INDEX SUMMATION NOTATION

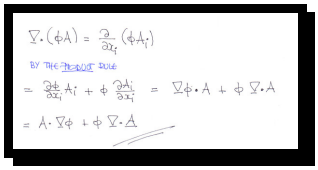
Question 1

Use index summation notation to prove the validity of the following vector identity

$$\nabla\cdot(\varphi\mathbf{A})\equiv\nabla\varphi\cdot\mathbf{A}+\varphi(\nabla\cdot\mathbf{A}),$$

where $\varphi=\varphi(x,y,z)$ is a smooth scalar function and $\mathbf{A}=\mathbf{A}(x,y,z)$ is a smooth vector function.

proof



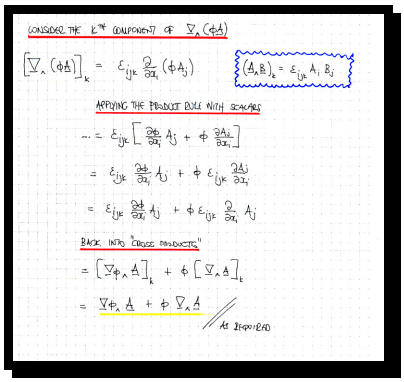
Question 2

Use index summation notation to prove the validity of the following vector identity

$$\nabla\wedge(\varphi\mathbf{A})\equiv\nabla\varphi\wedge\mathbf{A}+\varphi(\nabla\wedge\mathbf{A}),$$

where $\varphi=\varphi(x,y,z)$ is a smooth scalar function and $\mathbf{A}=\mathbf{A}(x,y,z)$ is a smooth vector function.

, proof



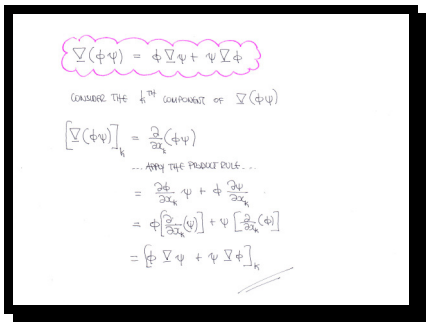
Question 3

Use index summation notation to prove the validity of the following vector identity

$$\nabla(\varphi\psi)\equiv\varphi\nabla\psi+\psi\nabla\varphi,$$

where $\varphi=\varphi(x,y,z)$ and $\psi=\psi(x,y,z)$ are smooth scalar functions.

proof



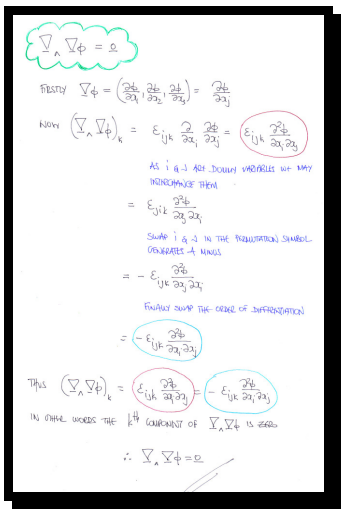
Question 4

Use index summation notation to prove the validity of the following vector identity

$$\nabla\wedge\nabla\varphi\equiv\mathbf{0},$$

where $\varphi=\varphi(x,y,z)$ is a smooth scalar function.

proof



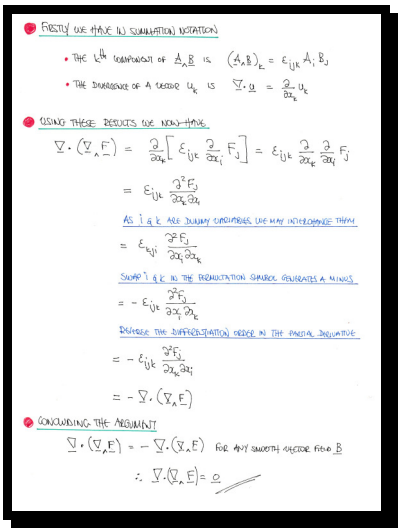
Question 5

Use index summation notation to prove the validity of the following vector identity

$$\nabla\cdot\left[\nabla\wedge\mathbf{F}\right]\equiv 0,$$

where $\mathbf{F}=\mathbf{F}\left(x,y,z\right)$ is a smooth vector function.

