IMPULSE FUNCTION

The Impulse Function / The Dirac Function

1. $\delta(t-c)=\begin{cases}\infty & t=c\\ 0 & t\neq c\end{cases}$, $\delta(t)=\begin{cases}\infty & t=0\\ 0 & t\neq 0\end{cases}$

2. $\delta(t)=\dfrac{1}{\pi}\lim\limits_{\varepsilon\to 0}\left[\dfrac{\varepsilon}{\varepsilon^2+t^2}\right]$

3. $\displaystyle\int_a^b \delta(t-c)\ dt=\begin{cases}1 & a\le c\le b\\ 0 & \text{otherwise}\end{cases}$

4. $\displaystyle\int_a^b f(t)\delta(t-c)\ dt=\begin{cases} f(c) & a\le c\le b\\ 0 & \text{otherwise}\end{cases}$

5. $\mathcal{L}\left[\delta(t-c)\right]=\mathrm{e}^{-cs}$

6. $\mathcal{L}\left[f(t)\delta(t-c)\right]=f(c)\mathrm{e}^{-cs}$

7. $\mathcal{F}\left[\delta(x)\right]=\dfrac{1}{\sqrt{2\pi}}$

8. $\mathcal{F}^{-1}\left[\delta(k)\right]=\dfrac{1}{\sqrt{2\pi}}$

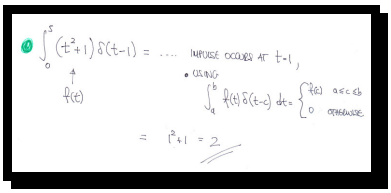
9. $\dfrac{d}{dt}\left[\mathrm{H}(t-c)\right]=\delta(t-c)$

Question 1

Evaluate the following integral

$$\int_0^5 \left(t^2+1\right)\delta\left(t-1\right)\ dt\,.$$

$\boxed{2}$

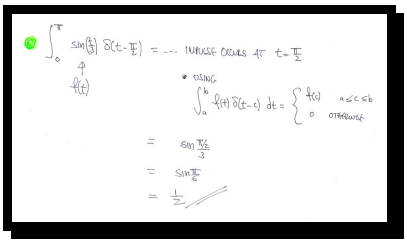


Question 2

Evaluate the following integral

$$\int_0^{\pi} \sin\left(\frac{1}{3}t\right)\delta\left(t-\frac{\pi}{2}\right)\ dt\,.$$

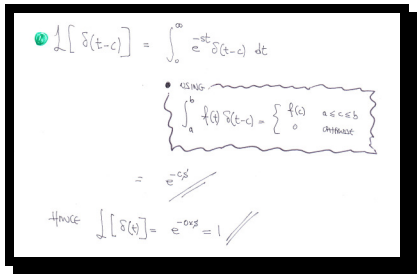
$\boxed{\frac{1}{2}}$



Question 3

Find the Laplace transform of $\delta(t-c)$, where c is a positive constant, and hence state the Laplace transform of $\delta(t)$.

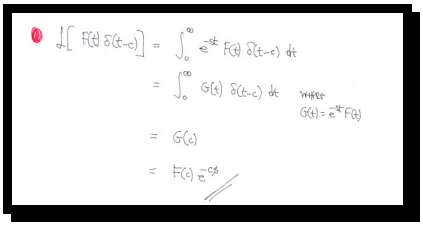
$$\boxed{\mathcal{L}\left[\delta(t-c)\right]=\mathrm{e}^{-cs}}, \quad \boxed{\mathcal{L}\left[\delta(t)\right]=1}$$



Question 4

Given that $F(t)$ is a piecewise continuous function defined for $t \geq 0$, find the Laplace transform of $F(t)\,\delta(t-c)$, where c is a positive constant.

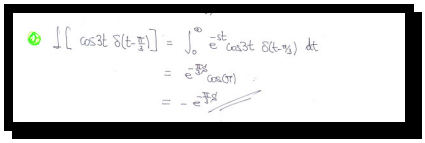
$$\boxed{\mathcal{L}\left[F(t)\,\delta(t-c)\right]=F(c)\mathrm{e}^{-cs}}$$



Question 5

Find the Laplace transform of $\cos 3t\ \delta\left(t-\frac{\pi}{3}\right)$.

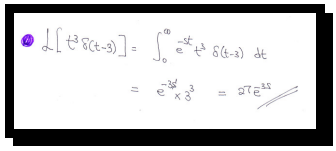
$$\mathcal{L}\left[\cos 3t\ \delta\left(t-\frac{\pi}{3}\right)\right]=-\mathrm{e}^{-\frac{1}{3}\pi s}$$



Question 6

Find the Laplace transform of $t^3\ \delta(t-3)$.

$$\mathcal{L}\left[t^3\ \delta(t-3)\right]=27\mathrm{e}^{-3s}$$



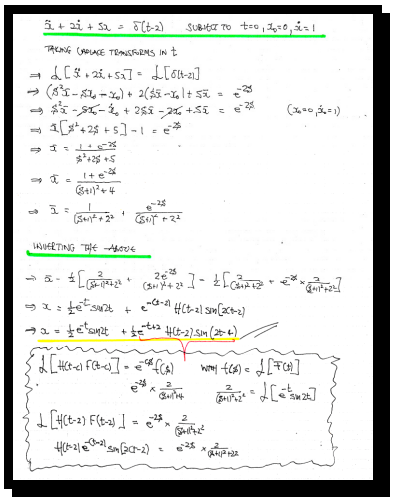
Question 7

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=\delta(t-2),$$

given further that $x=0$, $\frac{dx}{dt}=1$ at $t=0$.

