GAMMA FUNCTION

SUMMARY OF THE GAMMA FUNCTION

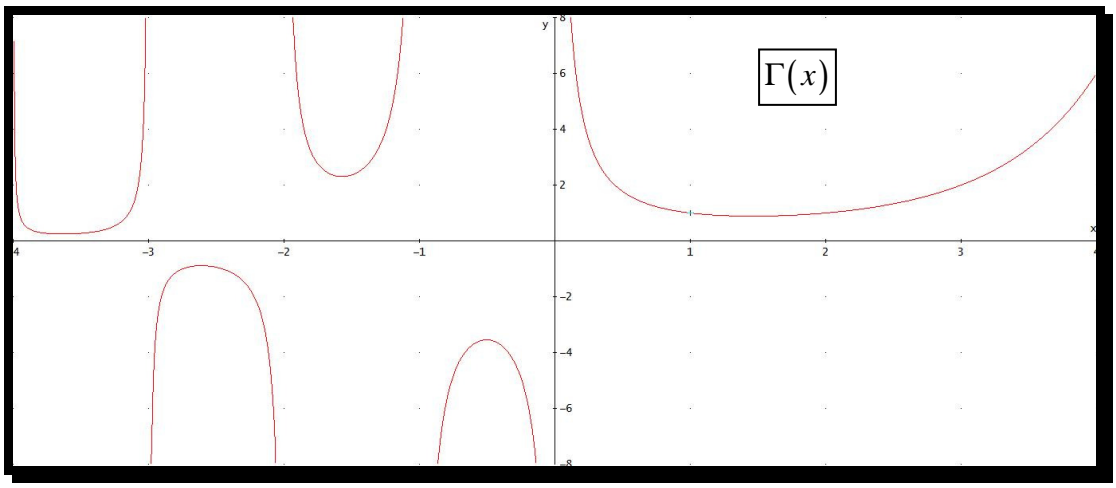
The Gamma function $\Gamma(x)$, is defined as

$$\Gamma(x) \equiv \int_0^{\infty} t^{x-1} \mathrm{e}^{-t} \ dt,$$

where $x \neq \ldots -4, -3, -2, -1, 0$.

Gamma function common rules and facts

- $\Gamma(x+1) \equiv x\Gamma(x) \qquad x \in \mathbb{R}$
- $\Gamma(n+1) \equiv n!, \ n \in \mathbb{N} \qquad \text{or} \qquad \Gamma(n) \equiv (n-1)!, \ n \in \mathbb{N}$
- $\Gamma(1) = 1, \ \Gamma(2) = 1, \ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\Gamma(0) = \pm\infty, \ \Gamma(-1) = \pm\infty, \ \Gamma(-2) = \pm\infty, \ \Gamma(-3) = \pm\infty$, etc
- $\Gamma(z)\Gamma(1-z) \equiv \dfrac{\pi}{\sin \pi z}, \ z \in \mathbb{C}$
- $\Gamma'(x) = \Gamma(x)\left[-\gamma + \dfrac{1}{x} + \displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{x+k}\right)\right]$



Question 1

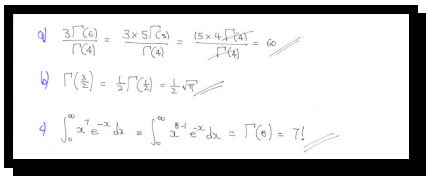
Evaluate each of the following expressions, leaving the final answer in exact simplified form.

a) $\dfrac{3\Gamma(6)}{\Gamma(4)}$.

b) $\Gamma\left(\dfrac{3}{2}\right)$.

c) $\displaystyle\int_0^{\infty} x^7 \, \mathrm{e}^{-x} \, dx$.

V, $\boxed{60}$, $\boxed{\dfrac{1}{2}\sqrt{\pi}}$, $\boxed{7!=5040}$



Question 2

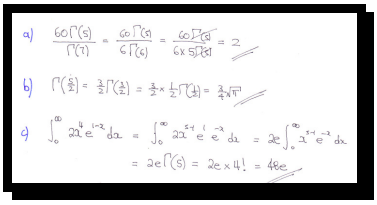
Evaluate each of the following expressions, leaving the final answer in exact simplified form.

a) $\dfrac{60\Gamma(5)}{\Gamma(7)}$.

b) $\Gamma\left(\dfrac{5}{2}\right)$.

c) $\displaystyle\int_0^{\infty} 2x^4 \, \mathrm{e}^{1-x} \, dx$.

V, $\boxed{2}$, $\boxed{\dfrac{3}{4}\sqrt{\pi}}$, $\boxed{48\mathrm{e}}$



Question 3

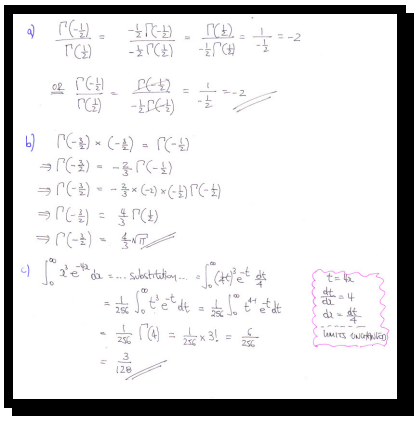
Evaluate each of the following expressions, leaving the final answer in exact simplified form.

a) $\dfrac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$.

b) $\Gamma\left(-\frac{3}{2}\right)$.

c) $\displaystyle\int_0^{\infty} x^3 \mathrm{e}^{-4x}\ dx$.

V, $\boxed{-2}$, $\boxed{\frac{4}{3}\sqrt{\pi}}$, $\boxed{\frac{3}{128}}$

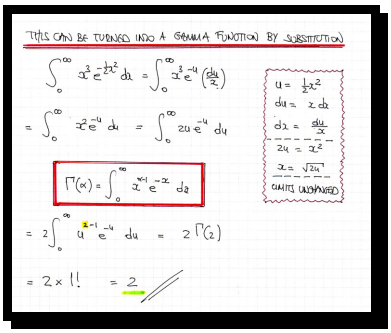


Question 4

By using techniques involving the Gamma function, find the value of

$$\int_0^{\infty} x^3 \mathrm{e}^{-\frac{1}{2}x^2} \, dx \, .$$

V, [ ], 2

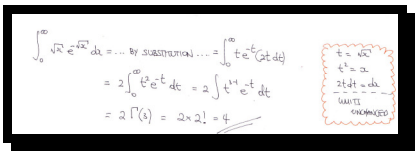


Question 5

By using techniques involving the Gamma function, find the value of

$$\int_0^{\infty} \sqrt{x} \, \mathrm{e}^{-\sqrt{x}} \, dx \, .$$

V, 4

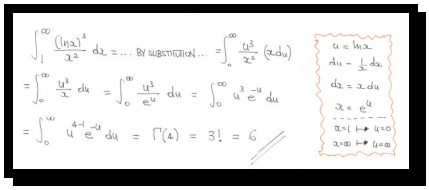


Question 6

By using techniques involving the Gamma function, find the value of

$$\int_1^{\infty} \frac{(\ln x)^3}{x^2}\, dx\,.$$

V, $\boxed{6}$



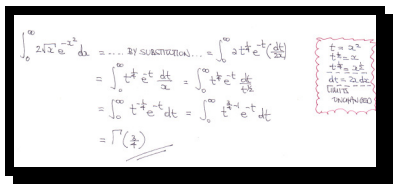
Question 7

By using techniques involving the Gamma function, find the exact value of

$$\int_0^{\infty} 2\sqrt{x}\, \mathrm{e}^{-x^2}\ \ dx\,.$$

Give the answer in the form $\Gamma(k)$, where k is a rational constant.

V, $\boxed{\Gamma\left(\frac{3}{4}\right)}$



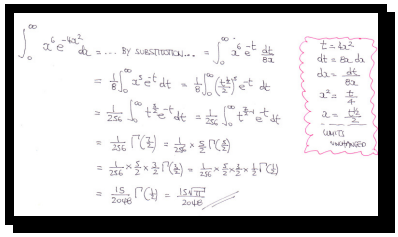
Question 8

By using techniques involving the Gamma function, find the exact value of

$$\int_0^{\infty} x^6 \mathrm{e}^{-4x^2} \, dx .$$

Give the answer in the form $k\sqrt{\pi}$, where k is a rational constant.

V, $\dfrac{15\sqrt{\pi}}{2048}$



Question 9

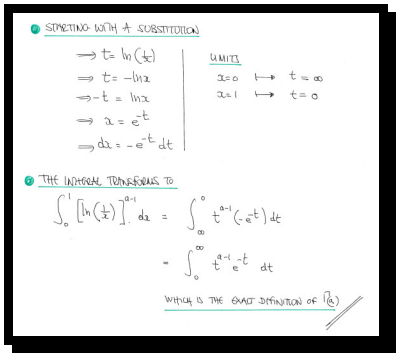
By using techniques involving the Gamma function, find the exact value of

$$\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{a-1} \, dx \, ,$$

where $a \neq 1, \ 0, -1, -2, -3, \ldots$

Give the answer in terms of a Gamma function.

V, [ ], $\boxed{\Gamma(a)}$

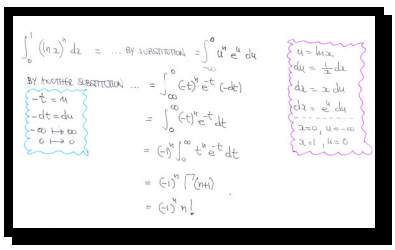


Question 10

By using techniques involving the Gamma function, show that

$$\int_0^1 [\ln x]^n \, dx = (-1)^n \, n! \, , \quad n \in \mathbb{N} \, .$$

proof

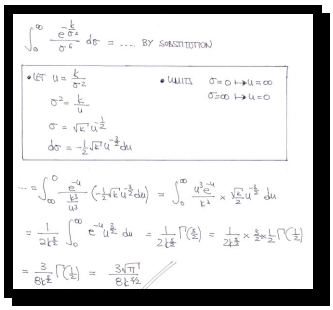


Question 11

By using techniques involving the Gamma function, show that

$$\int_0^{\infty} \frac{e^{-\frac{k}{\sigma^2}}}{\sigma^6}\, d\sigma = \frac{3\sqrt{\pi}}{8k^{\frac{5}{2}}}, \quad k \neq 0.$$

proof



Question 12

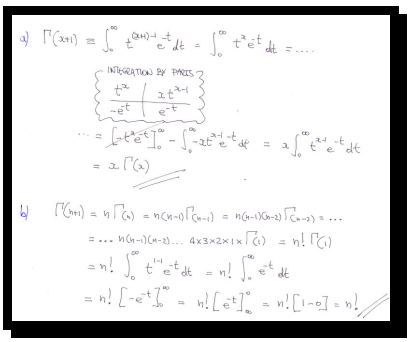
a) Show clearly that

$$\Gamma(x+1) \equiv x\,\Gamma(x), \; x \neq 0, \; x \notin -\mathbb{N}.$$

b) Hence show further that

$$\Gamma(n+1) \equiv n!, \; n \in \mathbb{N}.$$

proof

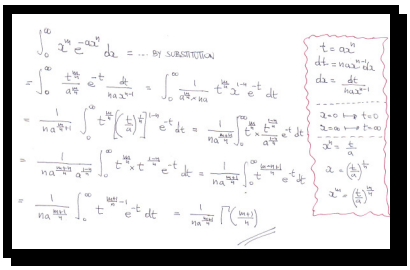


Question 13

By using techniques involving the Gamma function, show that

$$\int_0^{\infty} x^m \mathrm{e}^{-ax^n} \, dx = \frac{\Gamma\left(\frac{m+1}{n}\right)}{n a^{\frac{m+1}{n}}} \ .$$

