FOURIER TRANSFORM

Fourier Transform Summary

Definitions

- $\mathcal{F}\left[f(x)\right]=\hat{f}(k)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}f(x)\mathrm{e}^{-\mathrm{i}kx}\ dx$
- $\mathcal{F}^{-1}\left[\hat{f}(k)\right]=f(x)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}f(k)\mathrm{e}^{\mathrm{i}kx}\ dk$

Useful Results

- $\mathcal{F}\left[f'(x)\right]=\mathrm{i}k\,\hat{f}(k)$
- $\mathcal{F}\left[x\,f(x)\right]=\mathrm{i}\dfrac{d}{dk}\left[\hat{f}(k)\right]$

Shift Results

- $\mathcal{F}\left[f(x+c)\right]=\mathrm{e}^{\mathrm{i}kc}\,\hat{f}(k)$
- $\mathcal{F}^{-1}\left[\hat{f}(k+c)\right]=\mathrm{e}^{-\mathrm{i}cx}f(x)$

Convolution Theorem

$$\mathcal{F}\left\{[f*g](x)\right\}\ =\ \sqrt{2\pi}\,\mathcal{F}\left[f(x)\right]\mathcal{F}\left[g(x)\right]$$

where $[f*g](x)=\displaystyle\int_{-\infty}^{\infty}f(x-y)\,g(y)\ dy$

Parseval's Theorem

$$\int_{-\infty}^{\infty}h(y)\,g(y)\ dy\ =\ \int_{-\infty}^{\infty}\overline{\hat{h}}(k)\,\hat{g}(k)\ dk\quad\text{or}\quad\int_{-\infty}^{\infty}\left|h(y)\right|^2\ dy\ =\ \int_{-\infty}^{\infty}\left|\hat{h}(k)\right|^2\ dk$$

FINDING FOURIER TRANSFORMS and INVERSES

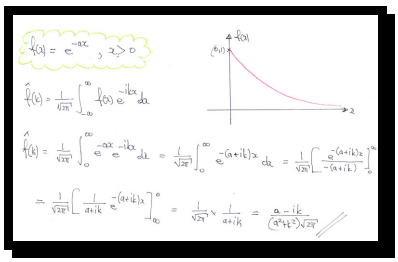
Question 1

$$f(x)=\mathrm{e}^{-ax}\ ,\ x>0\ ,$$

where a is a positive constant.

Find the Fourier transform of $f(x)$.

$$\hat{f}(k)=\frac{a-\mathrm{i}k}{\left(a^2+k^2\right)\sqrt{2\pi}}$$



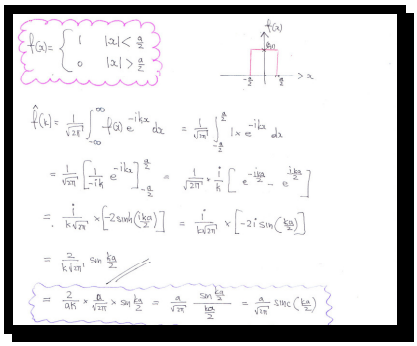
Question 2

$$f(x)=\begin{cases}1 & |x|<\frac{1}{2}a\\ 0 & |x|>\frac{1}{2}a\end{cases},$$

where a is a positive constant.

Find the Fourier transform of $f(x)$.

$$\boxed{\hat{f}(k)=\frac{2}{k\sqrt{2\pi}}\sin\left(\frac{1}{2}ka\right)=\frac{a}{\sqrt{2\pi}}\operatorname{sinc}\left(\frac{1}{2}ka\right)}$$

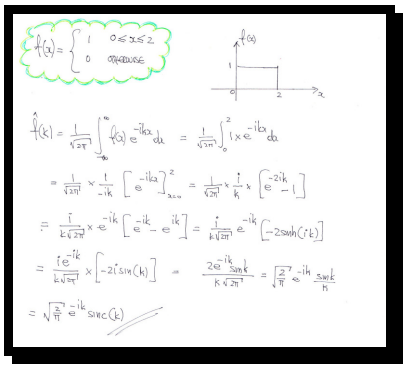


Question 3

$$f(x)=\begin{cases}1 & 0\le x\le 2\\ 0 & \text{otherwise}\end{cases}.$$

Find the Fourier transform of $f(x)$.

$$\boxed{\hat{f}(k)=\sqrt{\frac{2}{\pi}}\,\mathrm{e}^{-\mathrm{i}k}\,\operatorname{sinc}k}$$



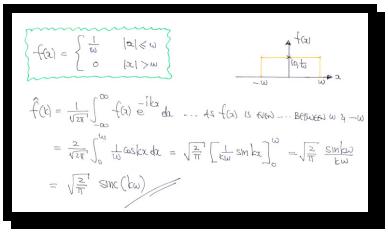
Question 4

$$f(x)=\begin{cases}\dfrac{1}{\omega} & |x|\le\omega\\ 0 & |x|>\omega\end{cases}$$

where ω is a positive constant.

Find the Fourier transform of $f(x)$.

$$\boxed{\hat{f}(k)=\sqrt{\frac{2}{\pi}}\operatorname{sinc}\omega}$$



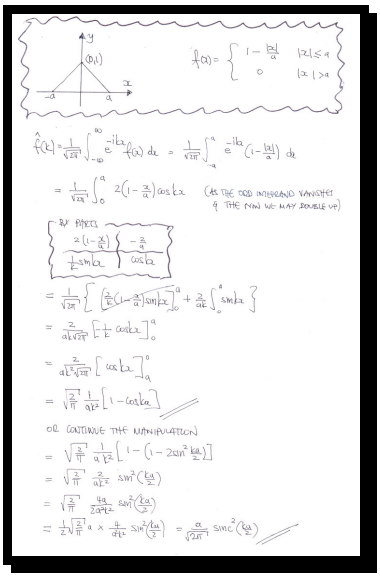
Question 5

The function $f(x)$ is defined in terms of the positive constant a, by

$$f(x)=\begin{cases}1-\dfrac{|x|}{a} & |x|\le a\\ 0 & |x|>a\end{cases}$$

Find the Fourier transform of $f(x)$.

$$\boxed{\mathcal{F}\left[f(x)\right]=\hat{f}(k)=\sqrt{\frac{2}{\pi}}\frac{1}{ak^2}\left[1-\cos(ak)\right]=\frac{a}{\sqrt{2\pi}}\operatorname{sinc}^2\left(\frac{1}{2}ka\right)}$$



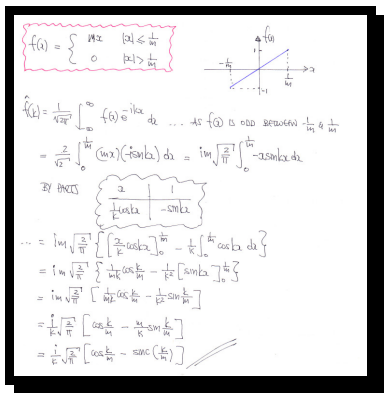
Question 6

$$f(x)=\begin{cases} mx & |x|\le \dfrac{1}{m} \\ 0 & |x|>\dfrac{1}{m} \end{cases}.$$

where m is a positive constant.

Find the Fourier transform of $f(x)$.

$$\boxed{\hat{f}(k)=\frac{\mathrm{i}}{k}\sqrt{\frac{2}{\pi}}\left[\cos\left(\frac{k}{m}\right)-\operatorname{sinc}\left(\frac{k}{m}\right)\right]}$$

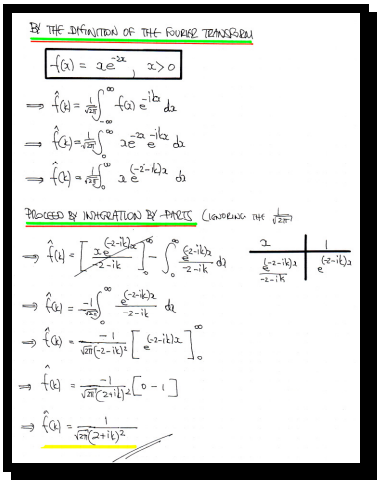


Question 7

$$f(x)=x\mathrm{e}^{-2x},\ x>0.$$

Find, by direct integration, the Fourier transform of $f(x)$.