, proof

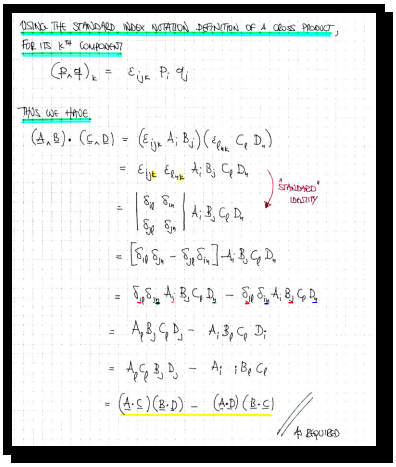


Question 6

Use index summation notation to prove the validity of the following vector identity.

$$(\mathbf{A}\wedge\mathbf{B})\cdot(\mathbf{C}\wedge\mathbf{D})\equiv(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}).$$

[ ], [proof]

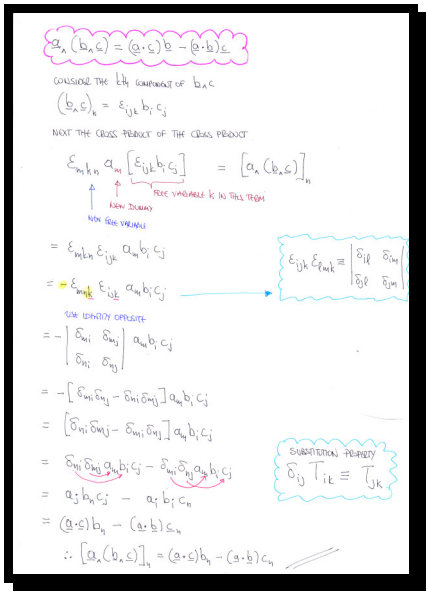


Question 7

Use index summation notation to prove the validity of the following vector identity

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})\equiv(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}\,.$$

proof



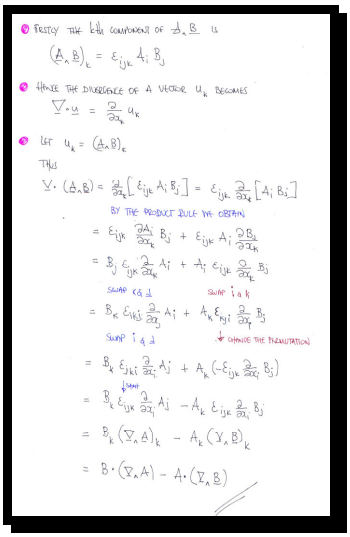
Question 8

Use index summation notation to prove the validity of the following vector identity

$$\nabla\cdot\left[\mathbf{A}\wedge\mathbf{B}\right]\equiv\mathbf{B}\cdot\left(\nabla\wedge\mathbf{A}\right)-\mathbf{A}\cdot\left(\nabla\wedge\mathbf{B}\right),$$

where $\mathbf{A}=\mathbf{A}\left(x,y,z\right)$ and $\mathbf{B}=\mathbf{B}\left(x,y,z\right)$ are smooth vector functions.

proof



Question 9

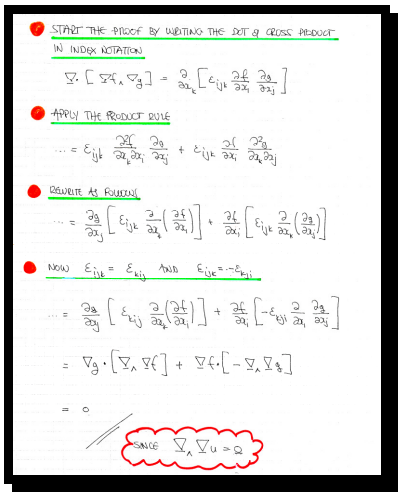
Use index summation notation to prove the validity of the following vector identity

$$\nabla\cdot\left[\nabla f\wedge\nabla g\right]\equiv 0,$$

where $f=f\left(x,y,z\right)$ and $g=g\left(x,y,z\right)$ are smooth scalar functions.