proof

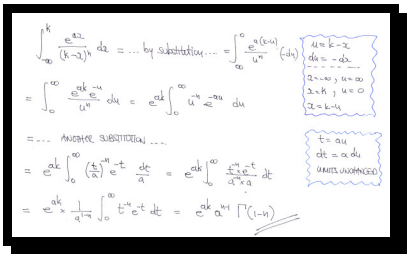


Question 14

By using techniques involving the Gamma function, show that

$$\int_{-\infty}^{k} \frac{\mathrm{e}^{ax}}{(k-x)^n} \, dx = \mathrm{e}^{ak} \, a^{n-1} \, \Gamma(1-n) \ , \ 0 < n < 1 \, .$$

proof



Question 15

The gamma function is defined as

$$\Gamma(x)=\int_0^{\infty} \mathrm{e}^{-t}\, t^{x-1}\ dt\,, \qquad x\neq 0,-1,\ -2,\ -3,\ ...$$

i. Show that for $x>0$

$$\Gamma(x+1)=x\,\Gamma(x)\,.$$

ii. Express the integral

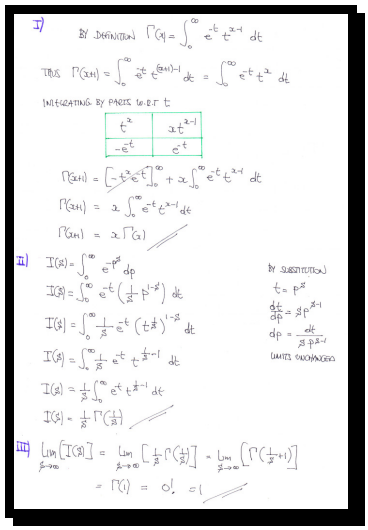
$$I(s)=\int_0^{\infty} \mathrm{e}^{-p^s}\ dp\ , \qquad s>0\,,$$

in terms of Gamma functions.

iii. Deduce that

$$\lim_{s\to\infty}\left[I(s)\right]=1\,.$$

$$\boxed{I(s)=\frac{1}{s}\Gamma\left(\frac{1}{s}\right)}$$

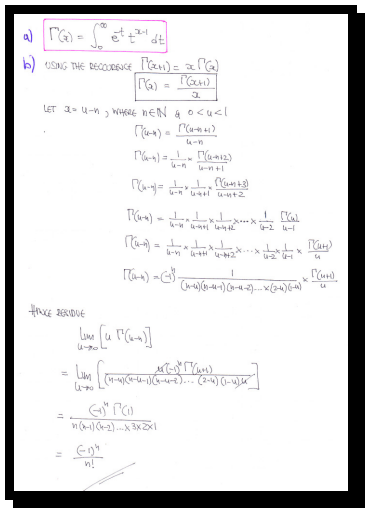


Question 16

a) Write down the definition of the Gamma function $\Gamma(x)$, for $\mathrm{Re}(x)>0$.

b) Use the standard recurrence relation of the Gamma function to extent $\Gamma(x)$, for $\mathrm{Re}(x)\leq 0$, and hence find the residues at the simple poles at $x=-n$, $n=0,\,1,\,2,\,3\,...$

$$\boxed{\frac{(-1)^n}{n!}}$$



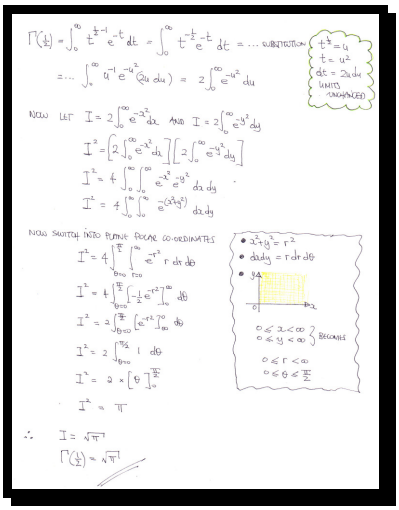
Question 17

$$I = 2\int_0^{\infty} \mathrm{e}^{-x^2}\ dx \quad \text{and} \quad I = 2\int_0^{\infty} \mathrm{e}^{-y^2}\ dy\,.$$

By finding an expression for I, show that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\,.$$

proof



Question 18

It is given that the following integral converges

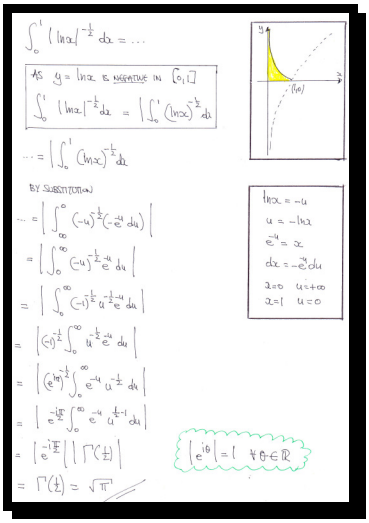
$$\int_0^1 \left|\ln x\right|^{-\frac{1}{2}} \, dx \, .$$

Evaluate the above integral by a suitable substitution introducing Gamma Functions.

You may assume that

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi} \, .$$

$\sqrt{\pi}$



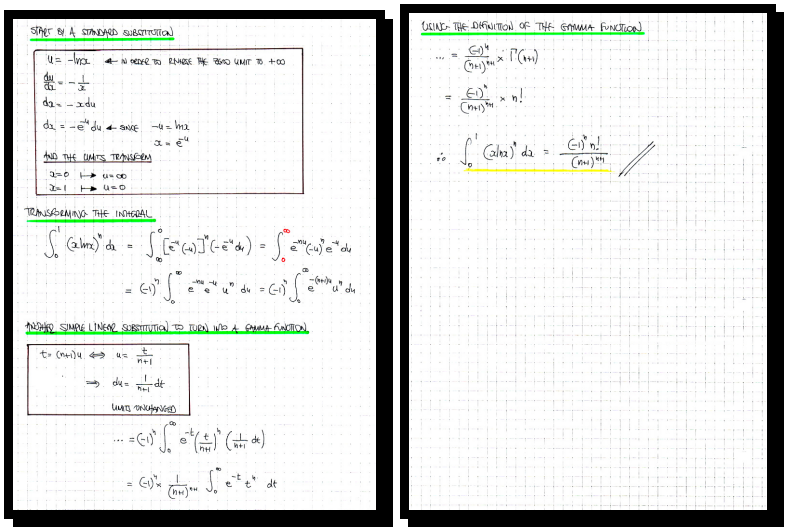
Question 19

It is given that the following integral converges

$$\int_0^1 \left(x \ln x\right)^n \, dx \, , \; n \in \mathbb{N} \, .$$

Evaluate the above integral by a suitable substitution introducing Gamma Functions.

$$\square \, , \quad \boxed{\frac{(-1)^n \, n!}{(n+1)^{n+1}}}$$



Question 20

It is given that the following integral converges

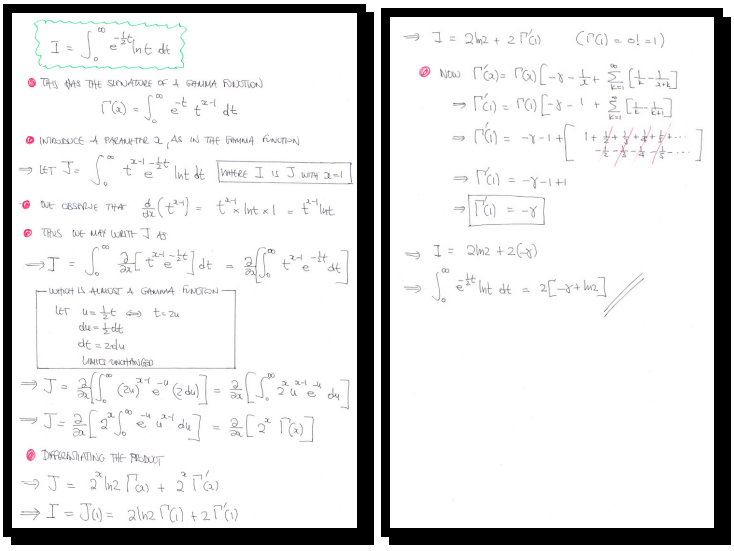
$$\int_0^{\infty} \mathrm{e}^{-\frac{1}{2}t} \ln t \ \ dt \, .$$

Evaluate the above integral by introducing a new parametric term in the integrand and carrying out a suitable differentiation under the integral sign.

You may assume that

$$\Gamma'(x) = \Gamma(x)\left[-\gamma + \frac{1}{x} + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{x+k} \right) \right].$$

$$\boxed{2(-\gamma + \ln 2)}$$

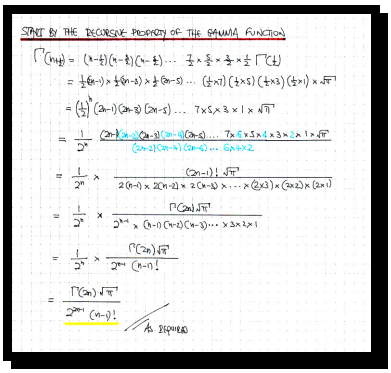


Question 21

Prove the validity of Legendre's duplication formula for the Gamma function, which states that

$$\Gamma\left(n+\frac{1}{2}\right)\equiv\frac{\Gamma(2n)\sqrt{\pi}}{2^{2n-1}\Gamma(n)},\ n\in\mathbb{N}.$$

, proof



Question 22

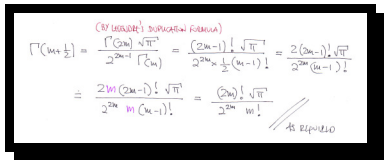
Legendre's duplication formula for the Gamma function, states that

$$\Gamma\left(m+\frac{1}{2}\right)\equiv\frac{\Gamma(2m)\sqrt{\pi}}{2^{2m-1}\Gamma(m)},\ m\in\mathbb{N}.$$

Show that

$$\Gamma\left(m+\frac{1}{2}\right)\equiv\frac{(2m)!\sqrt{\pi}}{2^{2m}\ m!},\ m\in\mathbb{N}$$

proof



Question 23

$$I(\lambda,x)=\int_0^{\infty} \mathrm{e}^{-\lambda t}\, t^{x-1} \ln t \;\; dt\,.$$

a) By carrying a suitable differentiation over the integral sign, show that

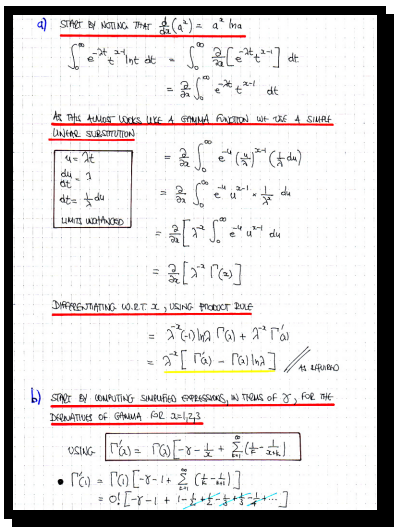
$$I(\lambda,x)=\lambda^{-x}\left[\Gamma'(x)-\Gamma(x)\ln\lambda\right].$$

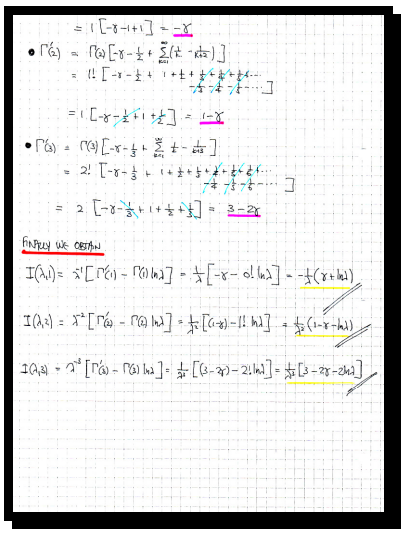
b) Find simplified expressions for $I(\lambda,1)$, $I(\lambda,2)$ and $I(\lambda,3)$.