$\square$, $\hat{f}(k)=\dfrac{1}{(2+\mathrm{i}k)^2\sqrt{2\pi}}$



Question 8

The triangle function $\Lambda_n(x)$ is defined as

$$\Lambda_n(x)=\begin{cases}\dfrac{1}{n^2}(n+x) & -n<x<0\\ \dfrac{1}{n^2}(n-x) & 0<x<n\\ 0 & \text{otherwise}\end{cases}$$

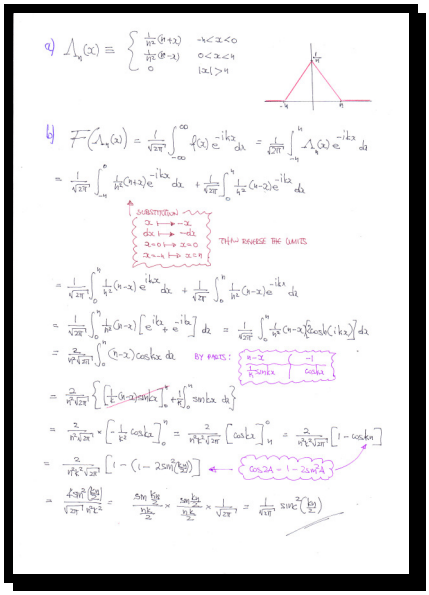
where n is a positive constant.

a) Sketch the graph of $\Lambda_n(x)$.

b) Show that the Fourier transform of $\Lambda_n(x)$ is

$$\frac{1}{\sqrt{2\pi}}\operatorname{sinc}^2\left(\frac{1}{2}kn\right).$$

proof



Question 9

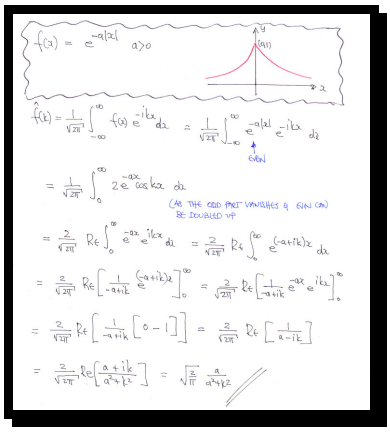
The function f is defined by

$$f(x)=\mathrm{e}^{-a|x|},$$

where a is a positive constant.

Find the Fourier transform of $f(x)$.

$$\mathcal{F}\left[\mathrm{e}^{-a|x|}\right]=\hat{f}(k)=\sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+k^2}$$



Question 10

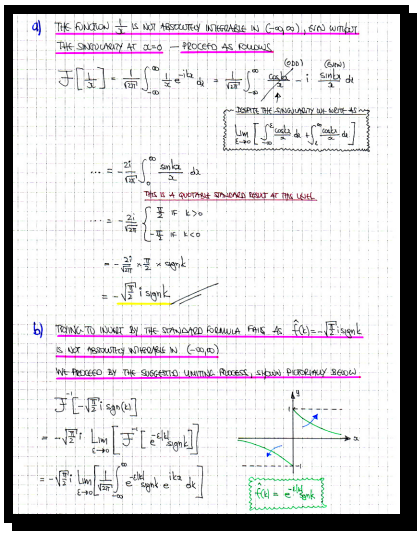
The function f is defined by

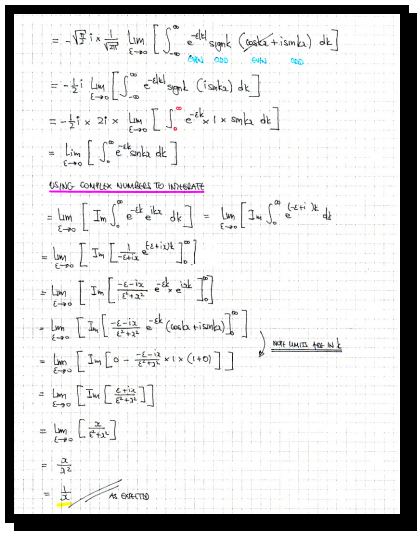
$$f(x)=\frac{1}{x},\ x\neq 0.$$

a) Determine the Fourier transform of $f(x)$, assuming without proof any standard results about $\displaystyle\int_0^{\infty}\frac{\sin ax}{x}\,dx$.

b) By introducing the converging factor $\mathrm{e}^{-\varepsilon|x|}$ and letting $\varepsilon\to 0$, invert the answer of part **(a)** to obtain f.

$$\square,\quad \boxed{\mathcal{F}\left[\frac{1}{x}\right]=\hat{f}(k)=-\mathrm{i}\sqrt{\frac{\pi}{2}}\,\operatorname{sign}(k)}$$





Question 11

The impulse function $\delta(x)$ is defined by

$$\delta(x)=\begin{cases}\infty & x=0\\ 0 & x\neq 0\end{cases}$$

a) Determine

i. … $\mathcal{F}\left[\delta(x)\right]$.

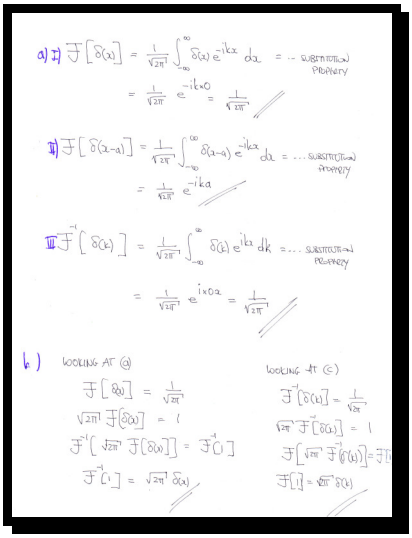
ii. … $\mathcal{F}\left[\delta(x-a)\right]$, where a is a positive constant.

iii. … $\mathcal{F}^{-1}\left[\delta(k)\right]$.

b) Use the above results to deduce $\mathcal{F}[1]$ and $\mathcal{F}^{-1}[1]$.

$$\boxed{\mathcal{F}\left[\delta(x)\right]=\frac{1}{\sqrt{2\pi}}},\ \boxed{\mathcal{F}\left[\delta(x-a)\right]=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}ka}},\ \boxed{\mathcal{F}^{-1}\left[\delta(k)\right]=\frac{1}{\sqrt{2\pi}}},$$

$$\boxed{\mathcal{F}[1]=\sqrt{2\pi}\,\delta(k)},\ \boxed{\mathcal{F}^{-1}[1]=\sqrt{2\pi}\,\delta(x)}$$



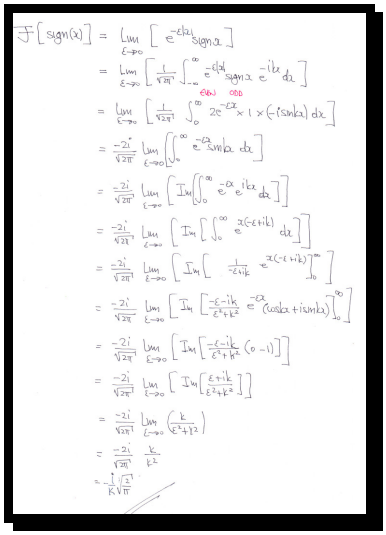
Question 12

The signum function $\text{sign}(x)$ is defined by

$$\text{sign}(x)=\begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases}$$

By introducing the converging factor $e^{-\varepsilon|x|}$ and letting $\varepsilon \to 0$, determine the Fourier transform of $\text{sign}(x)$.

$$\boxed{\mathcal{F}\left[\text{sign}(x)\right]=-\frac{\mathrm{i}}{k}\sqrt{\frac{1}{\pi}}}$$



Question 13

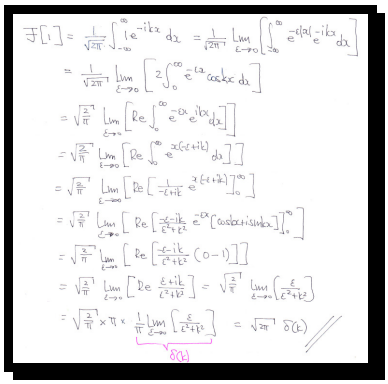
The Unit function $\mathrm{U}(x)$ is defined by

$$\mathrm{U}(x)=1.$$

By introducing the converging factor $\mathrm{e}^{-\varepsilon|x|}$ and letting $\varepsilon\to 0$, determine the Fourier transform of $\mathrm{U}(x)$.

You may assume that $\delta(t)=\dfrac{1}{\pi}\lim\limits_{\varepsilon\to 0}\left[\dfrac{\varepsilon}{\varepsilon^2+t^2}\right]$.

$$\boxed{\mathcal{F}\left[\mathrm{U}(x)\right]=\sqrt{2\pi}\,\delta(k)}$$



Question 14

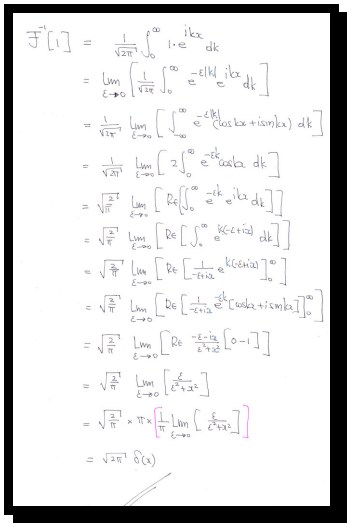
The Unit function $\mathrm{U}(x)$ is defined by

$$\mathrm{U}(x)=1.$$

By introducing the converging factor $\mathrm{e}^{-\varepsilon|k|}$ and letting $\varepsilon\to 0$, find $\mathcal{F}^{-1}\left[\mathrm{U}(k)\right]$.

