HEAVISIDE FUNCTION

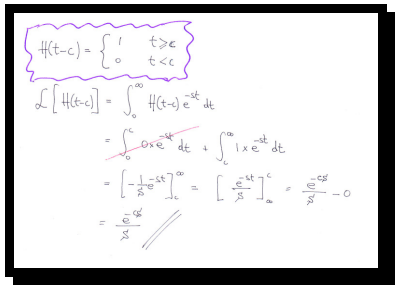
Question 1

The Heaviside function $\mathrm{H}(t)$ is defined as

$$\mathrm{H}(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$$

Determine the Laplace transform of $\mathrm{H}(t-c)$.

$$\mathcal{L}\left(\mathrm{H}(t-c)\right)=\frac{\mathrm{e}^{-cs}}{s}$$



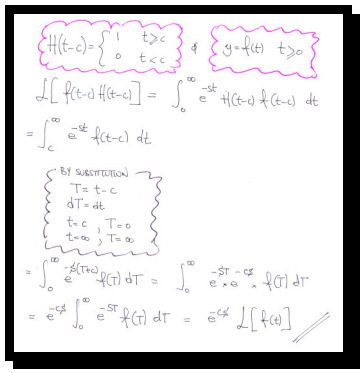
Question 2

The Heaviside step function $\mathrm{H}(t)$ is defined as

$$\mathrm{H}(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$$

Determine the Laplace transform of $\mathrm{H}(t-c)\,f(t-c)$, where $f(t)$ is a continuous or piecewise continuous function defined for $t\geq 0$.

$$\boxed{\mathcal{L}\left(\mathrm{H}(t-c)\,f(t-c)\right)=\mathrm{e}^{-cs}\,\mathcal{L}\left(f(t)\right)}$$



Question 3

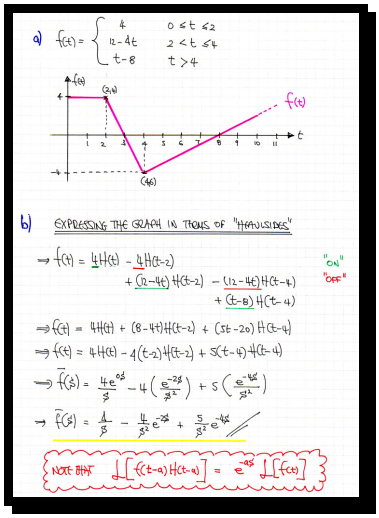
The piecewise continuous function $f(t)$ is defined as

$$f(t)=\begin{cases} 4 & 0\le t\le 2 \\ 12-4t & 2<t\le 4 \\ t-8 & t>4 \end{cases}$$

a) Sketch the graph of $f(t)$.

b) Express $f(t)$ in terms of the Heaviside step function, and hence find the Laplace transform of $f(t)$.

$$f(t)=4\mathrm{H}(t)-4(t-2)\mathrm{H}(t-2)+5(t-4)\mathrm{H}(t-4), \quad \mathcal{L}\{f(t)\}=\frac{8}{s}-\frac{4\mathrm{e}^{-2s}}{s^2}+\frac{5\mathrm{e}^{-4s}}{s^2}$$



Question 4

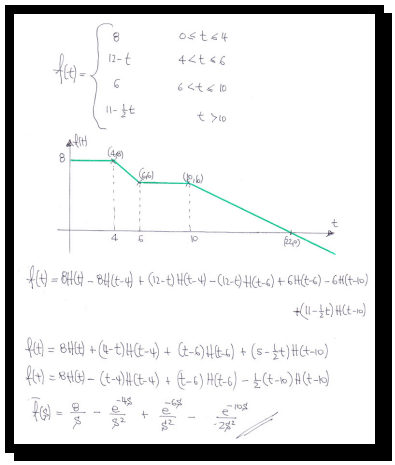
The piecewise continuous function $f(t)$ is defined as

$$f(t)=\begin{cases} 8 & 0\le t\le 4 \\ 12-t & 4<t\le 6 \\ 6 & 6<t\le 10 \\ 11-\frac{1}{2}t & t>10 \end{cases}$$