You may assume that $\Gamma'(x)=\Gamma(x)\left[-\gamma+\dfrac{1}{x}+\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{x+k}\right)\right]$.

$\square$, $\boxed{I(\lambda,1)=-\dfrac{1}{\lambda}\left[\gamma+\ln\lambda\right]}$, $\boxed{I(\lambda,2)=\dfrac{1}{\lambda^2}\left[1-\gamma-\ln\lambda\right]}$,

$\boxed{I(\lambda,3)=\dfrac{1}{\lambda^3}\left[3-2\gamma-2\ln\lambda\right]}$





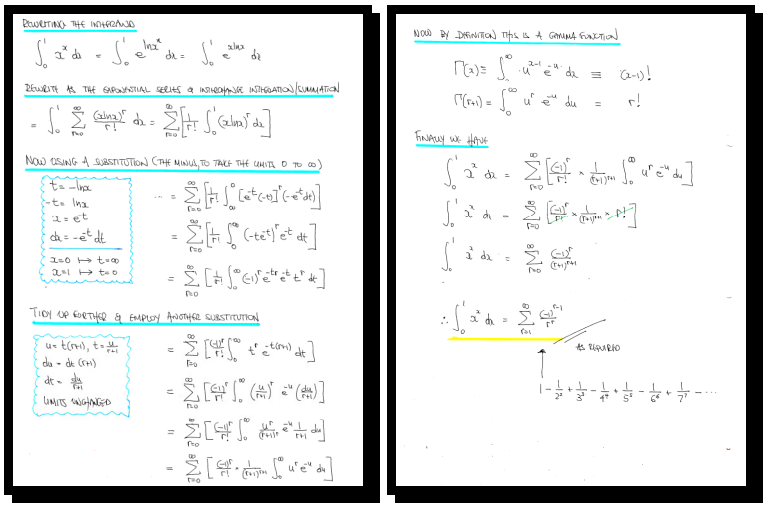
Question 24

By considering standard power series expansions and using Gamma functions, show

$$\int_0^1 x^x \, dx = \sum_{r=1}^{\infty}\left[\frac{(-1)^{r-1}}{r^r}\right].$$

You may assume that integration and summation commute.

, proof



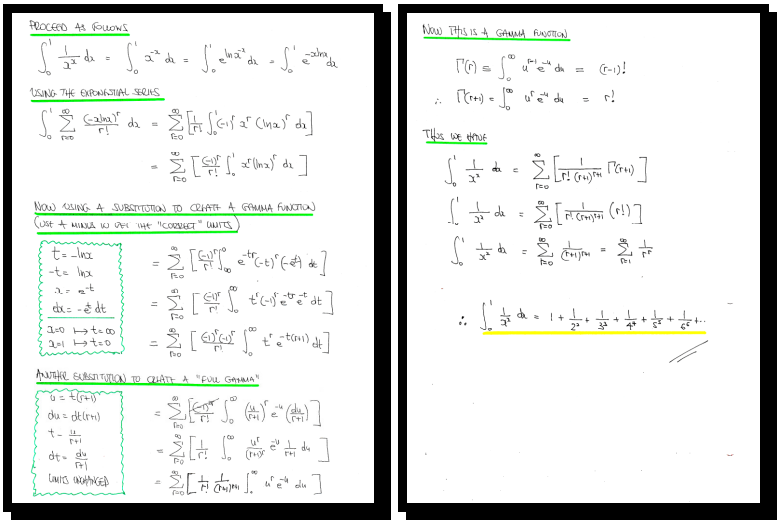
Question 25

By considering standard power series expansions and using Gamma functions, show

$$\int_0^1 \frac{1}{x^x}\, dx = 1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\frac{1}{5^5}+\ldots$$

You may assume that integration and summation commute.

[ ], [proof]



Question 26

$$I=\int_1^{\infty}\frac{(\ln x)^3}{x^2(x-1)}\,dx\,.$$

Show by the means of partial fractions and a suitable substitution that

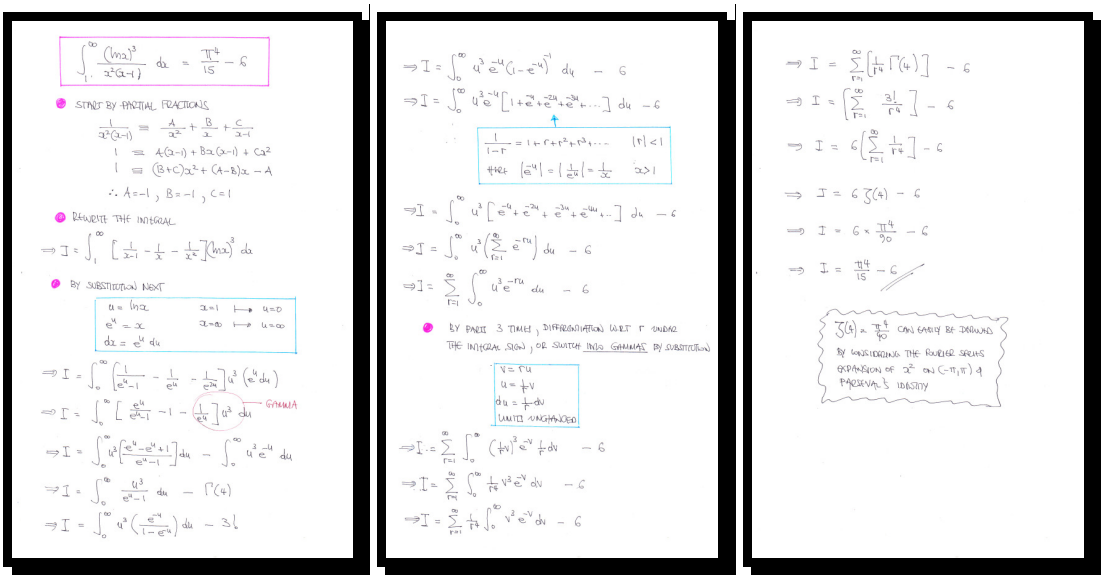
$$I=-6+\int_0^{\infty}u^3\mathrm{e}^{-u}\left(1-\mathrm{e}^{-u}\right)^{-1}\,du\,,$$

and hence show that $I=\dfrac{\pi^4}{15}-6$.

You may assume without proof that

$$\zeta(4)=\sum_{r=1}^{\infty}\left[\frac{1}{r^4}\right]=\frac{\pi^4}{90}.$$

proof

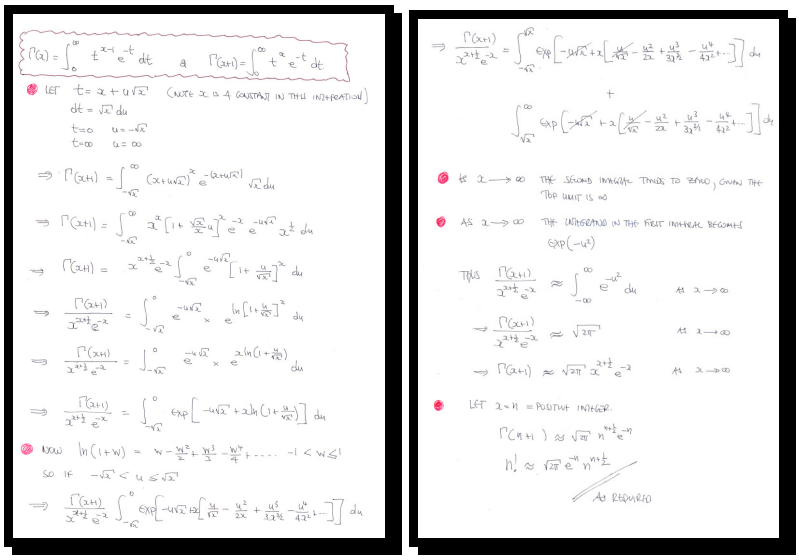


Question 27

By using the substitution $u = x + u\sqrt{x}$ in the integral definition of $\Gamma(x+1)$, show that for large values of n

$$n! \approx \sqrt{2\pi}\, \mathrm{e}^{-n}\, n^{n+\frac{1}{2}}, \quad n \in \mathbb{N}.$$

proof



BETA FUNCTION

SUMMARY OF THE BETA FUNCTION

The Beta function $\mathrm{B}(m,n)$, is defined as

$$\mathrm{B}(m,n) \equiv \int_0^1 x^{m-1}(1-x)^{n-1}\ dx .$$

Alternative definitions of $\mathrm{B}(m,n)$ are

$$\mathrm{B}(m,n) \equiv \int_0^{\frac{\pi}{2}} 2\sin^{2m-1}\theta\ \cos^{2n-1}\theta\, d\theta .$$

$$\mathrm{B}(m,n) \equiv \int_0^{\infty} \frac{x^{m-1}}{(x+1)^{m+n}}\ dx$$

Beta function common rules and facts

- $\mathrm{B}(m,n) \equiv \mathrm{B}(n,m)$
- $\mathrm{B}(m,n) \equiv \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, or $\mathrm{B}(m,n) \equiv \dfrac{(m-1)!(n-1)!}{(m+n+1)!}$, $m \in \mathbb{N}$, $n \in \mathbb{N}$

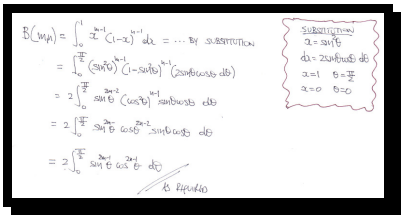
  or $\mathrm{B}(m,n) \equiv \dfrac{(m-1)(n-1)}{(m+n-1)(m+n-2)}\mathrm{B}(m-1,n-1)$

