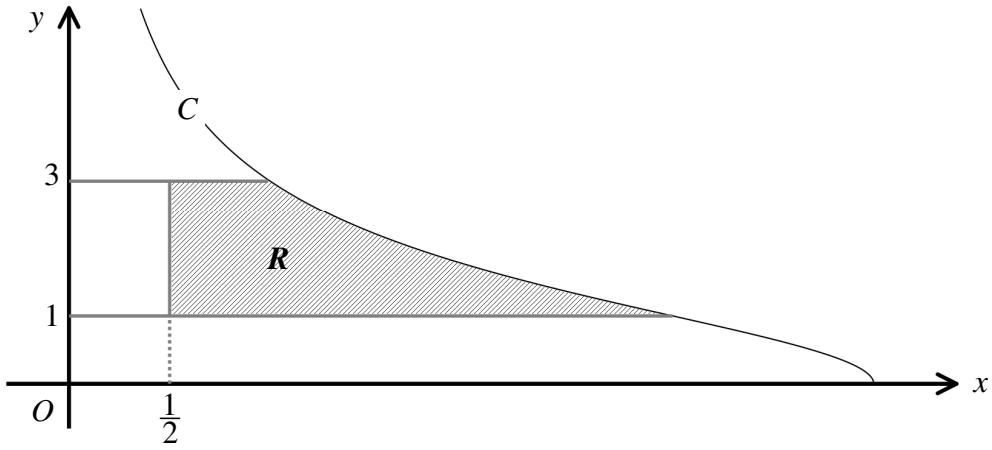
The density ρ at any point in the sphere is inversely proportional to the distance of this point from the centre of the sphere

Show that the moment of inertia of this sphere about one of its diameters is $\frac{1}{3}ma^2$

proof



Question 46



The figure above shows the curve C with parametric equations

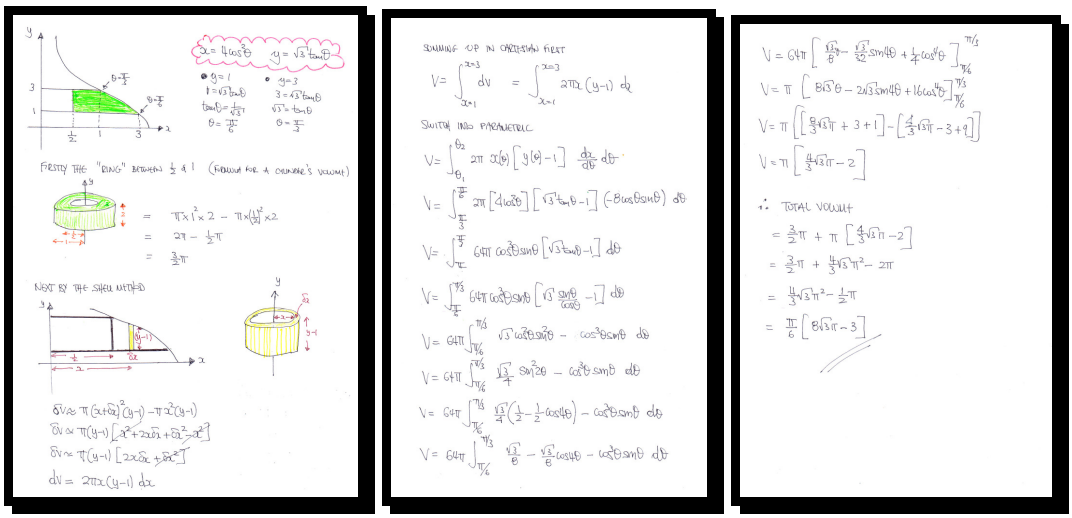
$$x = 4\cos^2\theta, \quad y = \sqrt{3}\tan\theta, \quad 0 \le \theta < \frac{\pi}{2}.$$

The finite region R shown shaded in the figure, bounded by C, and the straight lines with equations $y = 1$, $y = 3$ and $x = \frac{1}{2}$.

Use integration in parametric to find an exact value for the volume of the solid formed when R is fully revolved about the y axis.