You may assume that $\delta(t)=\dfrac{1}{\pi}\lim\limits_{\varepsilon\to 0}\left[\dfrac{\varepsilon}{\varepsilon^2+t^2}\right]$.

$$\boxed{\mathcal{F}^{-1}\left[\mathrm{U}(k)\right]=\sqrt{2\pi}\,\delta(x)}$$



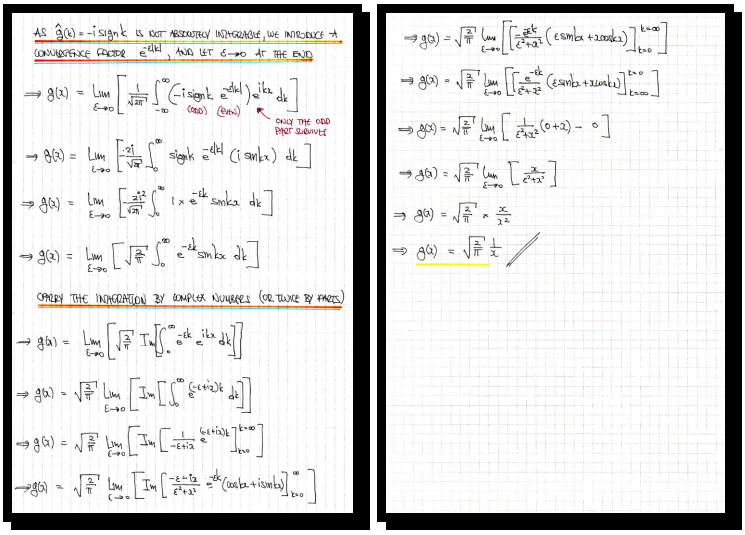
Question 15

The function $\mathrm{g}(x)$ has Fourier transform given by

$$\hat{g}(k)=-\mathrm{i}\,\mathrm{sign}(k).$$

By introducing the converging factor $\mathrm{e}^{-\varepsilon|k|}$ and letting $\varepsilon\to 0$, find $\mathcal{F}^{-1}\left[\hat{g}(k)\right]$.

$\square$, $\mathcal{F}^{-1}\left[\hat{g}(k)\right]=\sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{x}$



Question 16

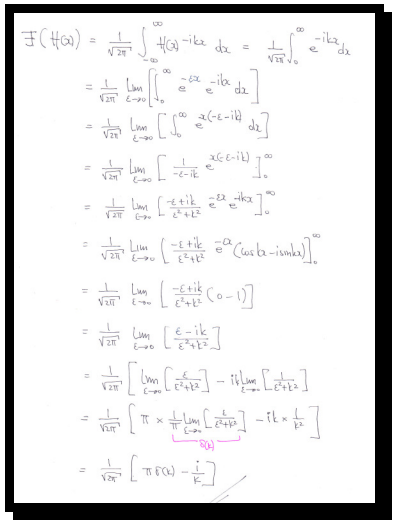
The Heaviside function $\mathrm{H}(x)$ is defined by

$$\mathrm{H}(x)=\begin{cases} 1 & x\geqslant 0 \\ 0 & x<0 \end{cases}$$

By introducing the converging factor $\mathrm{e}^{-\varepsilon x}$ and letting $\varepsilon\to 0$, determine the Fourier transform of $\mathrm{H}(x)$.

You may assume that $\delta(t)=\dfrac{1}{\pi}\lim\limits_{\varepsilon\to 0}\left[\dfrac{\varepsilon}{\varepsilon^2+t^2}\right]$.

$$\boxed{\mathcal{F}\left[\mathrm{H}(x)\right]=\frac{1}{\sqrt{2\pi}}\left[\pi\delta(k)-\frac{\mathrm{i}}{k}\right]}$$



Question 17

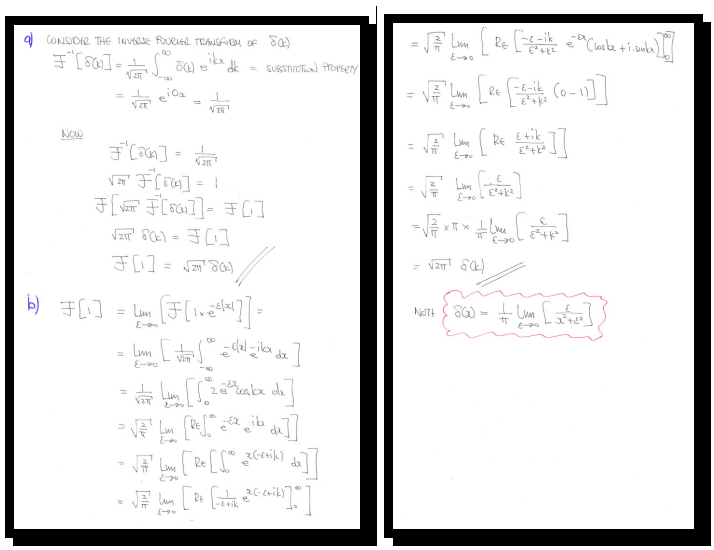
The impulse function $\delta(x)$ is defined by

$$\delta(x)=\begin{cases}\infty & x=0\\ 0 & x\neq 0\end{cases}$$

a) Determine the inverse Fourier transform of the impulse function $\mathcal{F}^{-1}\left[\delta(k)\right]$, and use it to deduce the Fourier transform of $f(x)=1$.

b) Find directly the Fourier transform of $f(x)=1$, by introducing the converging factor $\mathrm{e}^{-\varepsilon|x|}$ and letting $\varepsilon\to 0$.

$$\boxed{\mathcal{F}[1]=\sqrt{2\pi}\,\delta(k)}$$



Question 18

The function f is defined by

$$f(x)=\text{sign}(x)=\begin{cases}1 & x>0\\ -1 & x<0\end{cases}$$

a) By introducing the converging factor $\mathrm{e}^{-\varepsilon|x|}$ and letting $\varepsilon\to 0$, find the Fourier transform of f.

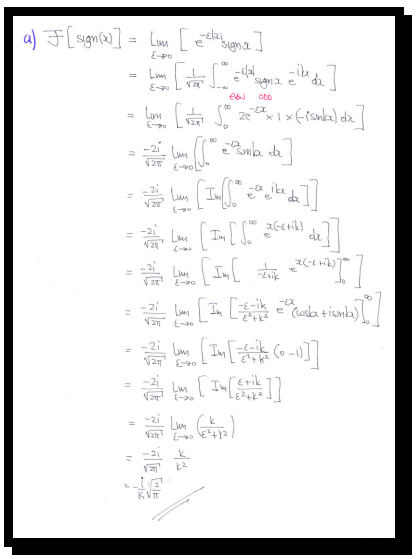
b) By introducing the converging factor $\mathrm{e}^{-\varepsilon|x|}$ and letting $\varepsilon\to 0$, find the Fourier transform of $g(x)=1$.

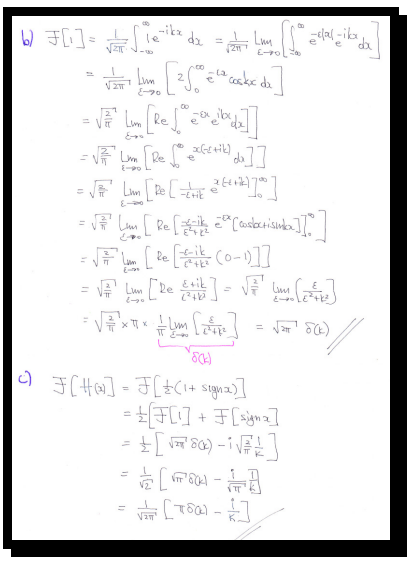
You may assume that $\delta(t)=\dfrac{1}{\pi}\lim\limits_{\varepsilon\to 0}\left[\dfrac{\varepsilon}{\varepsilon^2+t^2}\right]$.

c) Hence determine the Fourier transform of the Heaviside function $\mathrm{H}(x)$,

$$\mathrm{H}(x)=\begin{cases}1 & x\geq 0\\ 0 & x<0\end{cases}$$

$$\boxed{\mathcal{F}[\text{sign}(x)]=-\frac{\mathrm{i}}{k}\sqrt{\frac{1}{\pi}}},\quad \boxed{\mathcal{F}[1]=\sqrt{2\pi}\,\delta(k)},\quad \boxed{\mathcal{F}[\mathrm{H}(x)]=\frac{1}{\sqrt{2\pi}}\left[\pi\delta(k)-\frac{\mathrm{i}}{k}\right]}$$