Express $f(t)$ in terms of the Heaviside step function, and hence find the Laplace transform of $f(t)$.

$$\boxed{f(t)=8\mathrm{H}(t)-(t-4)\mathrm{H}(t-4)+(t-4)\mathrm{H}(t-6)-\frac{1}{2}(t-10)\mathrm{H}(t-10)},$$

$$\boxed{\mathcal{L}\big(f(t)\big)=\frac{8}{s}-\frac{\mathrm{e}^{-4s}}{s^2}+\frac{\mathrm{e}^{-6s}}{s^2}-\frac{\mathrm{e}^{-10s}}{2s^2}}$$



Question 5

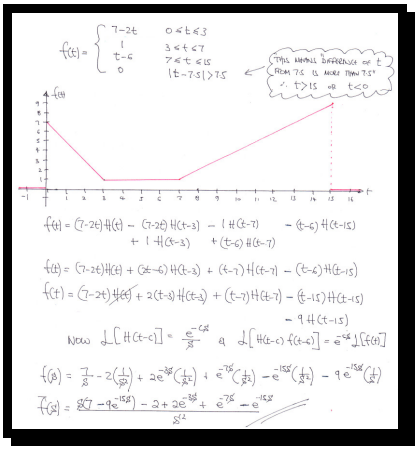
The piecewise continuous function $f(t)$ is defined as

$$f(t)=\begin{cases} 7-2t & 0<t\le 3 \\ 1 & 3<t\le 7 \\ t-6 & 7<t\le 15 \\ 0 & |t-7.5|>7.5 \end{cases}$$

Express $f(t)$ in terms of the Heaviside step function, and hence find the Laplace transform of $f(t)$.

$$\boxed{f(t)=(7-2t)\mathrm{H}(t)+2(t-3)\mathrm{H}(t-3)+(t-7)\mathrm{H}(t-7)-(t-15)\mathrm{H}(t-15)-9\mathrm{H}(t-15)},$$

$$\boxed{\mathcal{L}\{f(t)\}=\frac{s\left(7-9\mathrm{e}^{-15s}\right)-2+2\mathrm{e}^{-3s}+\mathrm{e}^{-7s}-\mathrm{e}^{-15s}}{s^2}}$$



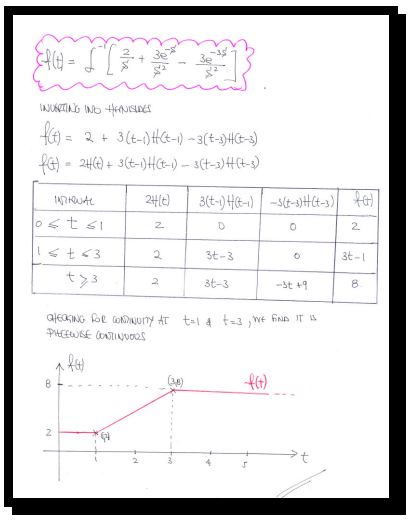
Question 6

The piecewise continuous function $f(t)$ is defined as

$$f(t)=\mathcal{L}^{-1}\left(\frac{2}{s}+\frac{3\mathrm{e}^{-s}}{s^2}-\frac{3\mathrm{e}^{-3s}}{s^2}\right).$$

Sketch the graph of $f(t)$.

graph



Question 7

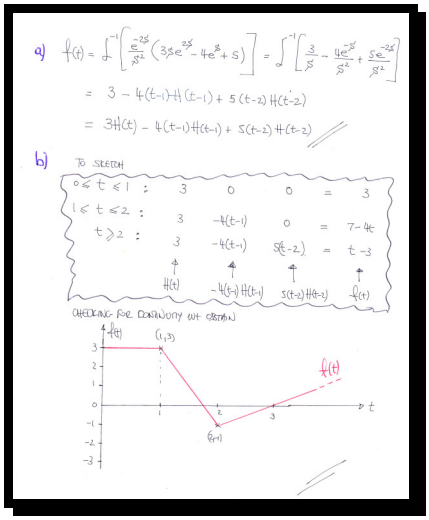
The piecewise continuous function $f(t)$ is defined as

$$f(t)=\mathcal{L}^{-1}\left[\frac{\mathrm{e}^{-2s}}{s^2}\left(3s\,\mathrm{e}^{2s}-4\mathrm{e}^{s}+5\right)\right].$$

a) Determine an expression for $f(t)$.