Question 1

Show clearly that

$$\mathrm{B}(m,n)=2\int_0^{\frac{\pi}{2}}(\sin\theta)^{2m-1}(\sin\theta)^{2n-1}\ d\theta\,.$$

proof

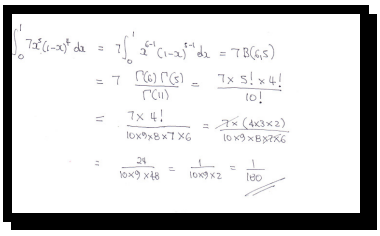


Question 2

By using techniques involving the Beta function, find the exact value of

$$\int_0^1 7x^5(1-x)^4\ dx\,.$$

$$\frac{1}{180}$$

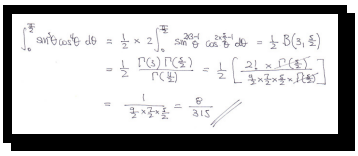


Question 3

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\frac{\pi}{2}} \sin^5\theta\cos^4\theta \ \ d\theta .$$

$$\boxed{\frac{8}{315}}$$

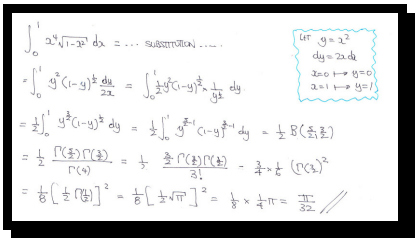


Question 4

By using techniques involving the Beta function, find the exact value of

$$\int_0^1 x^4\sqrt{1-x^2} \ \ dx .$$

$$\boxed{\frac{\pi}{32}}$$

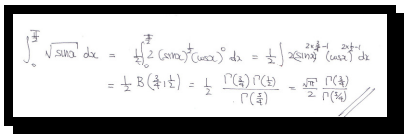


Question 5

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\ \ dx\,.$$

Express the value of the above integral in terms of Gamma functions, in their simplest form

$$\boxed{\frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{2\Gamma\left(\frac{5}{4}\right)}}$$

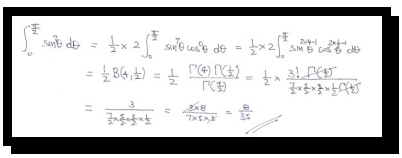


Question 6

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\frac{\pi}{2}} \sin^7\theta\ \ d\theta\,.$$

$$\boxed{\frac{8}{35}}$$

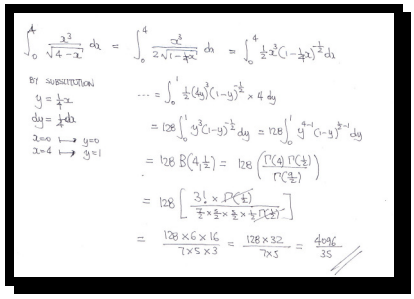


Question 7

By using techniques involving the Beta function, find the exact value of

$$\int_0^4 \frac{x^3}{\sqrt{4-x}}\, dx\,.$$

$$\boxed{\frac{4096}{35}}$$

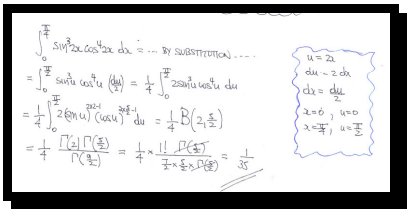


Question 8

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\frac{\pi}{4}} \sin^2 2x \cos^4 2x \;\; dx\,.$$

$$\boxed{\frac{1}{35}}$$

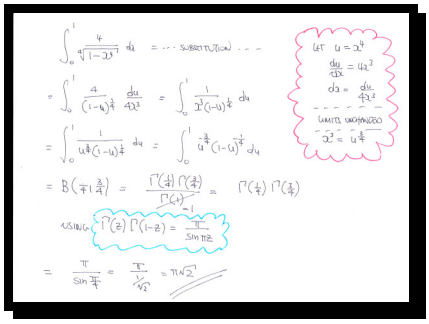


Question 9

By using techniques involving the Beta function, find the exact value of

$$\int_0^1 \frac{4}{\sqrt[4]{1-x^4}}\,dx\,.$$

$$\boxed{\pi\sqrt{2}}$$

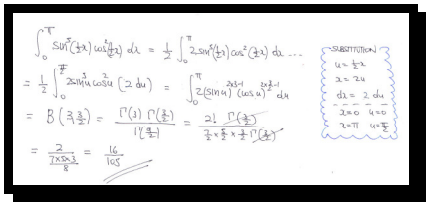


Question 10

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\pi} \sin^5\left(\tfrac{1}{2}x\right)\cos^2\left(\tfrac{1}{2}x\right)\,dx\,.$$

$$\boxed{\frac{16}{105}}$$



Question 11

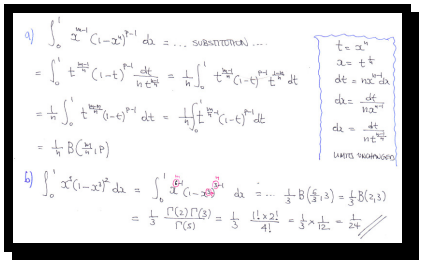
a) Show clearly that

$$\int_0^1 x^{m-1}\left(1-x^n\right)^{p-1}\ dx=\frac{1}{n}\mathrm{B}\left(\frac{m}{n},p\right),\ n\neq 0\,.$$

b) Hence, find the exact value of

$$\int_0^1 x^5\left(1-x^3\right)^2\ dx\,.$$

$$\boxed{\frac{1}{24}}$$

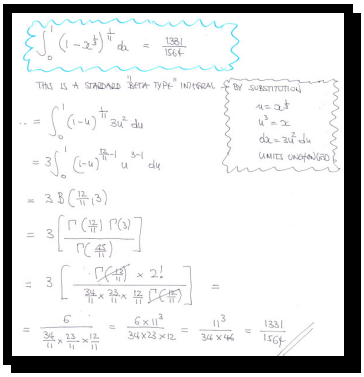


Question 12

By using techniques involving the Beta function, find the exact value of

$$\int_0^1 \sqrt[11]{1-\sqrt[3]{x}}\ dx\,.$$

$$\boxed{\frac{1331}{1564}}$$



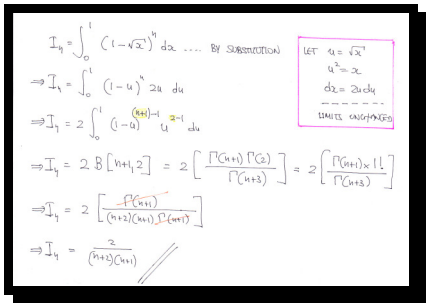
Question 13

$$I_n = \int_0^1 \left(1-\sqrt{x}\right)^n \, dx \, .$$

Use Beta functions to show that

$$I_n = \frac{2}{(n+2)(n+1)} \, .$$

proof



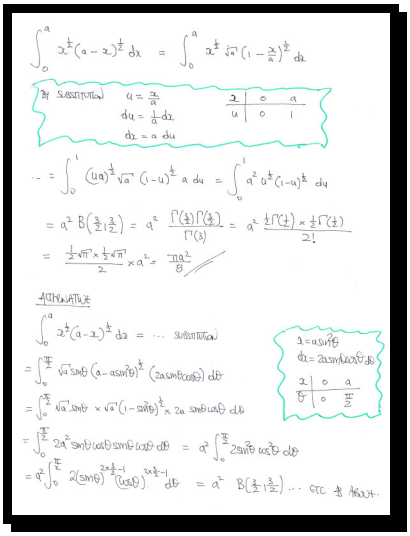
Question 14

$$\int_0^a x^{\frac{1}{2}}\sqrt{a-x}\ dx .$$

Find an exact simplified value for the integral above, by using a suitable substitution to transform into a Beta function.

You may assume that a is a positive constant.

$$\boxed{\frac{1}{8}\pi a^2}$$



Question 15

$$I_{m,n}=\int_0^{\frac{\pi}{2}}\sin^m\theta\cos^n\theta\ d\theta\ ,\ m\in\mathbb{N}\ ,\ n\in\mathbb{N}\ .$$

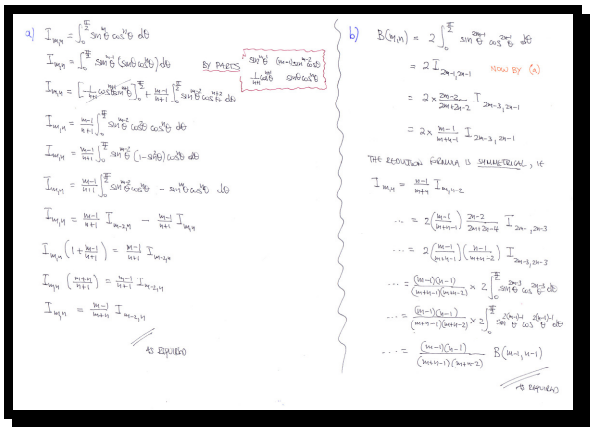
a) Show clearly that

$$I_{m,n}=\frac{m-1}{m+n}I_{m-2,n}$$

b) Hence, show further that

$$\mathrm{B}\left(m,n\right)=\frac{\left(m-1\right)\left(n-1\right)}{\left(m+n-1\right)\left(m+n-2\right)}\mathrm{B}\left(m-1,n-1\right).$$

proof

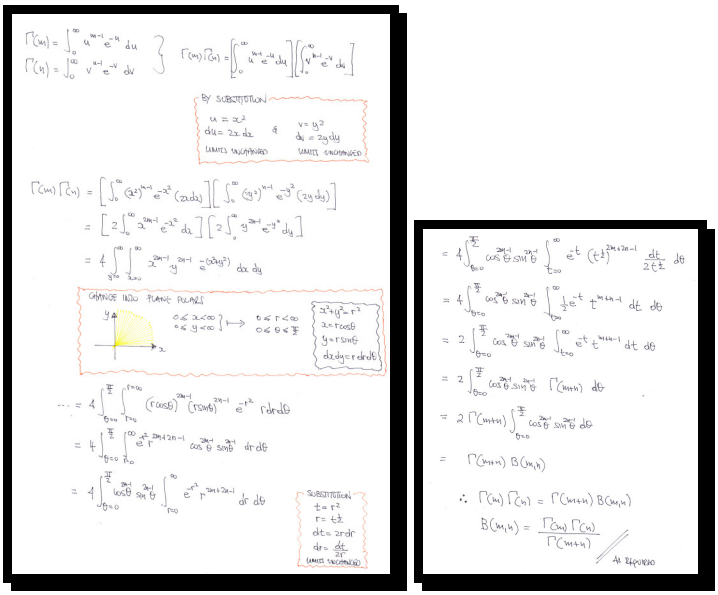


Question 16

Show clearly that

$$\mathrm{B}(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

proof

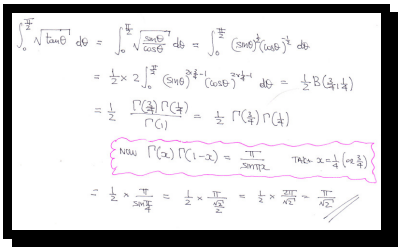


Question 17

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\frac{\pi}{2}}\sqrt{\tan\theta}\ \ d\theta\,.$$