Any additional results used must be clearly stated.

[ ], [proof]



Question 10

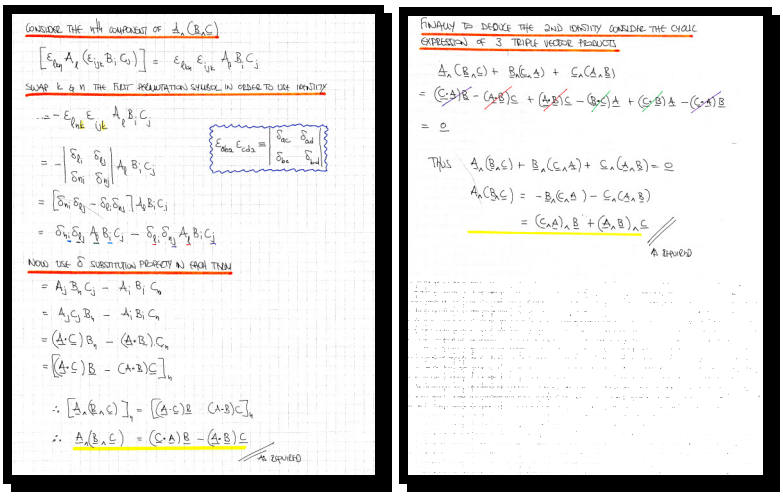
Use index summation notation to prove that

$$\mathbf{A}\wedge(\mathbf{B}\wedge\mathbf{C})\equiv(\mathbf{C}\cdot\mathbf{A})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C},$$

and hence deduce that

$$\mathbf{A}\wedge(\mathbf{B}\wedge\mathbf{C})\equiv(\mathbf{C}\wedge\mathbf{A})\wedge\mathbf{B}+(\mathbf{A}\wedge\mathbf{B})\wedge\mathbf{C}$$

, proof



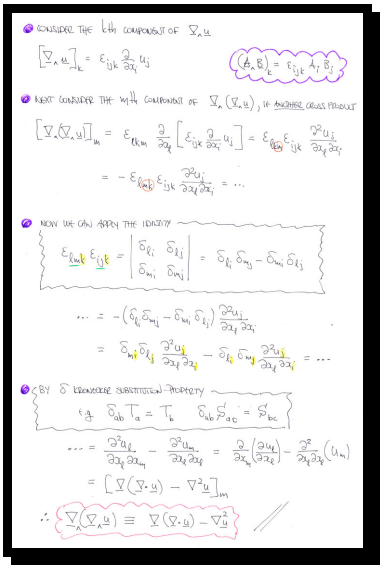
Question 11

Use index summation notation to prove the validity of the following vector identity

$$\nabla \wedge \left[\nabla \wedge \mathbf{u} \right] \equiv \nabla \left(\nabla \cdot \mathbf{u} \right) - \nabla^2 \mathbf{u} ,$$

where $\mathbf{u} = \mathbf{u}\left(x, y, z \right)$ is a smooth vector function.

proof



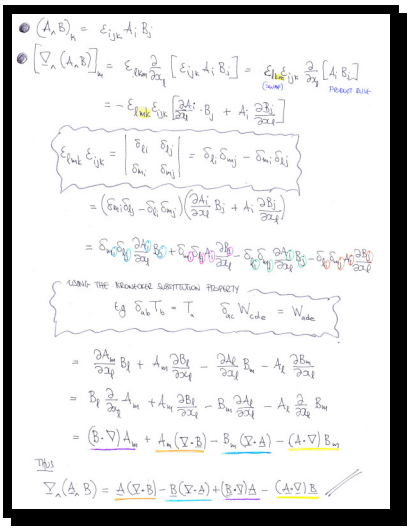
Question 12

Use index summation notation to prove the validity of the following vector identity

$$\nabla \wedge \left[\mathbf{A} \wedge \mathbf{B} \right] \equiv \mathbf{A}\left(\nabla \cdot \mathbf{B} \right) - \mathbf{B}\left(\nabla \cdot \mathbf{A} \right) + \left(\mathbf{B} \cdot \nabla \right)\mathbf{A} - \left(\mathbf{A} \cdot \nabla \right)\mathbf{B} ,$$

where $\mathbf{A} = \mathbf{A}\left(x, y, z \right)$ and $\mathbf{B} = \mathbf{B}\left(x, y, z \right)$ are smooth vector functions.