b) Sketch the graph of $f(t)$.

$$\boxed{f(t)=3\mathrm{H}(t)-4(t-1)\mathrm{H}(t-1)+5(t-2)\mathrm{H}(t-2)}$$



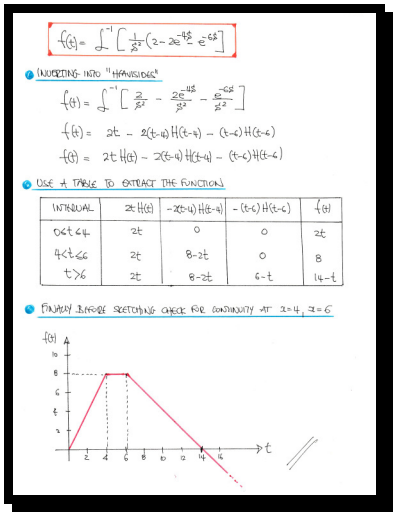
Question 8

The piecewise continuous function $f(t)$ is defined as

$$f(t)=\mathcal{L}^{-1}\left[\frac{1}{s^2}\left(2-2\mathrm{e}^{-4s}-\mathrm{e}^{-6s}\right)\right].$$

Sketch the graph of $f(t)$.

[ ], [graph]



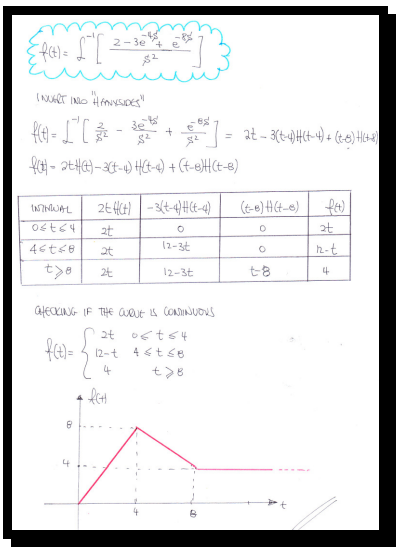
Question 9

The piecewise continuous function $f(t)$ is defined as

$$f(t)=\mathcal{L}^{-1}\left[\frac{2-3\mathrm{e}^{-4s}+\mathrm{e}^{-8s}}{s^2}\right].$$

Sketch the graph of $f(t)$.

graph



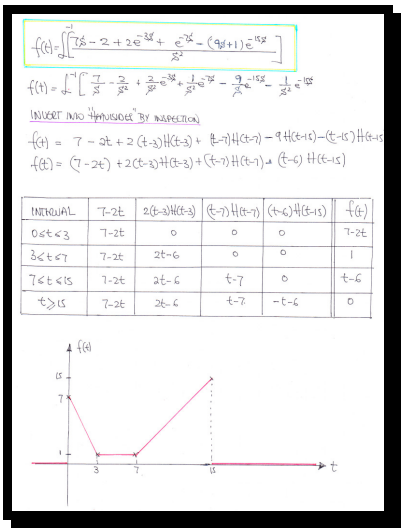
Question 10

The piecewise continuous function $f(t)$ is defined as

$$f(t)=\mathcal{L}^{-1}\left[\frac{s\left(7-9\mathrm{e}^{-15s}\right)-2+2\mathrm{e}^{-3s}+\mathrm{e}^{-7s}-\mathrm{e}^{-15s}}{s^2}\right].$$

Sketch the graph of $f(t)$.