$$\frac{\pi}{\sqrt{2}}$$

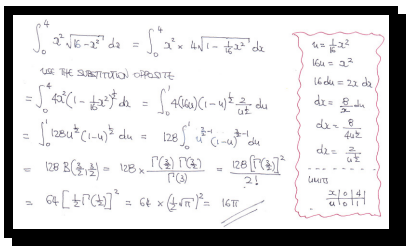


Question 18

By using techniques involving the Beta function, find the exact value of

$$\int_0^4 x^2\sqrt{16-x^2}\ \ dx\,.$$

$\boxed{16\pi}$

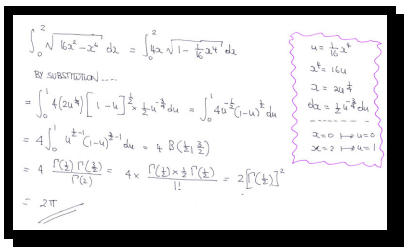


Question 19

By using techniques involving the Beta function, find the exact value of

$$\int_0^2 \sqrt{16x^2-x^6}\ \ dx\,.$$

$\boxed{2\pi}$

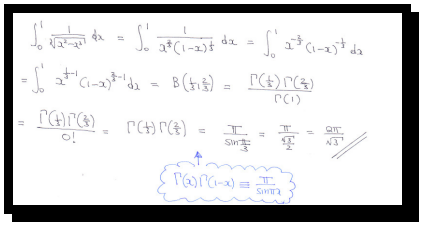


Question 20

By using techniques involving the Beta function, find the exact value of

$$\int_0^1 \frac{1}{\sqrt[3]{x^2-x^3}}\,dx\,.$$

$$\boxed{\frac{2\pi}{\sqrt{3}}}$$

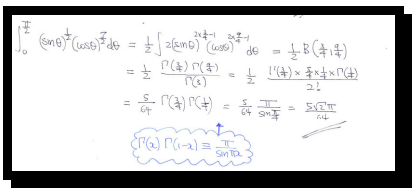


Question 21

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{1}{2}}(\cos\theta)^{\frac{7}{2}}\,d\theta\,.$$

$$\boxed{\frac{5\pi\sqrt{2}}{64}}$$

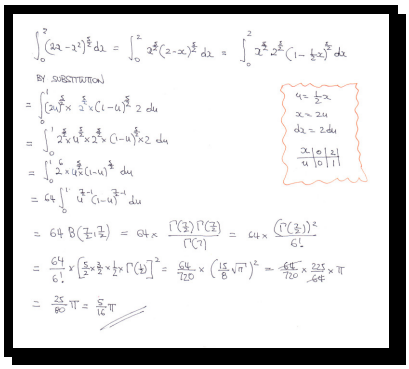


Question 22

By using techniques involving the Beta function, find the exact value of

$$\int_0^2 \left(2x-x^2\right)^{\frac{5}{2}} \, dx \, .$$

$$\boxed{\frac{5}{16}\pi}$$

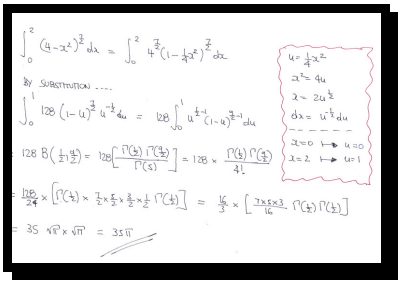


Question 23

By using techniques involving the Beta function, find the exact value of

$$\int_0^2 \left(4-x^2\right)^{\frac{7}{2}} \, dx \, .$$

$$\boxed{35\pi}$$

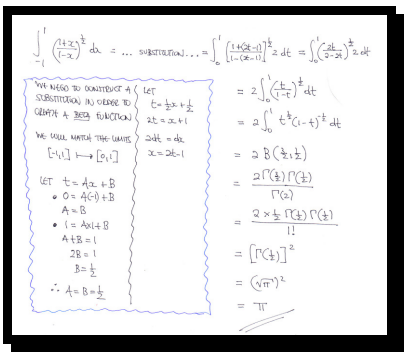


Question 24

By using techniques involving the Beta function, find the exact value of

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\ dx\,.$$

$$\boxed{\pi}$$

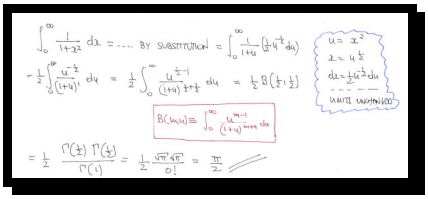


Question 25

By using techniques involving the Beta function, find the exact value of

$$\int_{0}^{\infty} \frac{1}{1+x^2}\ dx\,.$$

$$\boxed{\frac{\pi}{2}}$$

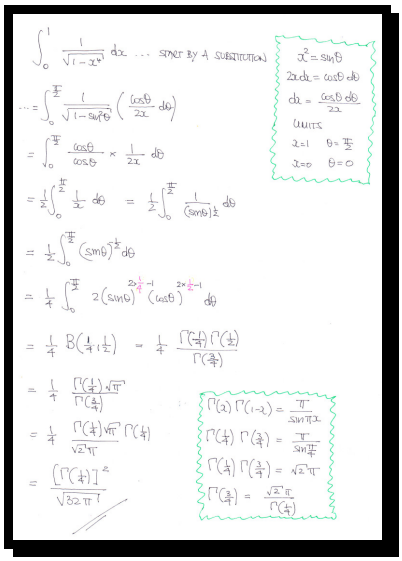


Question 26

Use the substitution $x^2 = \sin\theta$ to show that

$$\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx = \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{\sqrt{32\pi}}.$$

proof

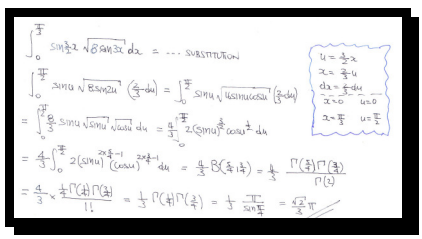


Question 27

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\frac{\pi}{3}} \sin\left(\frac{3}{2}x\right)\sqrt{8\sin 3x}\,\,dx\,.$$

$$\frac{\sqrt{2}}{3}\pi$$

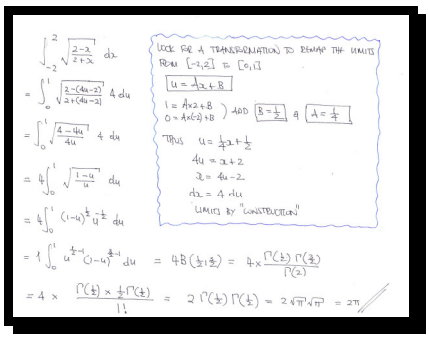


Question 28

By using techniques involving the Beta function, find the exact value of

$$\int_{-2}^{2} \sqrt{\frac{2-x}{2+x}}\, dx\,.$$

$$\boxed{2\pi}$$

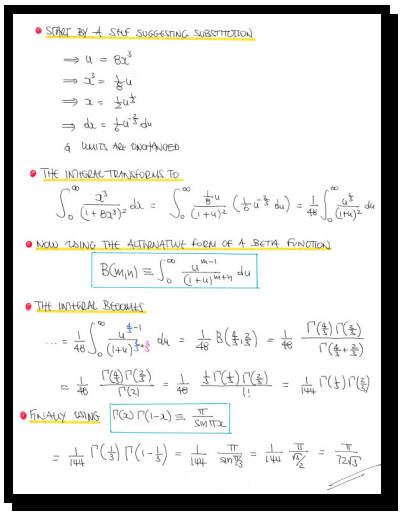


Question 29

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\infty} \frac{x^3}{\left(1+8x^3\right)^2}\, dx\,.$$

$\square$, $\boxed{\dfrac{\pi}{72\sqrt{3}}}$

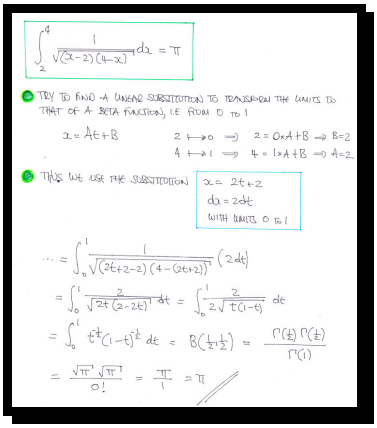


Question 30

By using techniques involving the Beta function, find the exact value of

$$\int_2^4 \frac{1}{\sqrt{(x-2)(4-x)}}\,dx\,.$$

$\boxed{\pi}$

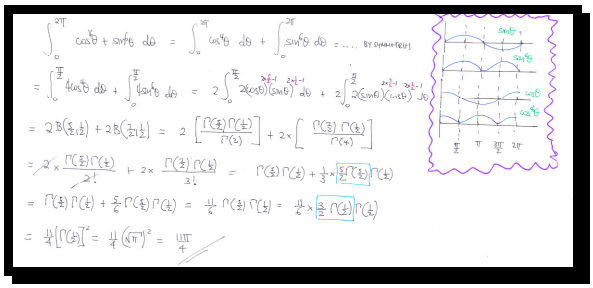


Question 31

By using techniques involving the Beta function, find the exact value of

$$\int_0^{2\pi} \cos^4\theta+\sin^6\theta\ \ d\theta\,.$$

$\boxed{\dfrac{11\pi}{4}}$

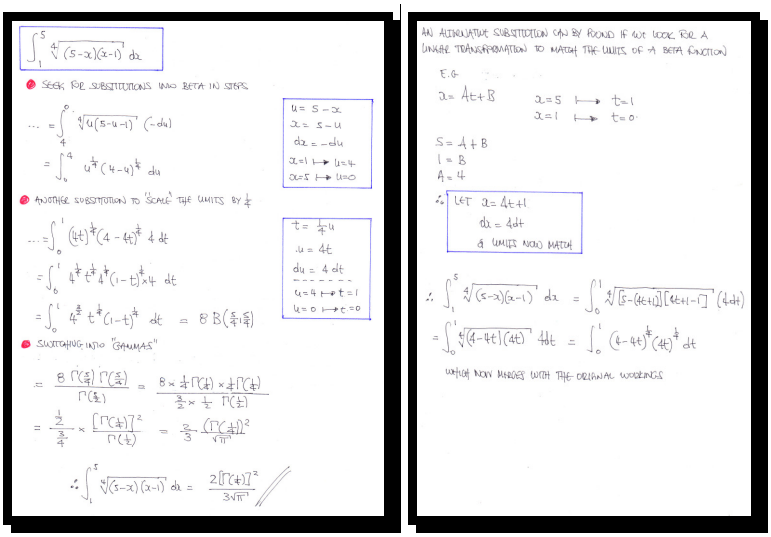


Question 32

By using techniques involving the Beta function, show that

$$\int_1^5 \sqrt[4]{(5-x)(x-1)}\ dx = \frac{2\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{3\sqrt{\pi}}.$$