Question 19

The Fourier transforms of the functions $f(x)$ and $g(x)$ are

$$\hat{f}(k)=\delta(k) \quad \text{and} \quad \hat{g}(k)=\frac{1}{\mathrm{i}k},$$

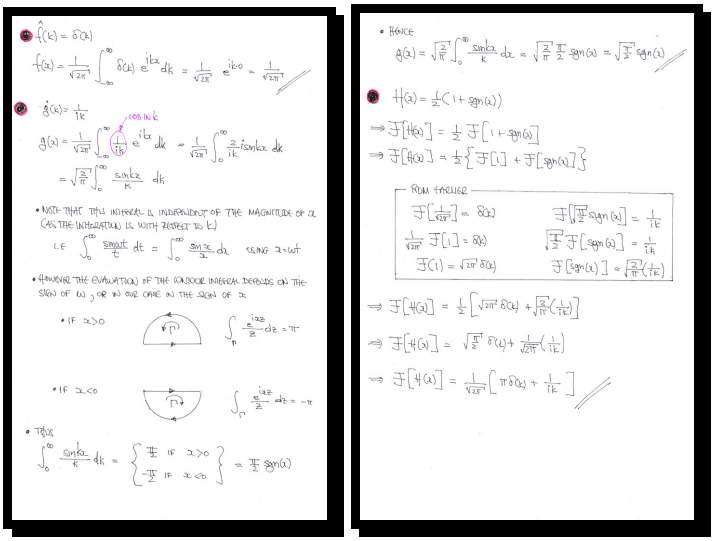
where $\delta(x)$ denotes the impulse function.

Find simplified expressions for $f(x)$ and $g(x)$, and use them to show that

$$\mathcal{F}\left[\mathrm{H}(x)\right]=\frac{1}{\sqrt{2\pi}}\left[\pi\delta(k)+\frac{1}{\mathrm{i}k}\right],$$

where $\mathrm{H}(x)$ denotes the Heaviside function.

$$\boxed{f(x)=\frac{1}{\sqrt{2\pi}}}, \quad \boxed{g(x)=\frac{1}{2}\pi\,\mathrm{sgn}(x)}$$



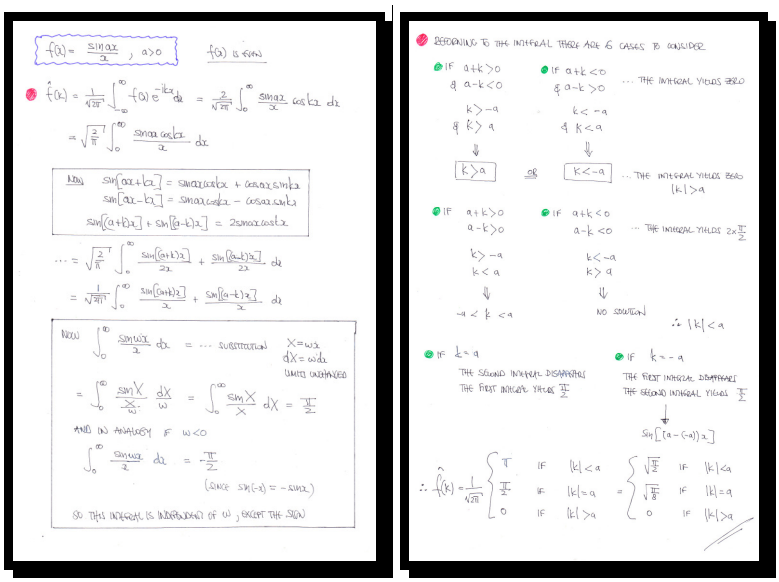
Question 20

The function f is defined by

$$f(x)=\frac{\sin ax}{x},\ a>0.$$

Find the Fourier transform of $f(x)$, stating clearly any results used.

$$\mathcal{F}\left[\frac{\sin ax}{x}\right]=\begin{cases}\sqrt{\dfrac{\pi}{2}} & |k|<a\\ \sqrt{\dfrac{\pi}{8}} & |k|=a\\ 0 & |k|>a\end{cases}$$

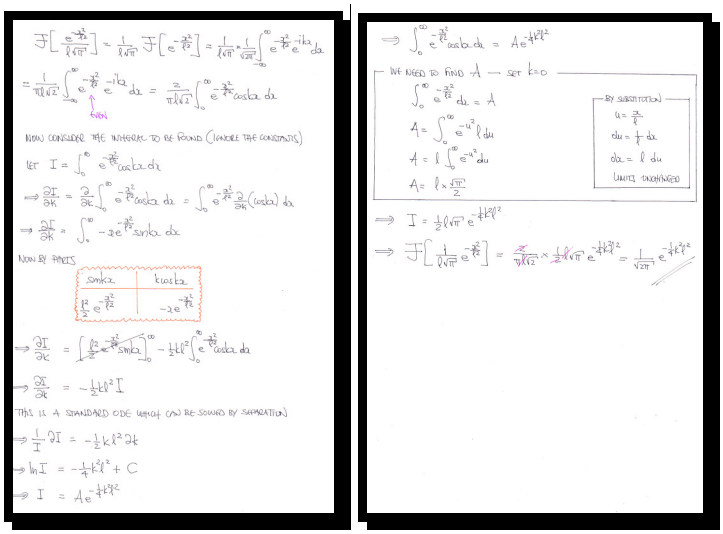


Question 21

Given that l is a non zero constant, show that

$$\mathcal{F}\left[\frac{\exp\left(-\frac{x^2}{l^2}\right)}{l\sqrt{\pi}}\right]=\hat{f}(k)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{k^2l^2}{4}\right).$$

proof



Question 22

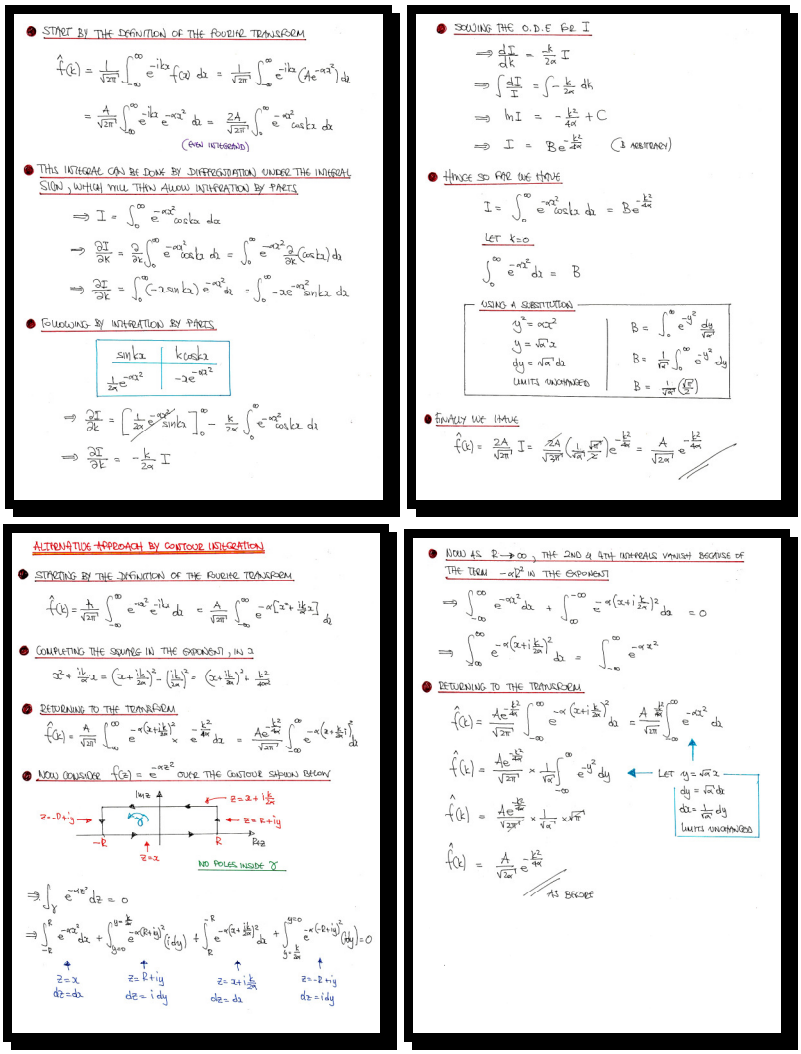
The Gaussian function $f(x)$ is defined by

$$f(x)=A\mathrm{e}^{-\alpha x^2},$$

where A and α are positive constants.

Find the Fourier transform of $f(x)$.