[*you may only use the shell method in parametric in this question*]

$$\boxed{V = \frac{\pi}{6}\left[8\pi\sqrt{3} - 3\right]}$$



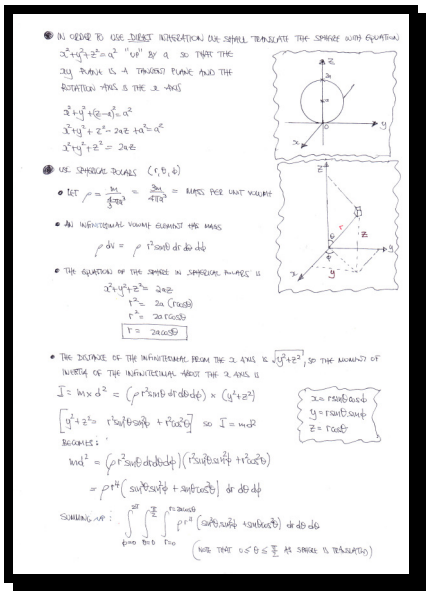
Question 47

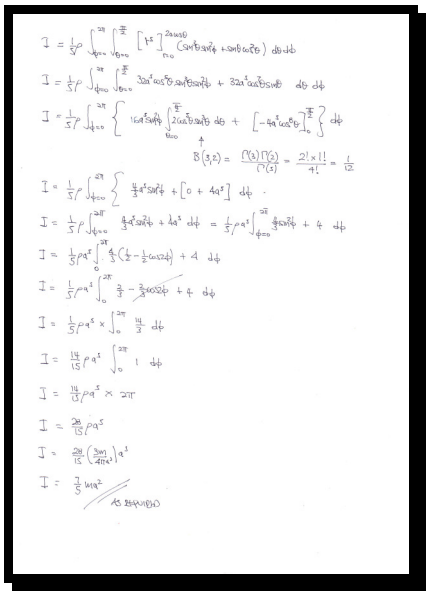
A solid uniform sphere has mass M and radius a.

Use spherical polar coordinates, (r,θ,φ), and direct calculus methods, to show that the moment of inertial of this sphere about one of its tangents is $\frac{7}{5}Ma^2$.

*You may **not** use any standard rules or standard results about moments of inertia in this question apart from the definition of moment of inertia.*

proof





Question 48

$$\mathbf{F}(x,y) \equiv \left(-\frac{y}{x^2+y^2}\right)\mathbf{i} + \left(\frac{x}{x^2+y^2}\right)\mathbf{j}.$$

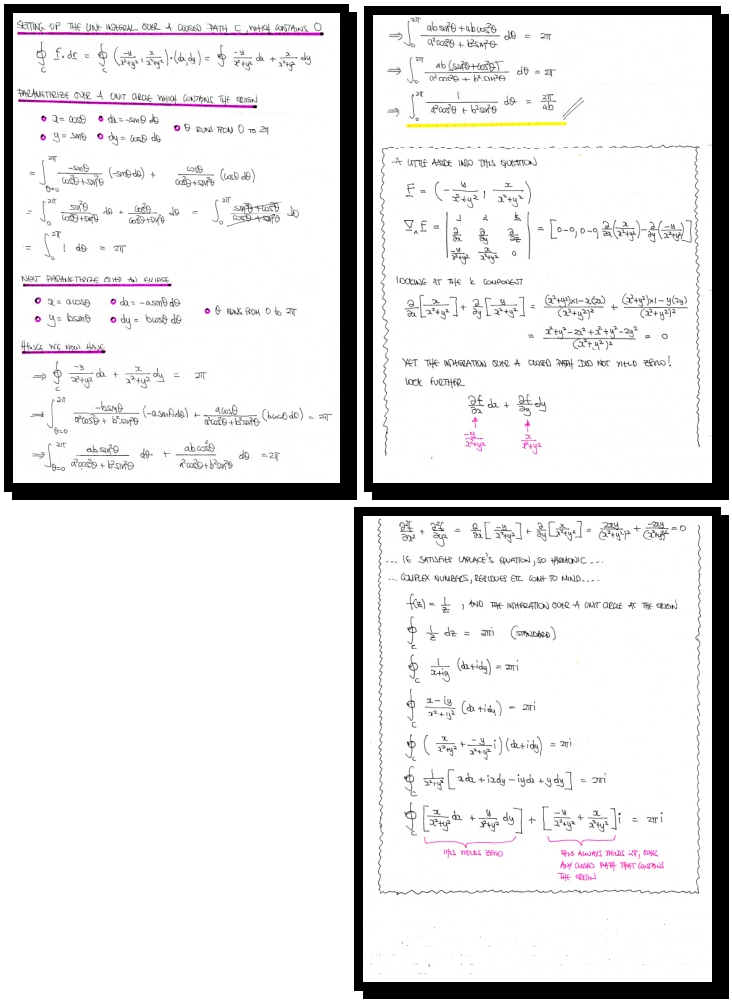
By considering the line integral of $\mathbf{F}$ over two different suitably parameterized closed paths, show that

$$\int_0^{2\pi} \frac{1}{a^2\cos^2\theta + b^2\sin^2\theta}\, d\theta = \frac{2\pi}{ab},$$

where a and b are real constants.