proof



Question 33

$$\mathrm{B}(m,n) \equiv \int_0^1 x^{m-1}(1-x)^{n-1}\ dx\,.$$

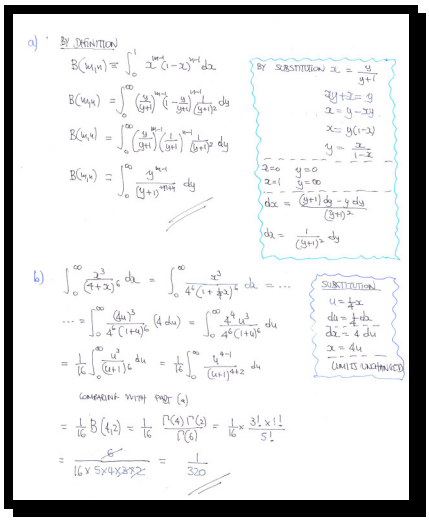
a) Use the substitution $x = \dfrac{y}{y+1}$ to show that

$$\mathrm{B}(m,n) \equiv \int_0^{\infty} \frac{y^{m-1}}{(y+1)^{m+n}}\ dy\,.$$

b) Hence find the exact value of

$$\int_0^{\infty} \frac{x^3}{(4+x)^6}\ dx\,.$$

$$\boxed{\frac{1}{320}}$$



Question 34

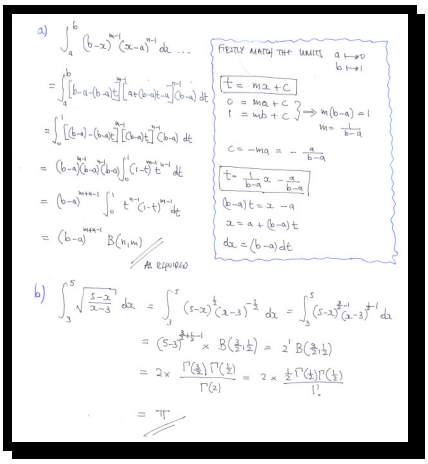
a) Show clearly that

$$\int_a^b (b-x)^{m-1}(x-a)^{n-1}\,dx = (b-a)^{m+n-1}\,\mathrm{B}(m,n).$$

b) Hence find the exact value of

$$\int_3^5 \sqrt{\frac{5-x}{x-3}}\,dx.$$

π

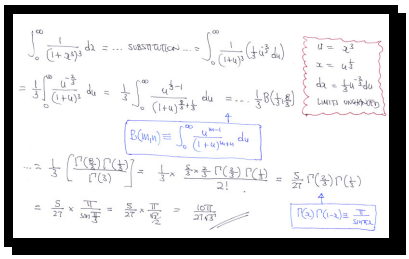


Question 35

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\infty} \frac{1}{\left(1+x^3\right)^3}\, dx\,.$$

$$\boxed{\frac{10\pi}{27\sqrt{3}}}$$

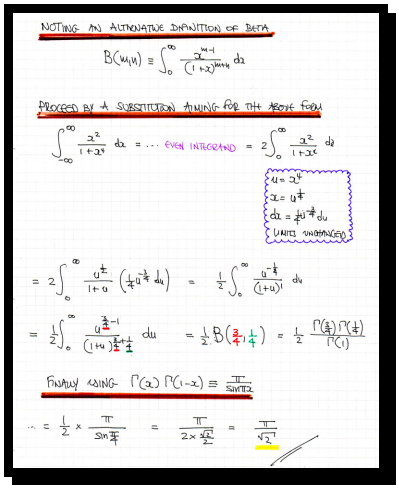


Question 36

By using techniques involving the Beta function, find the exact value of

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx\,.$$

$$\boxed{}\,, \qquad \boxed{\frac{\pi}{\sqrt{2}}}$$

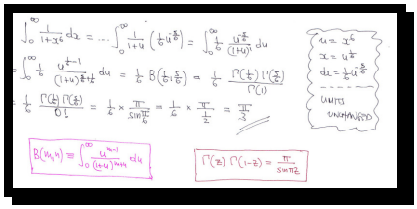


Question 37

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\infty} \frac{1}{1+x^6}\, dx\,.$$

$$\boxed{\frac{\pi}{3}}$$

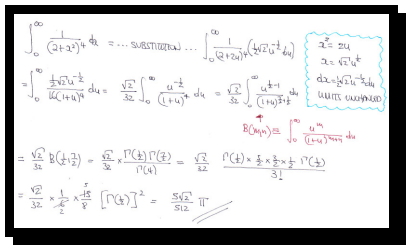


Question 38

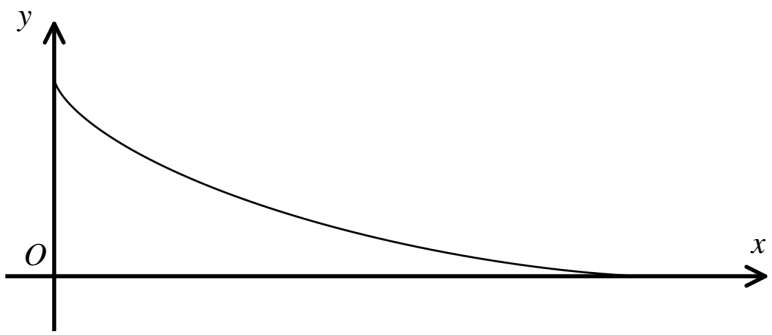
By using techniques involving the Beta function, find the exact value of

$$\int_0^{\infty} \frac{1}{\left(2+x^2\right)^4}\, dx\,.$$

$$\boxed{\frac{5\pi\sqrt{2}}{512}}$$



Question 39



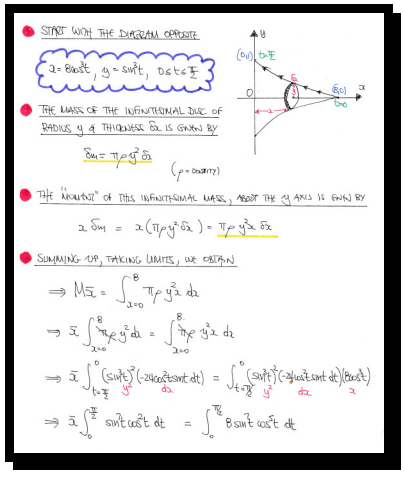
The figure above shows the curve with parametric equations

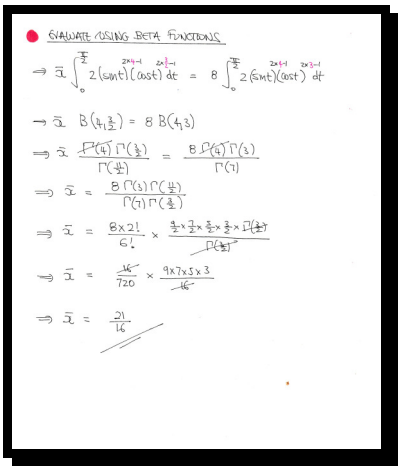
$$x = 8\cos^3 t, \quad y = \sin^3 t, \quad 0 \leq t \leq \frac{1}{2}\pi .$$

The finite region bounded by the curve and the coordinate axes is revolved fully about the x axis, forming a solid of revolution S.

Determine the x coordinate of the centre of mass of S.

$\boxed{}$, $\boxed{\overline{x} = \frac{21}{16}}$



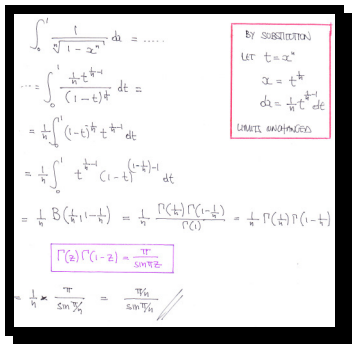


Question 40

By using techniques involving the Beta function, show that

$$\int_0^1 \frac{1}{\sqrt[n]{1-x^n}}\,dx = \frac{\frac{\pi}{n}}{\sin\left(\frac{\pi}{n}\right)}.$$

proof

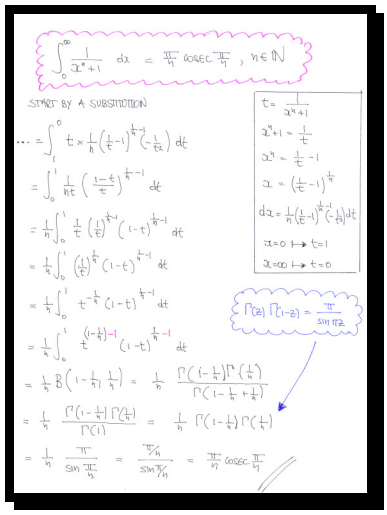


Question 41

By using techniques involving the Beta function, show that

$$\int_0^1 \frac{1}{x^n+1}\,dx = \frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right).$$

proof



Question 42

$$\mathrm{B}(m,n) \equiv \int_0^1 x^{m-1}(1-x)^{n-1}\ dx\,.$$

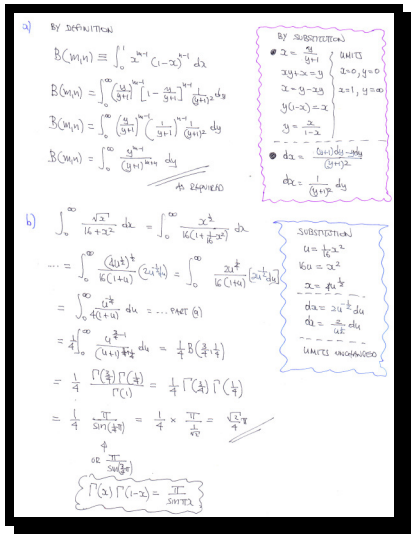
a) Use the substitution $x = \dfrac{y}{y+1}$ to show that

$$\mathrm{B}(m,n) \equiv \int_0^{\infty} \frac{y^{m-1}}{(y+1)^{m+n}}\ dy\,.$$

b) Hence find the exact value of

$$\int_0^{\infty} \frac{\sqrt{x}}{16+x^2}\ dx\,.$$

$$\boxed{\frac{\pi\sqrt{2}}{4}}$$

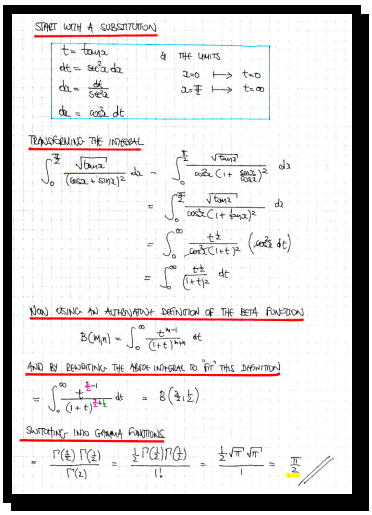


Question 43

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{(\cos x+\sin x)^2}\, dx\,.$$