$\square$, $\boxed{\mathcal{F}\left[A\mathrm{e}^{-\alpha x^2}\right]=\hat{f}(k)=\dfrac{A}{\sqrt{2\alpha}}\exp\left(-\dfrac{k^2}{4\alpha}\right)}$



Question 23

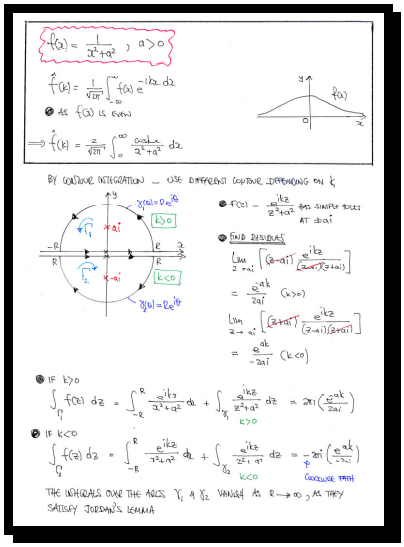
The function f is defined by

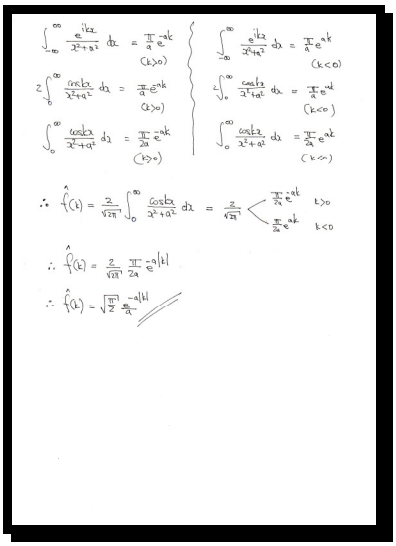
$$f(x)=\frac{1}{x^2+a^2},$$

where a is a positive constant.

Use contour integration to find the Fourier transform of $f(x)$.

$$\mathcal{F}\left[\frac{1}{x^2+a^2}\right]=\hat{f}(k)=\sqrt{\frac{\pi}{2}}\,\frac{\mathrm{e}^{-a|k|}}{a}$$





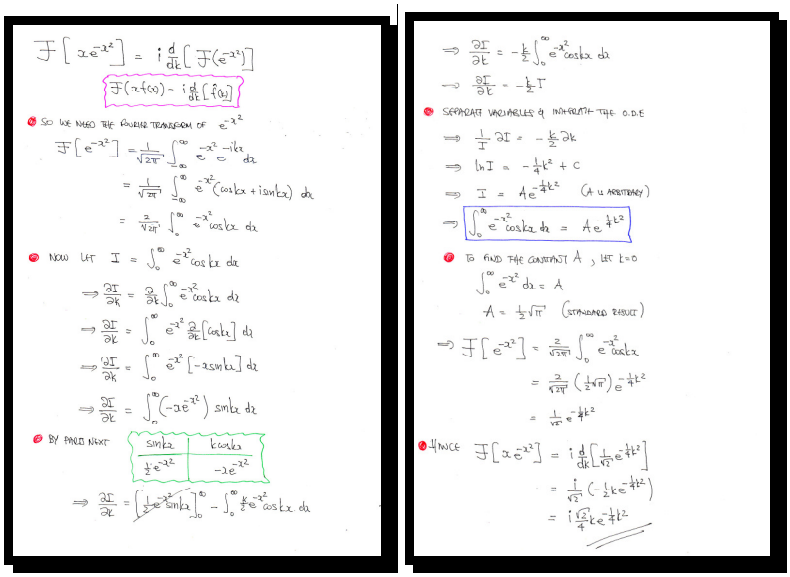
Question 24

The function f is defined by

$$f(x)=xe^{-x^2},\ x\in\mathbb{R}.$$

Find the Fourier transform of $f(x)$, stating clearly any results used.

$$\boxed{\mathcal{F}\left[xe^{-x^2}\right]=\frac{1}{4}k\sqrt{2}\,e^{-\frac{1}{4}k^2}}$$



Question 25

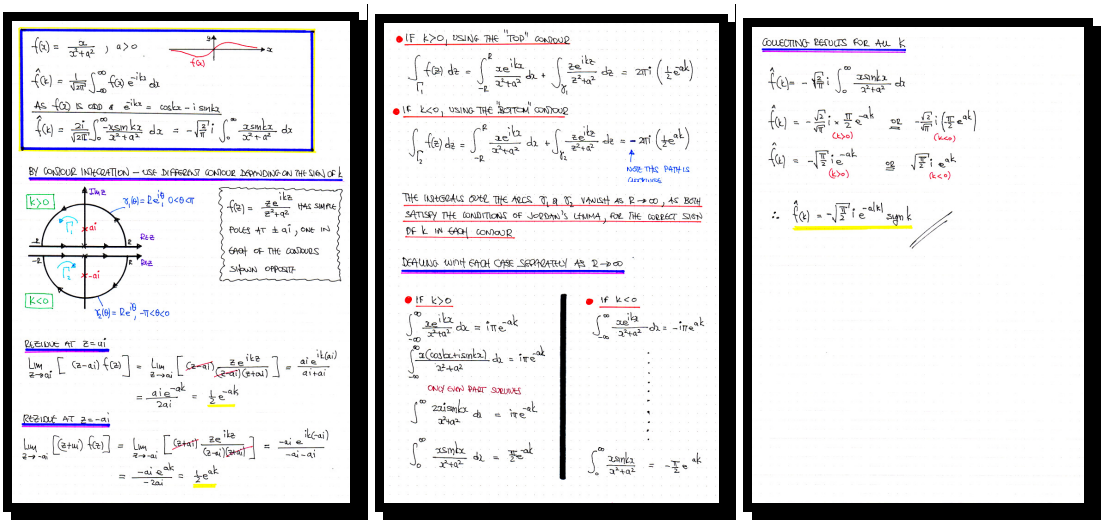
The function f is defined by

$$f(x)=\frac{x}{x^2+a^2},$$

where a is a positive constant.

Use contour integration to find the Fourier transform of $f(x)$.

$\square$, $\boxed{\mathcal{F}\left[\frac{x}{x^2+a^2}\right]=\hat{f}(k)=-\mathrm{i}\sqrt{\frac{\pi}{2}}\,\frac{\mathrm{e}^{-a|k|}\operatorname{sign}k}{a}}$



Question 26

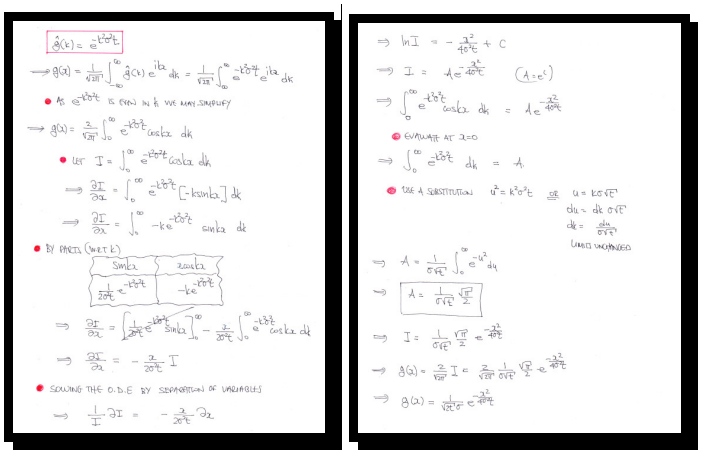
Find the inverse Fourier transform of

$$\hat{g}(k)=\mathrm{e}^{-k^2\sigma^2 t},$$

where σ and t are positive constants.

$$\boxed{\mathcal{F}^{-1}\left[\mathrm{e}^{-k^2\sigma^2 t}\right]=\frac{1}{\sqrt{2t}\,\sigma}\exp\left(-\frac{x^2}{4t\sigma^2}\right)}$$

proof



Question 27

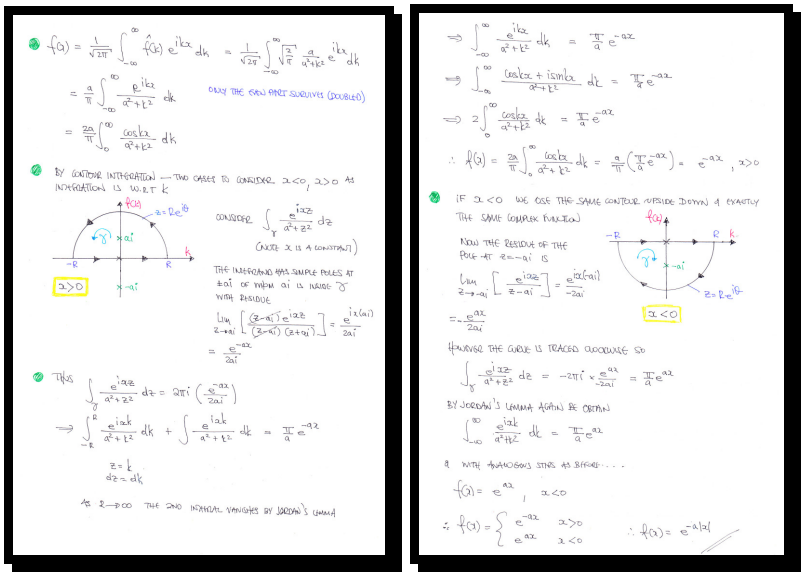
The Fourier transform $\hat{f}(k)$, of function $f(x)$ is

$$\hat{f}(k)=\sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+k^2},$$

where a is a positive constant.