proof



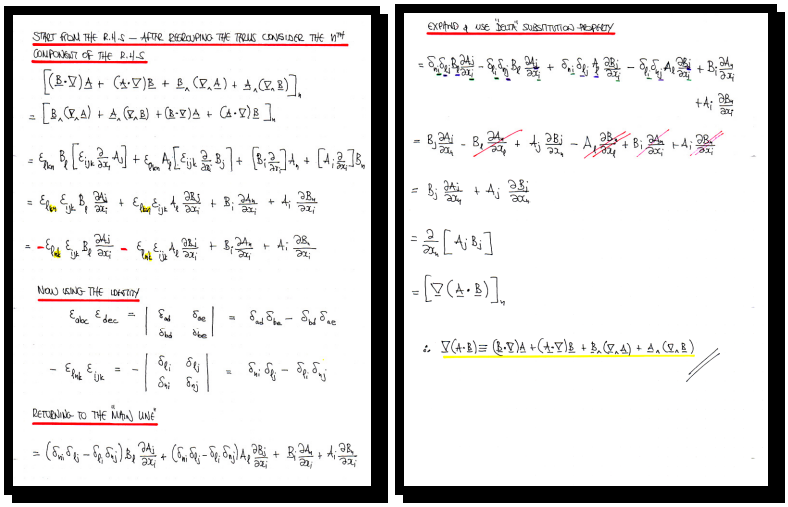
Question 13

Use index summation notation to prove the validity of the following vector identity

$$\nabla\left[\mathbf{A}\cdot\mathbf{B}\right]\equiv\left(\mathbf{B}\cdot\nabla\right)\mathbf{A}+\left(\mathbf{A}\cdot\nabla\right)\mathbf{B}+\mathbf{B}\wedge\left(\nabla\wedge\mathbf{A}\right)+\mathbf{A}\wedge\left(\nabla\wedge\mathbf{B}\right),$$

where $\mathbf{A}=\mathbf{A}\left(x,y,z\right)$ and $\mathbf{B}=\mathbf{B}\left(x,y,z\right)$ are smooth vector functions.

[ ], [proof]



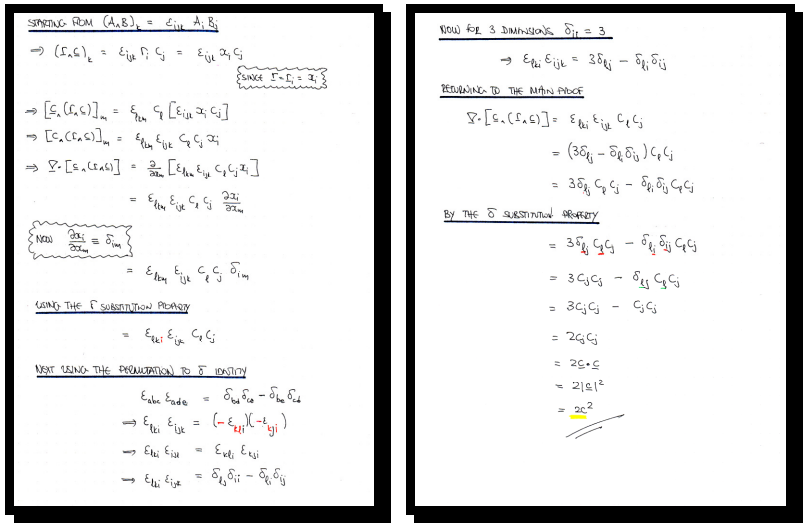
Question 14

Use index summation notation to prove the validity of the following vector identity

$$\nabla\cdot\left[\mathbf{c}\wedge\left(\mathbf{r}\wedge\mathbf{c}\right)\right]\equiv 2c^2,$$

where $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and $\mathbf{c}$ is a constant three dimensional vector and $c\equiv|\mathbf{c}|$.