graph



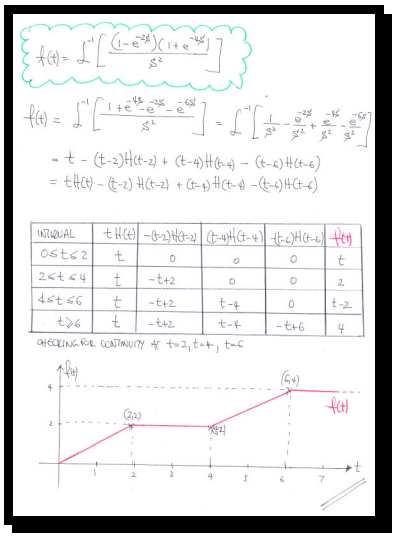
Question 11

The piecewise continuous function $f(t)$ is defined as

$$f(t)=\mathcal{L}^{-1}\left[\frac{\left(1-\mathrm{e}^{-2s}\right)\left(1+\mathrm{e}^{-4s}\right)}{s^2}\right].$$

Sketch the graph of $f(t)$.

graph



Question 12

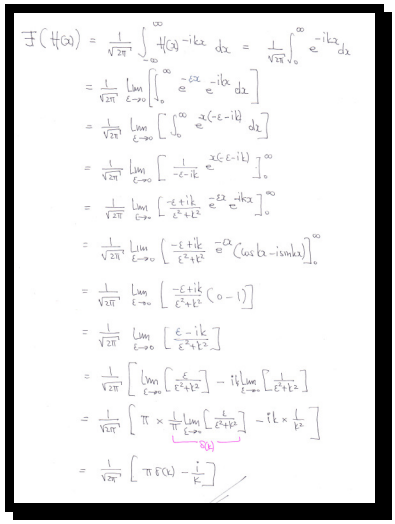
The Heaviside function $\mathrm{H}(x)$ is defined by

$$\mathrm{H}(x)=\begin{cases} 1 & x\geq 0 \\ 0 & x<0 \end{cases}$$

By introducing the converging factor $\mathrm{e}^{-\varepsilon x}$ and letting $\varepsilon\to 0$, determine the Fourier transform of $\mathrm{H}(x)$.

You may assume that $\delta(t)=\dfrac{1}{\pi}\lim\limits_{\varepsilon\to 0}\left[\dfrac{\varepsilon}{\varepsilon^2+t^2}\right]$.

$$\boxed{\mathcal{F}\left[\mathrm{H}(x)\right]=\frac{1}{\sqrt{2\pi}}\left[\pi\delta(k)-\frac{\mathrm{i}}{k}\right]}$$



Question 13

The function f is defined by

$$f(x)=\text{sign}(x)=\begin{cases}1 & x>0\\ -1 & x<0\end{cases}$$

a) By introducing the converging factor $\mathrm{e}^{-\varepsilon|x|}$ and letting $\varepsilon\to 0$, find the Fourier transform of f.

b) By introducing the converging factor $\mathrm{e}^{-\varepsilon|x|}$ and letting $\varepsilon\to 0$, find the Fourier transform of $g(x)=1$.

You may assume that $\delta(t)=\dfrac{1}{\pi}\lim\limits_{\varepsilon\to 0}\left[\dfrac{\varepsilon}{\varepsilon^2+t^2}\right]$.

c) Hence determine the Fourier transform of the Heaviside function $\mathrm{H}(x)$,

$$\mathrm{H}(x)=\begin{cases}1 & x\geq 0\\ 0 & x<0\end{cases}$$

$$\boxed{\mathcal{F}[\text{sign}(x)]=-\frac{\mathrm{i}}{k}\sqrt{\frac{1}{\pi}}},\quad \boxed{\mathcal{F}[1]=\sqrt{2\pi}\,\delta(k)},\quad \boxed{\mathcal{F}[\mathrm{H}(x)]=\frac{1}{\sqrt{2\pi}}\left[\pi\delta(k)-\frac{\mathrm{i}}{k}\right]}$$