Use contour integration to find an expression for $f(x)$.

$$\boxed{f(x)=\mathrm{e}^{a|x|}}$$



Question 28

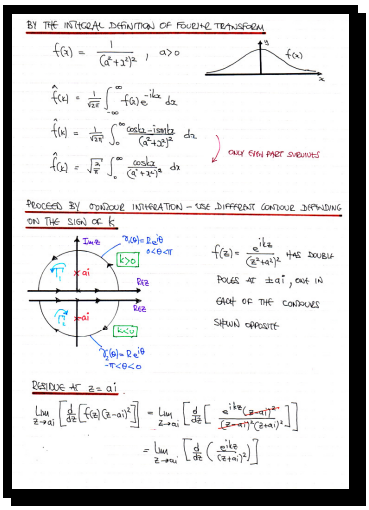
The function f is defined by

$$f(x)=\frac{1}{\left(x^2+a^2\right)^2},$$

where a is a positive constant.

Use contour integration to find the Fourier transform of $f(x)$.

$$\square, \quad \boxed{\mathcal{F}\left[\frac{1}{\left(x^2+a^2\right)^2}\right]=\hat{f}(k)=\sqrt{\frac{\pi}{8}}\,\frac{\left(1+a|k|\right)\mathrm{e}^{-a|k|}}{a^3}}$$


BY THE INTEGRAL DEFINITION OF FOURIER TRANSFORM

$f(x)=\frac{1}{(x^2+a^2)^2}$, $a>0$

$\hat{f}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$

$\hat{f}(k)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{\cos kx-i\sin kx}{(x^2+a^2)^2}\,dx$

ONLY EVEN PART SURVIVES

$\hat{f}(k)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{\cos kx}{(x^2+a^2)^2}\,dx$

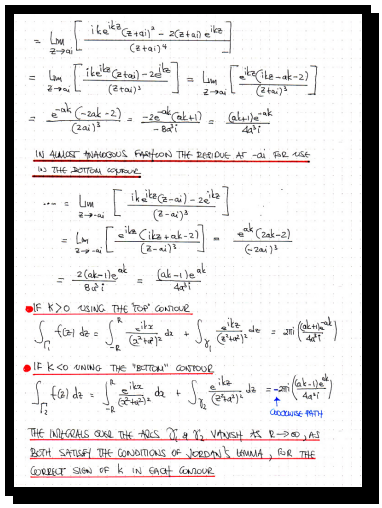
PROCEED BY CONTOUR INTEGRATION - USE DIFFERENT CONTOUR DEPENDING ON THE SIGN OF k

$f(z)=\frac{e^{ikz}}{(z^2+a^2)^2}$ HAS DOUBLE POLES AT $\pm ai$, ONLY IN EACH OF THE CONTOURS SHOWN OPPOSITE

RESIDUE AT $z=ai$

$\lim_{z\to ai}\left[\frac{d}{dz}\left[f(z)(z-ai)^2\right]\right]=\lim_{z\to ai}\left[\frac{d}{dz}\left[\frac{e^{ikz}(z-ai)^2}{(z-ai)^2(z+ai)^2}\right]\right]$

$=\lim_{z\to ai}\left[\frac{d}{dz}\left(\frac{e^{ikz}}{(z+ai)^2}\right)\right]$


$=\lim_{z\to ai}\left[\frac{ike^{ikz}(z+ai)^2-2(z+ai)e^{ikz}}{(z+ai)^4}\right]$

$=\lim_{z\to ai}\left[\frac{ike^{ikz}(z+ai)-2e^{ikz}}{(z+ai)^3}\right]=\lim_{z\to ai}\left[\frac{e^{ikz}(ikz-ak-2)}{(z+ai)^3}\right]$

$=\frac{e^{ak}(-2ak-2)}{(2ai)^3}=\frac{-2e^{ak}(ak+1)}{-8a^3i}=\frac{(ak+1)e^{ak}}{4a^3i}$

IN ALMOST ANALOGOUS FASHION THE RESIDUE AT $-ai$ FOR USE IN THE BOTTOM CONTOUR

$\cdots=\lim_{z\to -ai}\left[\frac{ike^{ikz}(z-ai)-2e^{ikz}}{(z-ai)^3}\right]$

$=\lim_{z\to -ai}\left[\frac{e^{ikz}(ikz+ak-2)}{(z-ai)^3}\right]=\frac{e^{ak}(2ak-2)}{(-2ai)^3}$

$=\frac{2(ak-1)e^{ak}}{8a^3i}=\frac{(ak-1)e^{ak}}{4a^3i}$

IF $k>0$ USING THE 'TOP' CONTOUR

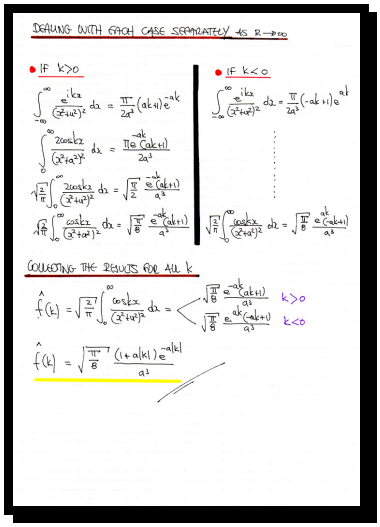
$\int_{\Gamma_1} f(z)\,dz=\int_{-R}^{R}\frac{e^{ikx}}{(x^2+a^2)^2}\,dx+\int_{\gamma_1}\frac{e^{ikz}}{(z^2+a^2)^2}\,dz=2\pi i\left(\frac{(ak+1)e^{ak}}{4a^3i}\right)$

IF $k<0$ USING THE 'BOTTOM' CONTOUR

$\int_{\Gamma_2} f(z)\,dz=\int_{-R}^{R}\frac{e^{ikx}}{(x^2+a^2)^2}\,dx+\int_{\gamma_2}\frac{e^{ikz}}{(z^2+a^2)^2}\,dz=-2\pi i\left(\frac{(ak-1)e^{ak}}{4a^3i}\right)$

ANTICLOCKWISE PATH

THE INTEGRALS OVER THE ARCS γ_1 & γ_2 VANISH AS $R\to\infty$, AS BOTH SATISFY THE CONDITIONS OF JORDAN'S LEMMA, FOR THE CORRECT SIGN OF k IN EACH CONTOUR


DEALING WITH EACH CASE SEPARATELY AS $R\to\infty$

IF $k>0$

$\int_{-\infty}^{\infty}\frac{e^{ikx}}{(x^2+a^2)^2}\,dx=\frac{\pi}{2a^3}(ak+1)e^{-ak}$

$\int_{0}^{\infty}\frac{2\cos kx}{(x^2+a^2)^2}\,dx=\frac{\pi e^{-ak}(ak+1)}{2a^3}$

$\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{2\cos kx}{(x^2+a^2)^2}\,dx=\sqrt{\frac{\pi}{2}}\frac{e^{-ak}(ak+1)}{a^3}$

$\sqrt{\frac{\pi}{8}}\int_{0}^{\infty}\frac{\cos kx}{(x^2+a^2)^2}\,dx=\sqrt{\frac{\pi}{8}}\frac{e^{-ak}(ak+1)}{a^3}$

IF $k<0$

$\int_{-\infty}^{\infty}\frac{e^{ikx}}{(x^2+a^2)^2}\,dx=\frac{\pi}{2a^3}(-ak+1)e^{ak}$

$\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{\cos kx}{(x^2+a^2)^2}\,dx=\sqrt{\frac{\pi}{8}}\frac{e^{ak}(-ak+1)}{a^3}$

COLLECTING THE RESULTS FOR ALL k

$\hat{f}(k)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{\cos kx}{(x^2+a^2)^2}\,dx=\begin{cases}\sqrt{\frac{\pi}{8}}\frac{e^{-ak}(ak+1)}{a^3} & k>0\\ \sqrt{\frac{\pi}{8}}\frac{e^{ak}(-ak+1)}{a^3} & k<0\end{cases}$

$\hat{f}(k)=\sqrt{\frac{\pi}{8}}\frac{(1+a|k|)e^{-a|k|}}{a^3}$

VARIOUS PROBLEMS on FOURIER TRANSFORMS

Question 1

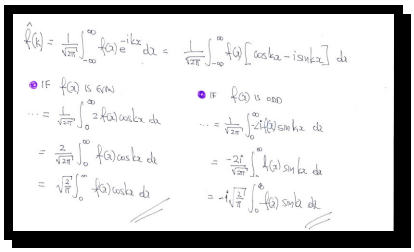
Find the Fourier transform of an arbitrary function $f(x)$ if

i. $f(x)$ is even.

ii. $f(x)$ is odd.