$\square$, $\boxed{\frac{\pi}{2}}$

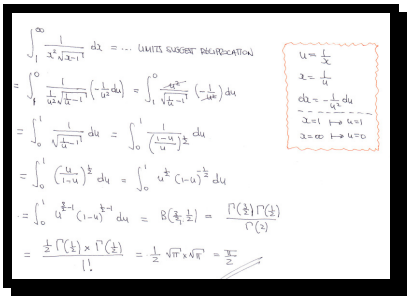


Question 44

By using techniques involving the Beta function, find the exact value of

$$\int_1^{\infty} \frac{1}{x^2\sqrt{x-1}}\, dx\,.$$

$$\boxed{\frac{\pi}{2}}$$



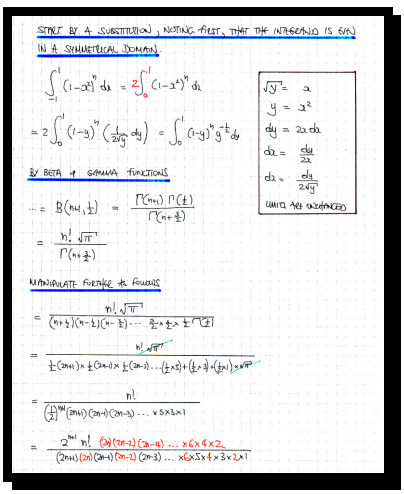
Question 45

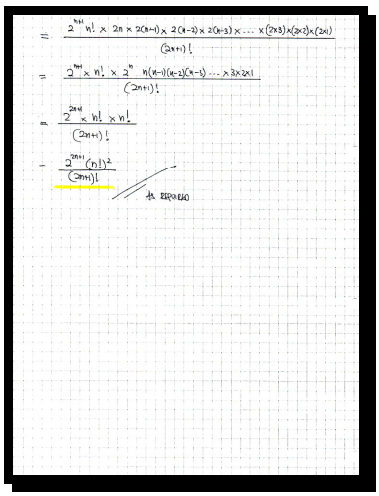
$$I=\int_{-1}^{1}\left(1-x^2\right)^n\ dx\ ,\ n\in\mathbb{N}\ .$$

Use techniques involving the Beta function to show that

$$I=\frac{2^{2n+1}\left(n!\right)^2}{\left(2n+1\right)!}\ .$$

$\boxed{}$, $\boxed{\text{proof}}$



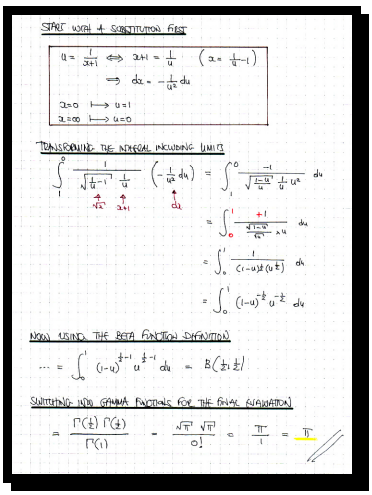


Question 46

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\infty} \frac{1}{(x+1)\sqrt{x}}\, dx\,.$$

$\square$, $\boxed{\pi}$

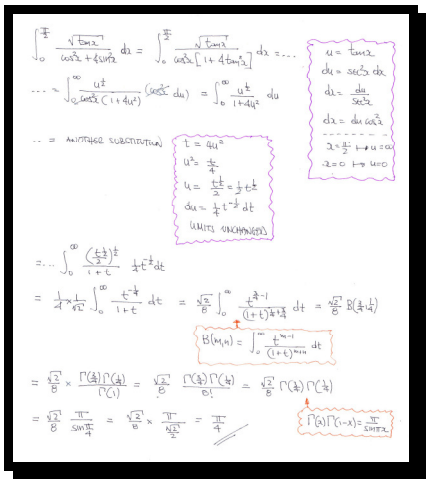


Question 47

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\cos^2 x + 4\sin^2 x}\, dx\,.$$

$$\frac{\pi}{4}$$



Question 48

$$\int_0^{\infty} \mathrm{e}^{-u^2} u^{2x-1} \, du \, .$$

a) Determine the value of the above integral in terms of Gamma functions.

b) Show that

$$\int_0^{\frac{1}{2}\pi} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} \, d\theta \equiv \frac{1}{2} \mathrm{B}(x,y) \, .$$

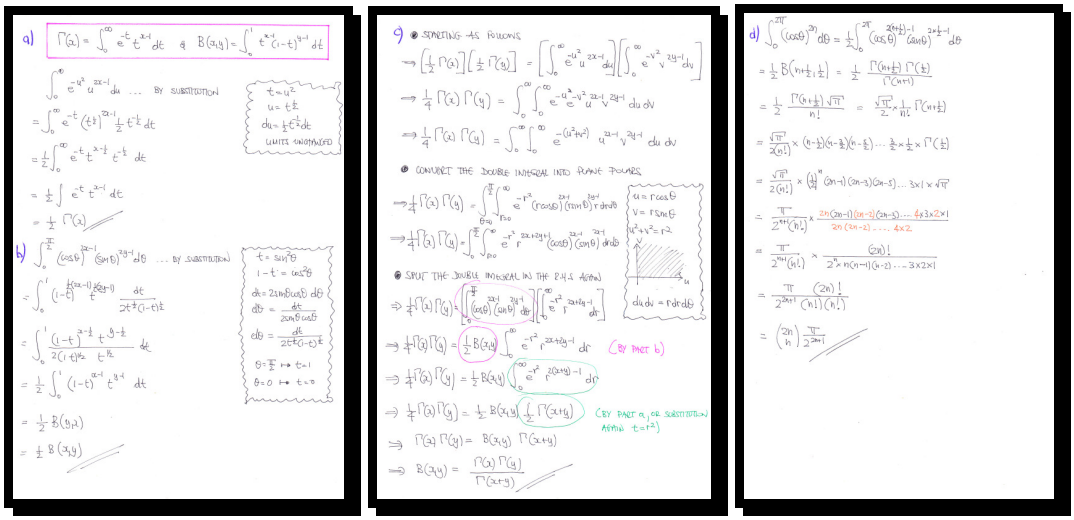
c) Use the results of part **(a)** and **(b)** to show further that

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \, .$$

d) Hence deduce that

$$\int_0^{\frac{1}{2}\pi} (\cos\theta)^{2n} \, d\theta = \binom{2n}{n} \frac{\pi}{2^{2n+1}} \, .$$

$\boxed{\frac{1}{2}\Gamma(x)}$



Question 49

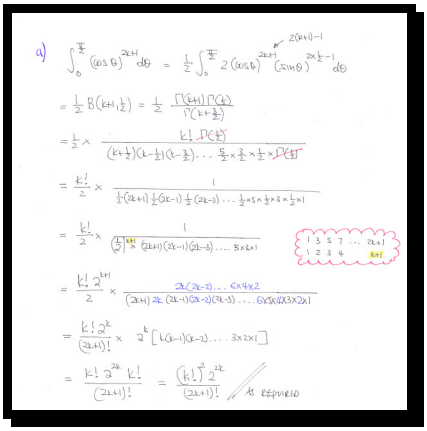
a) By using techniques involving the Beta function and the Gamma function, show that

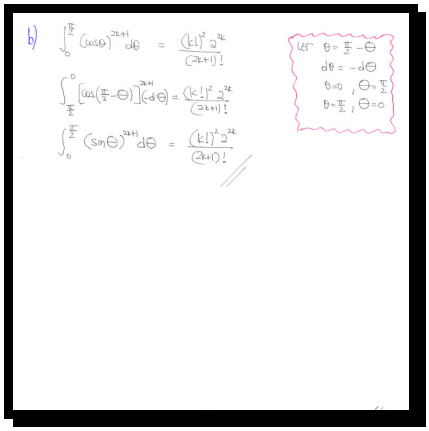
$$\int_0^{\frac{\pi}{2}} (\cos\theta)^{2k+1}\, d\theta = \frac{(k!)^2\, 2^{2k}}{(2k+1)!}.$$

b) Use a suitable substitution in the above integral to deduce an expression for

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{2k+1}\, d\theta.$$

$$\boxed{\int_0^{\frac{\pi}{2}} (\sin\theta)^{2k+1}\, d\theta = \int_0^{\frac{\pi}{2}} (\cos\theta)^{2k+1}\, d\theta = \frac{(k!)^2\, 2^{2k}}{(2k+1)!}}$$





Question 50

$$f(z)=\frac{z^{p-1}}{z+1},\ z\in\mathbb{C},\ p\in\mathbb{R},\ 0<p<1.$$

a) By integrating $f(z)$ over a keyhole contour with a branch cut along the positive x axis, show that

$$\int_0^\infty \frac{x^{p-1}}{1+x}\ dx=\frac{\pi}{\sin(p\pi)}.$$

b) Hence show further that

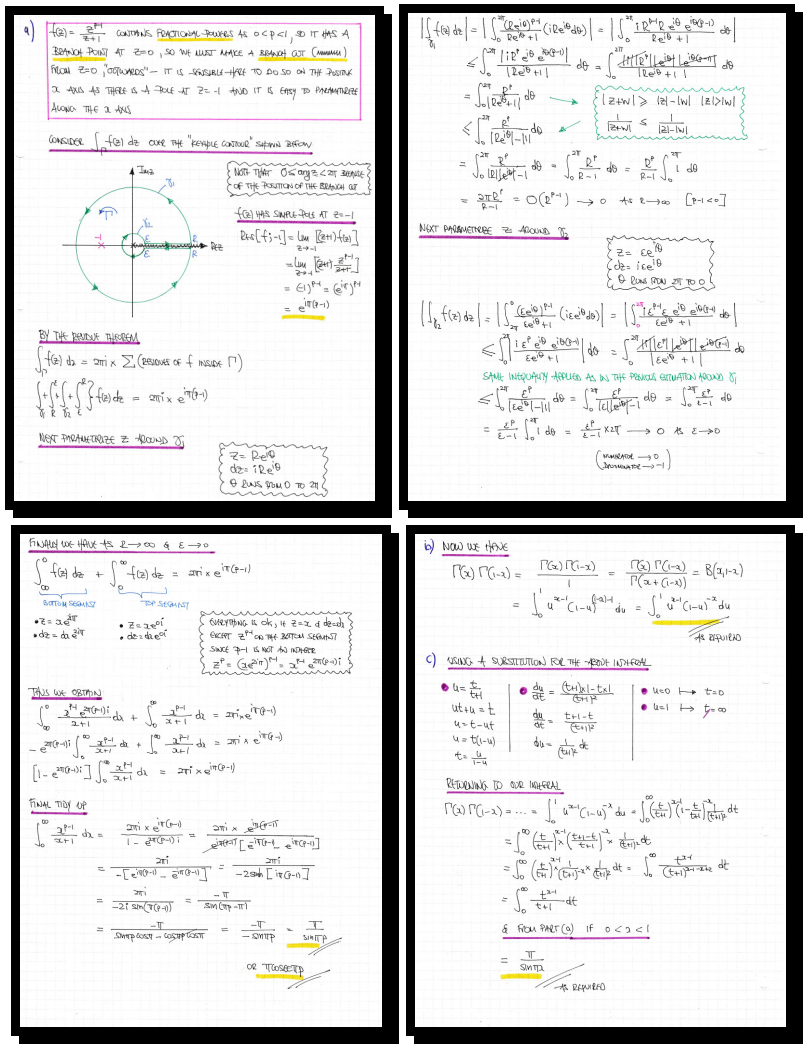
$$\Gamma(x)\Gamma(1-x)=\int_0^1 u^{x-1}(1-u)^{-x}\ du.$$

c) Use the substitution $u=\dfrac{t}{t+1}$ to deduce that

$$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin(\pi x)},\ 0<x<1.$$

V, [ ], proof

[solution overleaf]