You may assume without proof that the line integral of $\mathbf{F}$ yields the same value over any simple closed curve which contains the origin.

[ ], [proof]



Question 49

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

This implies that $\phi^2 - \phi - 1 = 0$, $\phi = \frac{1}{2}\left(1+\sqrt{5}\right) \approx 1.62$.

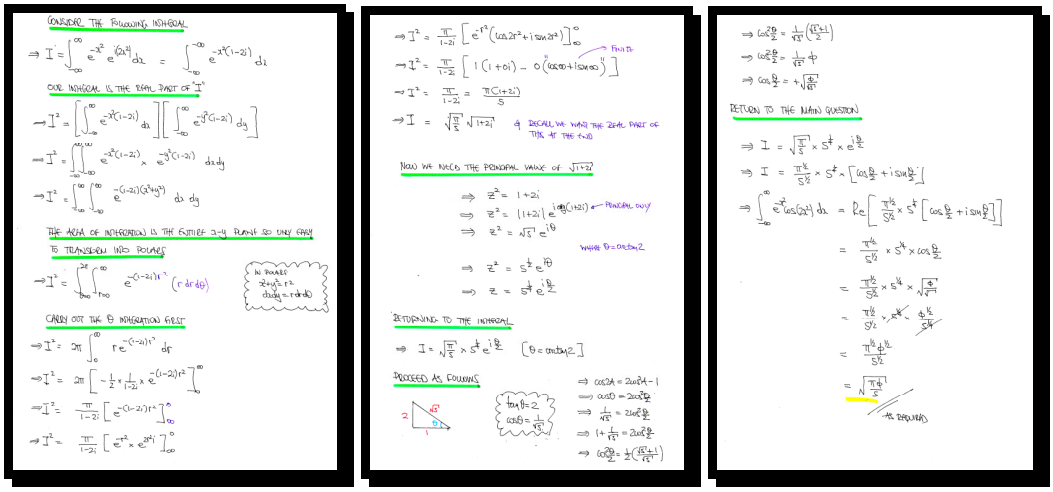
It is asserted that

$$I = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \cos\left(2x^2\right)\, dx = \sqrt{\frac{\pi\phi}{5}}\,.$$

By considering the real part of a suitable function, use double integration in plane polar coordinates to prove the validity of the above result.

You may assume the principal value in any required complex evaluation.

, proof



Question 50

The point $S\left[x_1, f\left(x_1\right)\right]$ and the point $T\left[x_2, f\left(x_2\right)\right]$ lie on the curve C with Cartesian equation $y = f\left(x\right)$.

The straight line L has equation $y = mx + c$, where m and c are constants.

The finite region R is bounded by C, L, and perpendicular straight line segments from S to L and from T to L.

A solid is formed by revolving R about L, by a complete turn.