Give the answers as a simplified integral form.

$$\hat{f}(k)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos kx\ dx \quad , \quad \hat{f}(k)=-\mathrm{i}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin kx\ dx$$

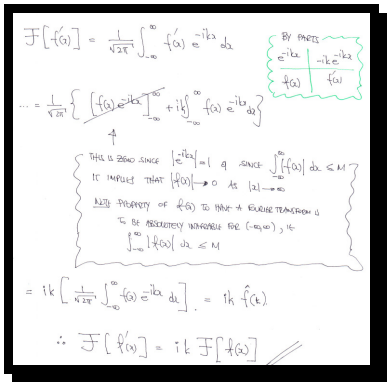


Question 2

Use the definition of the Fourier transform, of an absolutely integrable function $f(x)$, to show that

$$\mathcal{F}\left[f'(x)\right]=\mathrm{i}k\mathcal{F}\left[f(x)\right].$$

proof



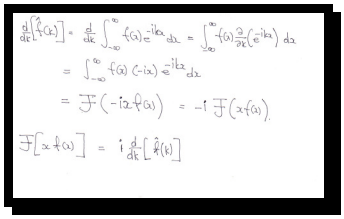
Question 3

The Fourier transform of an absolutely integrable function $f(x)$, is denoted by $\hat{f}(k)$.

Show that

$$\mathcal{F}\left[x f(x)\right] = \mathrm{i}\frac{d}{dk}\left[\hat{f}(k)\right].$$

proof

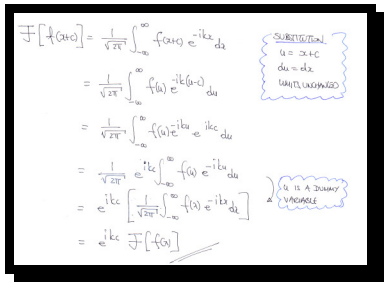


Question 4

Given that c is a constant show that

$$\mathcal{F}\left[f(x+c)\right] = \mathrm{e}^{\mathrm{i}kc}\,\mathcal{F}\left[f(x)\right].$$

proof



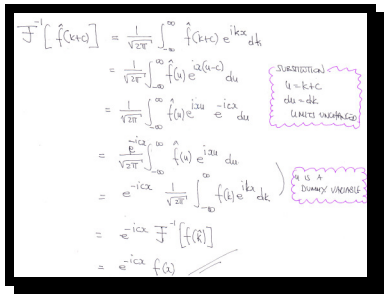
Question 5

Given that c is a constant show that

$$\mathcal{F}^{-1}\left[\hat{f}(k+c)\right] = \mathrm{e}^{\mathrm{i}cx} f(x),$$

where $\hat{f}(k) \equiv \mathcal{F}\left[f(x)\right]$

proof



Question 6

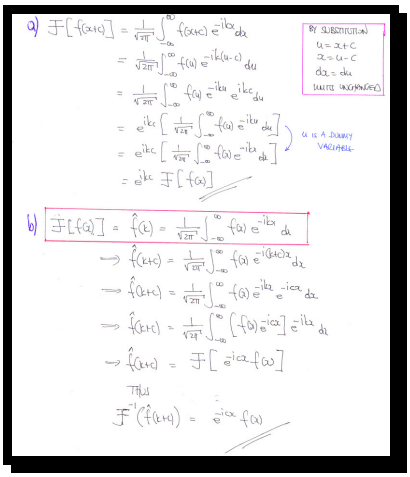
Given that c is a constant prove the validity of the two shift theorems

a) $\mathcal{F}\left[f\left(x+c\right)\right] = \mathrm{e}^{\mathrm{i}kc}\,\mathcal{F}\left[f\left(x\right)\right]$.

b) $\mathcal{F}^{-1}\left[\hat{f}\left(k+c\right)\right] = \mathrm{e}^{\mathrm{i}cx}\,f\left(x\right)$.

Note that $\hat{f}\left(k\right) \equiv \mathcal{F}\left[f\left(x\right)\right]$.

proof



Question 7

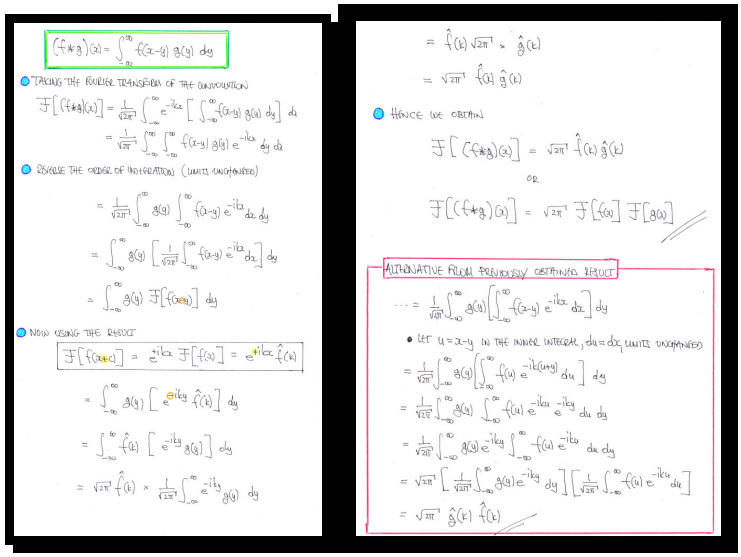
The convolution $[f*g](x)$, of two functions $f(x)$ and $g(x)$ is defined as

$$[f*g](x)=\int_{-\infty}^{\infty} f(x-y)\,g(y)\;dy\,.$$

Show that

$$\mathcal{F}\{[f*g](x)\} \;=\; \sqrt{2\pi}\,\mathcal{F}\big[f(x)\big]\mathcal{F}\big[g(x)\big] \;=\; \sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k)\,.$$

proof



Question 8

It is given that c is a constant and $\hat{f}(k) \equiv \mathcal{F}\left[f(x)\right]$.

a) Prove the validity of the inversion shift theorem

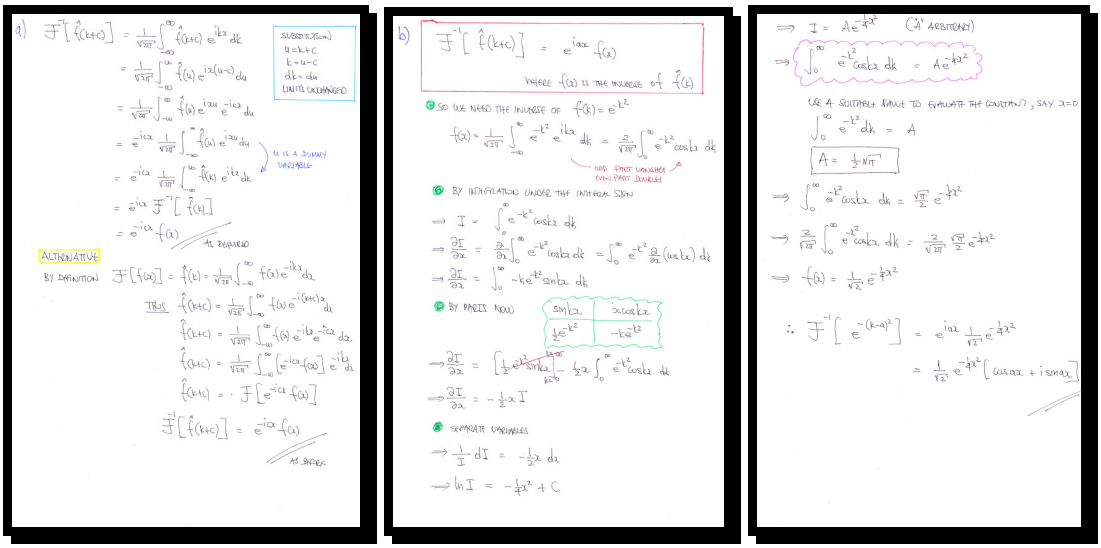
$$\mathcal{F}^{-1}\left[\hat{f}(k+c)\right] = \mathrm{e}^{\mathrm{i}cx} f(x).$$

b) Hence determine an expression for

$$\mathcal{F}^{-1}\left[\mathrm{e}^{-(k-a)^2}\right],$$

where a is a positive constant.