$\boxed{}$, $\boxed{x=\frac{1}{2}\mathrm{e}^{-t}\sin 2t+\frac{1}{2}\mathrm{e}^{2-t}\sin(2t-4)\mathrm{H}(t-2)}$



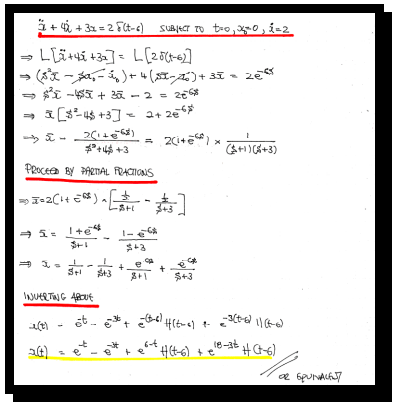
Question 8

Use Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+3x=2\delta(t-6),$$

given further that $x=0$, $\frac{dx}{dt}=2$ at $t=0$.

$\square$, $\boxed{x=\mathrm{e}^{-t}-\mathrm{e}^{-3t}+\mathrm{e}^{6-t}\,\mathrm{H}(t-6)+\mathrm{e}^{18-3t}\,\mathrm{H}(t-6)}$



Question 9

The function f is defined as

$$f(x)=\frac{1}{\pi}\left[\frac{\varepsilon}{\varepsilon^2+x^2}\right],$$

where ε is a positive parameter.

a) Show that $\lim_{\varepsilon\to 0}\left[f(x)\right]=\delta(x)$.

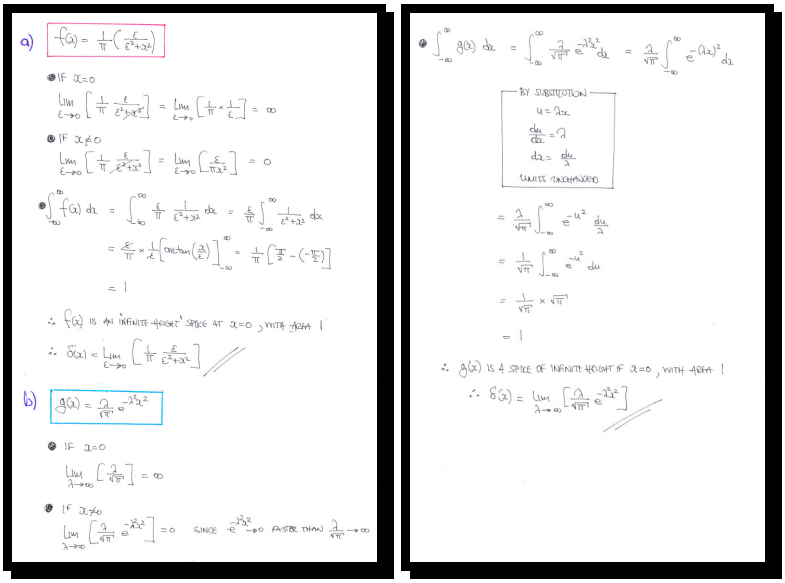
The function g is defined as

$$g(x)=\frac{\lambda}{\sqrt{\pi}}\mathrm{e}^{-\lambda^2x^2},$$

where λ is a positive parameter.

b) Show that $\lim_{\lambda\to\infty}\left[g(x)\right]=\delta(x)$.

proof



Question 10

The impulse function $\delta(x)$ is defined by

$$\delta(x)=\begin{cases}\infty & x=0\\ 0 & x\neq 0\end{cases}$$

a) Determine

i. … $\mathcal{F}\left[\delta(x)\right]$.

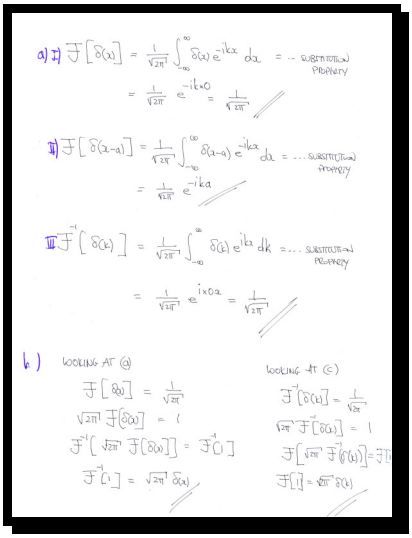
ii. … $\mathcal{F}\left[\delta(x-a)\right]$, where a is a positive constant.

iii. … $\mathcal{F}^{-1}\left[\delta(k)\right]$.

b) Use the above results to deduce $\mathcal{F}[1]$ and $\mathcal{F}^{-1}[1]$.

$$\boxed{\mathcal{F}\left[\delta(x)\right]=\frac{1}{\sqrt{2\pi}}},\ \boxed{\mathcal{F}\left[\delta(x-a)\right]=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}ka}},\ \boxed{\mathcal{F}^{-1}\left[\delta(k)\right]=\frac{1}{\sqrt{2\pi}}},$$

$$\boxed{\mathcal{F}[1]=\sqrt{2\pi}\,\delta(k)},\ \boxed{\mathcal{F}^{-1}[1]=\sqrt{2\pi}\,\delta(x)}$$



Question 11

The impulse function $\delta(x)$ is defined by

$$\delta(x)=\begin{cases}\infty & x=0\\ 0 & x\neq 0\end{cases}$$

a) Determine the inverse Fourier transform of the impulse function $\mathcal{F}^{-1}\left[\delta(k)\right]$, and use it to deduce the Fourier transform of $f(x)=1$.

b) Find directly the Fourier transform of $f(x)=1$, by introducing the converging factor $\mathrm{e}^{-\varepsilon|x|}$ and letting $\varepsilon\to 0$.

$$\boxed{\mathcal{F}[1]=\sqrt{2\pi}\,\delta(k)}$$