Question 51

$$\mathrm{B}(m,n)\equiv\int_0^1 x^{m-1}(1-x)^{n-1}\ dx\,.$$

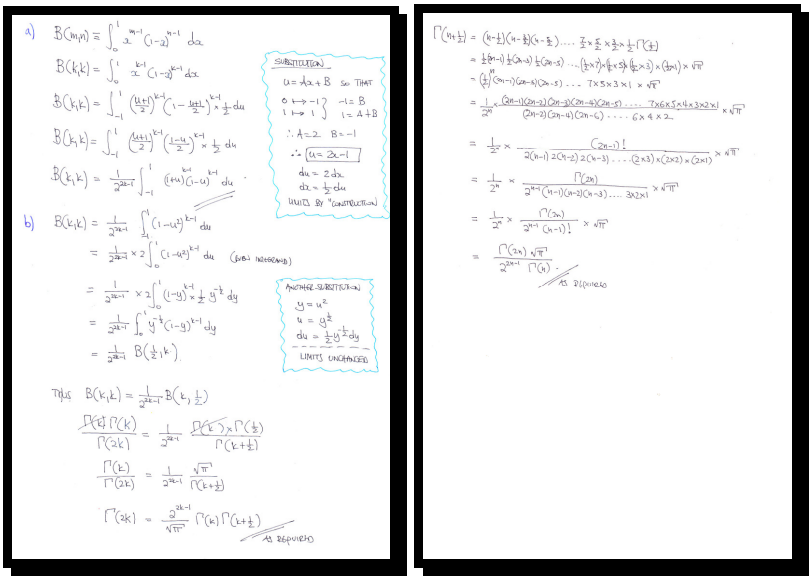
a) Use a suitable substitution to show that

$$\mathrm{B}(k,k)=\frac{1}{2^{2k-1}}\int_{-1}^{1}(1+u)^{k-1}(1-u)^{k-1}\ du\,.$$

b) Hence show further

$$\Gamma\left(k+\frac{1}{2}\right)=\frac{\Gamma(2k)\sqrt{\pi}}{2^{2k-1}\Gamma(k)}\,.$$

proof



Question 52

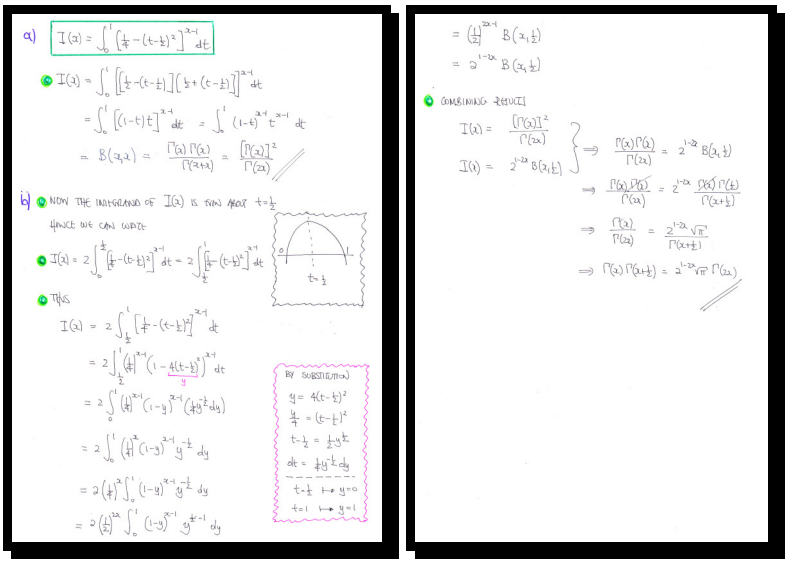
$$I(x)\equiv\int_0^1\left[\frac{1}{4}-\left(t-\frac{1}{2}\right)^2\right]^{x-1}\,dt\,.$$

a) Express $I(x)$ in terms of Gamma functions.

b) By considering the symmetry of the integrand about the line $t=\frac{1}{2}$ and using the substitution $y=4\left(t-\frac{1}{2}\right)^2$, show that

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)=2^{1-2x}\sqrt{\pi}\,\Gamma(2k)\,.$$

$$\boxed{I(x)=\frac{\Gamma(x)\Gamma(x)}{\Gamma(2x)}}$$



Question 53

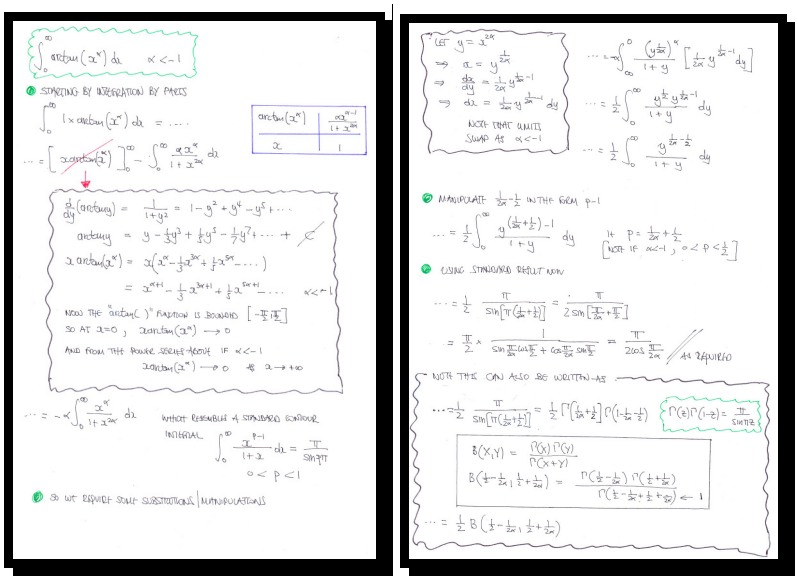
$$I(\alpha)=\int_0^{\infty} \arctan\left(x^{\alpha}\right)\, dx\,, \quad \alpha<-1\,.$$

By using integration by parts, and justifying all steps, show that

$$I(\alpha) = \frac{1}{2}\pi\sec\left(\frac{\pi}{2\alpha}\right) = \frac{1}{2}\mathrm{B}\left(\frac{1}{2}-\frac{1}{2\alpha},\frac{1}{2}+\frac{1}{2\alpha}\right).$$

You may assume that $\displaystyle\int_0^{\infty} \frac{x^{p-1}}{(x+1)}\, dx=\frac{\pi}{\sin p\pi}\,,\ 0<p<1\,.$

proof



Question 54

It is given that for any real constants x and y

$$\mathrm{B}(x+1,y+1)=\frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)}.$$

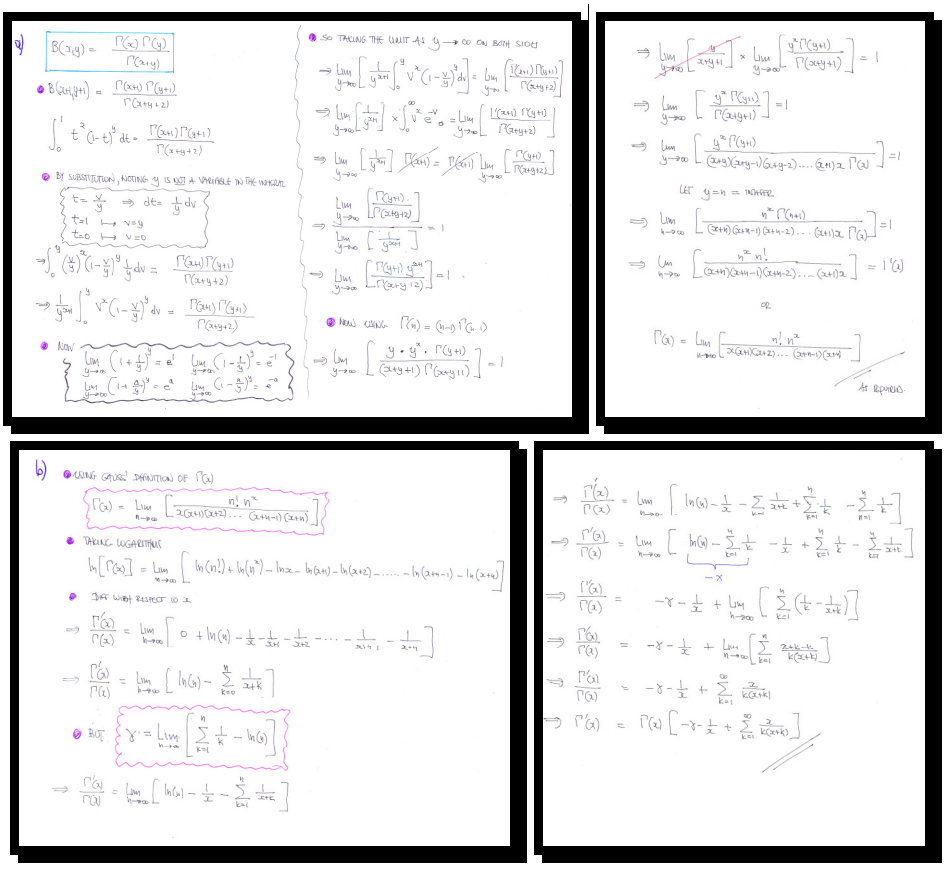
a) Use the integral definition of $\mathrm{B}(x+1,y+1)$, with a suitable substitution to derive Gauss' definition of the gamma function

$$\Gamma(x)=\lim_{n\to\infty}\left[\frac{n^x\, n!}{x(x+1)(x+2)...(x+n-1)(x+n)}\right].$$

b) Hence show

$$\Gamma'(x)=\Gamma(x)\left[-\gamma-\frac{1}{x}+\sum_{k=1}^{\infty}\frac{x}{k(x+k)}\right].$$

proof

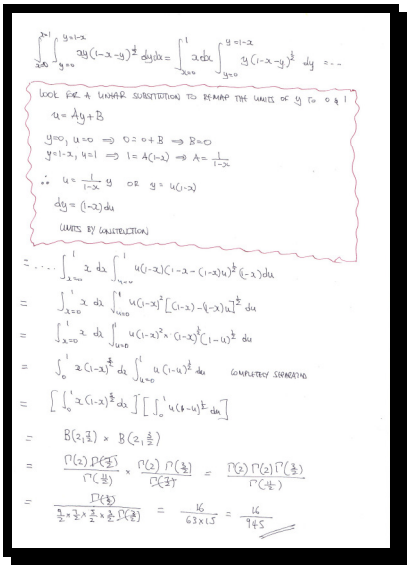


Question 55

$$I=\int_0^1\int_0^{1-x} xy\left(1-x-y\right)^{\frac{1}{2}}\,dydx\,.$$

Determine the exact value of I by transforming it into an expression involving Beta functions.

$$\boxed{\frac{16}{945}}$$



Question 56

A finite region R defined by the inequalities

$$x^3+y^3+z^3\le 1,\quad x\ge 0,\quad y\ge 0,\quad z\ge 0.$$

Show that the volume of R is

$$\left[\frac{1}{k}\Gamma\left(\frac{1}{k}\right)\right]^k,$$

where k is a positive integer to be found.

$$\boxed{k=3}$$