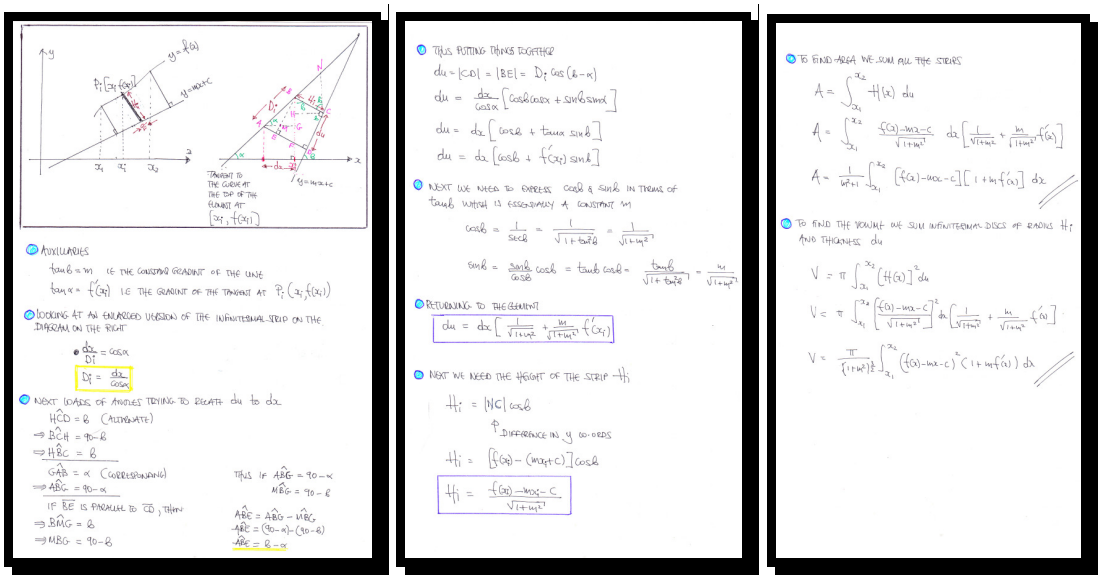
a) Show that the area of R is given by

$$\frac{1}{m^2+1}\int_{x_1}^{x_2}\left[f\left(x\right)-mx-c\right]\left[1+m\,f'\left(x\right)\right]\,dx\,.$$

b) Show that the volume of the solid of revolution is given by

$$\frac{\pi}{\left(m^2+1\right)^{\frac{3}{2}}}\int_{x_1}^{x_2}\left[f\left(x\right)-mx-c\right]^2\left[1+m\,f'\left(x\right)\right]\,dx\,.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 51

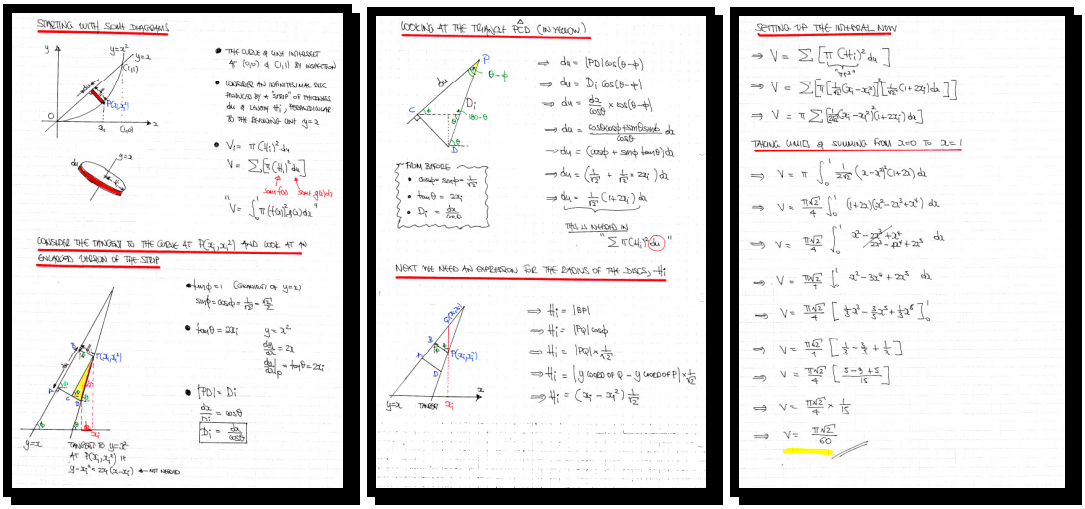
A curve C and a straight line L have respective equations

$$y = x^2 \quad \text{and} \quad y = x .$$

The finite region bounded by C and L is rotated around L by a full turn, forming a solid of revolution S.