[ ], [proof]



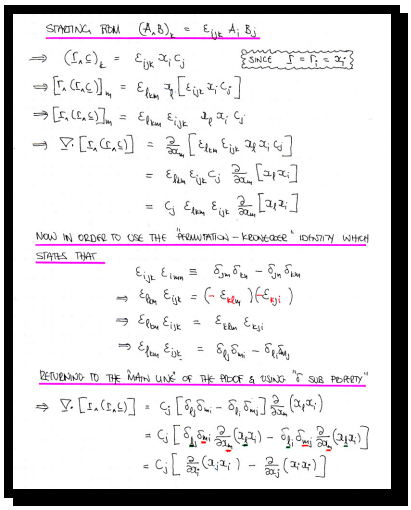
Question 15

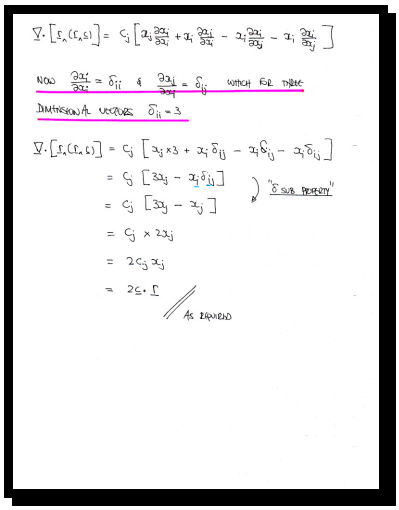
Use index summation notation to prove the validity of the following vector identity

$$\nabla\cdot\left[\mathbf{r}\wedge\left(\mathbf{r}\wedge\mathbf{c}\right)\right]\equiv 2\mathbf{r}\cdot\mathbf{c}\,,$$

where $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and $\mathbf{c}$ is a constant three dimensional vector.

[ ], [proof]





Question 16

The vector field $\mathbf{F}$ exists around the open surface S, with closed boundary C.

a) State Stokes' Integral Theorem for open surfaces, fully defining all the quantities involved.

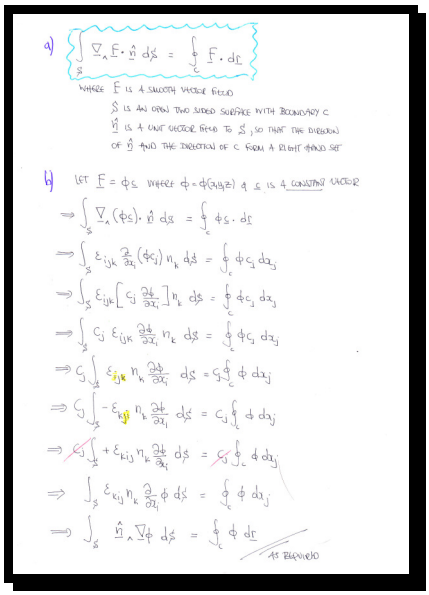
Let $\mathbf{c}$ be a constant vector and $\varphi = \varphi(x, y, z)$ a smooth scalar function.

b) By considering $\nabla \wedge (\mathbf{c}\varphi)$, use index summation notation in Stokes' Theorem to prove the validity of the following result

$$\int_S \hat{\mathbf{n}} \wedge \nabla \varphi \, dS = \oint_C \varphi \, d\mathbf{r},$$

where $\hat{\mathbf{n}}$ is a unit normal vector field to S.

proof



Question 17

The vector field $\mathbf{F}$ exists around the open surface S, with closed boundary C.

Let $\mathbf{c}$ be a constant vector and $\mathbf{A}=\mathbf{A}(x,y,z)$ a smooth vector function.

By considering $\mathbf{F}=\mathbf{c}\wedge\mathbf{A}$, use index summation notation in Stokes' Theorem to prove the validity of the following result

$$\int_S (\mathbf{dS}\wedge\nabla)\wedge\mathbf{A} \;=\; \oint_C d\mathbf{r}\wedge\mathbf{A}\,,$$

where $\mathbf{dS}=\hat{\mathbf{n}}\,dS$, $\hat{\mathbf{n}}$ is a unit normal vector field to S, forming a right hand set with the direction of C.

proof