Find, in exact form, the volume of S.

$$\boxed{}, \quad \boxed{\frac{\pi\sqrt{2}}{60}}$$



Question 52

A curve C has equation

$$y=(x-2)(6-x),\ x\in\mathbb{R}.$$

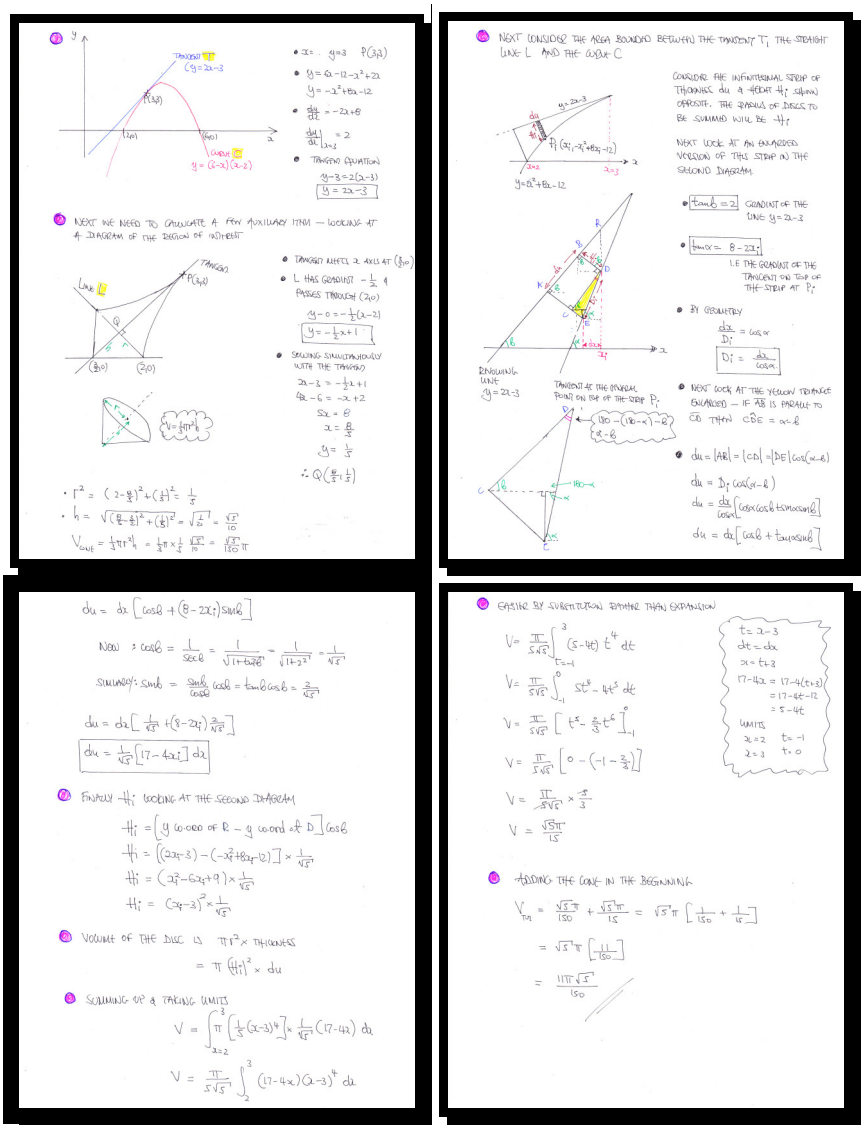
The straight line T is the tangent to C at the point where $x=3$.

The finite region R is bounded by C, T, and the x axis.

A solid S is formed by revolving R about T, by a complete turn.

Find, in exact form, the volume of S.

$$\boxed{},\ \boxed{\frac{11\pi\sqrt{5}}{150}}$$

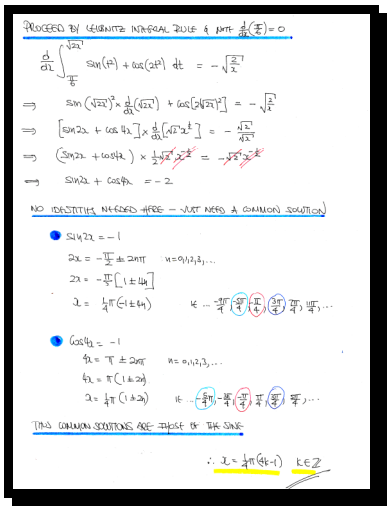


Question 53

Find the general solution of the following equation

$$\frac{d}{dx}\left[\int_{\frac{1}{6}\pi}^{\sqrt{2x}} \sin\left(t^2\right)+\cos\left(2t^2\right)\ dx\right] = -\sqrt{\frac{2}{x}}\,, \quad x\in\mathbb{R}\,.$$

$\square$, $\boxed{x=\frac{1}{4}\pi(4k-1) \quad k\in\mathbb{Z}}$



Created by T. Madas