$$\boxed{\mathcal{F}^{-1}\left[\mathrm{e}^{-(k-a)^2}\right] = \frac{1}{\sqrt{2}}\mathrm{e}^{-\frac{1}{4}x^2}\left[\cos ax + \mathrm{i}\sin ax\right]}$$



Question 9

The convolution theorem for two functions $f(x)$ and $g(x)$ asserts that

$$\mathcal{F}\{[f*g](x)\} = \sqrt{2\pi}\,\mathcal{F}\big[f(x)\big]\mathcal{F}\big[g(x)\big],$$

where

$$[f*g](x)=\int_{-\infty}^{\infty} f(x-y)\,g(y)\;dy\,.$$

a) Starting from the convolution theorem prove Parseval's Theorem

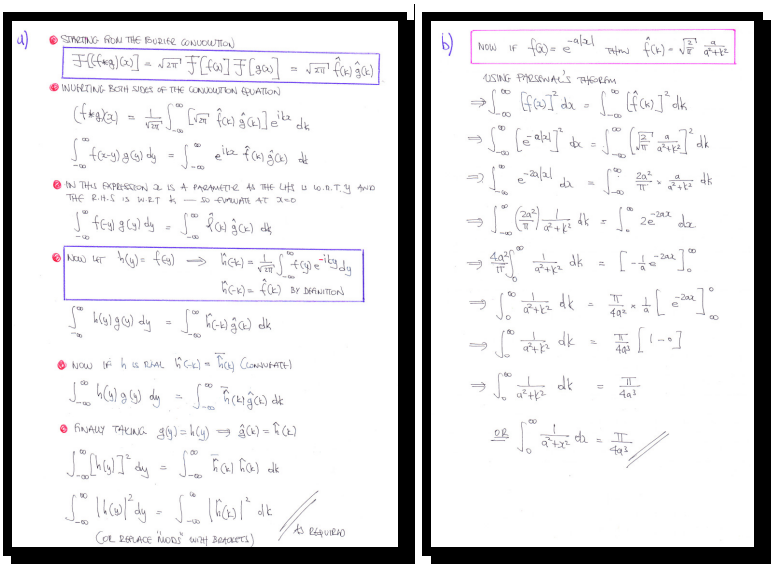
$$\int_{-\infty}^{\infty}\left|h(y)\right|^2\;dy \;=\; \int_{-\infty}^{\infty}\left|\hat{h}(k)\right|^2\;dk\,.$$

b) Use Parseval's Theorem to evaluate

$$\int_0^{\infty}\frac{1}{x^2+a^2}\;dx\,.$$

You may assume that if $f(x)=\mathrm{e}^{-a|x|}$, then $\hat{f}(k)=\sqrt{\dfrac{2}{\pi}}\,\dfrac{a}{a^2+k^2}$.

$$\boxed{\frac{\pi}{4a^3}}$$



Question 10

The convolution $[f*g](x)$, of two functions $f(x)$ and $g(x)$ is defined as

$$[f*g](x)=\int_{-\infty}^{\infty} f(x-y)\,g(y)\;dy\,.$$

a) Show that

$$\mathcal{F}\{[f*g](x)\} \;=\; \sqrt{2\pi}\,\mathcal{F}\big[f(x)\big]\mathcal{F}\big[g(x)\big] \;=\; \sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k)\,.$$

b) Hence prove Parseval's Theorem

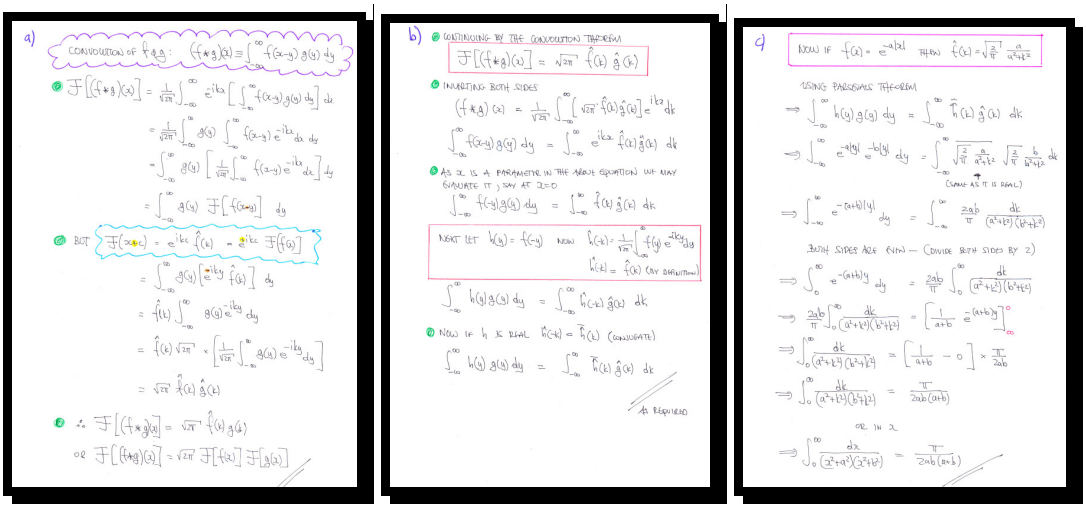
$$\int_{-\infty}^{\infty} h(y)\,g(y)\;dy \;=\; \int_{-\infty}^{\infty} \overline{\hat{h}}(k)\,\hat{g}(k)\;dk\,.$$

c) Use Parseval's Theorem to evaluate

$$\int_{0}^{\infty} \frac{1}{\left(x^2+a^2\right)\left(x^2+b^2\right)}\;dx\,.$$

You may assume that if $f(x)=\mathrm{e}^{-a|x|}$, then $\hat{f}(k)=\sqrt{\dfrac{2}{\pi}}\,\dfrac{a}{a^2+k^2}$.

$$\frac{\pi}{2ab(a+b)}$$



APPLICATIONS of FOURIER TRANSFORMS

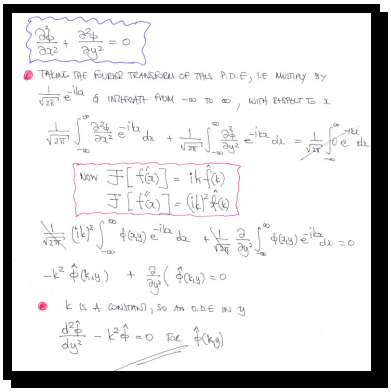
Question 1

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 .$$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation for $\hat{\varphi}(k, y)$, where $\hat{\varphi}(k, y)$ is the Fourier transform of $\varphi(x, y)$ with respect to x.

$$\boxed{\frac{d^2 \hat{\varphi}}{dx^2} - k^2 \hat{\varphi} = 0}$$



Question 2

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $y \geq 0$.

It is further given that

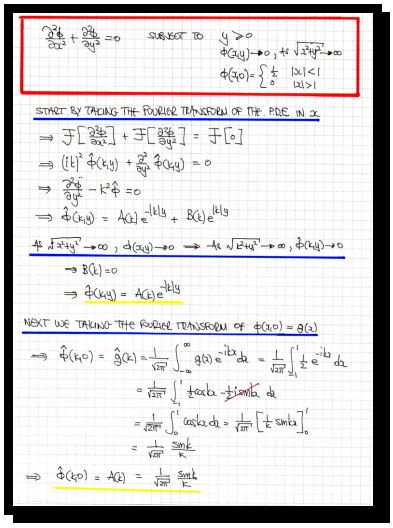
- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$
- $\varphi(x, 0) = \begin{cases} \frac{1}{2} & |x| < 1 \\ 0 & |x| > 1 \end{cases}$

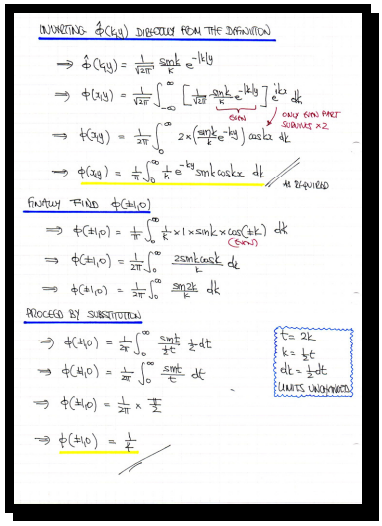
Use Fourier transforms to show that

$$\varphi(x, y) = \frac{1}{\pi} \int_0^{\infty} \frac{1}{k} \mathrm{e}^{-ky} \sin k \cos kx \ dk,$$

and hence deduce the value of $\varphi(\pm 1, 0)$.

$\square$, $\boxed{\varphi(\pm 1, 0) = \frac{1}{4}}$





Question 3

The Airy function $\mathrm{Ai}(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2}-xy=0\,.$$

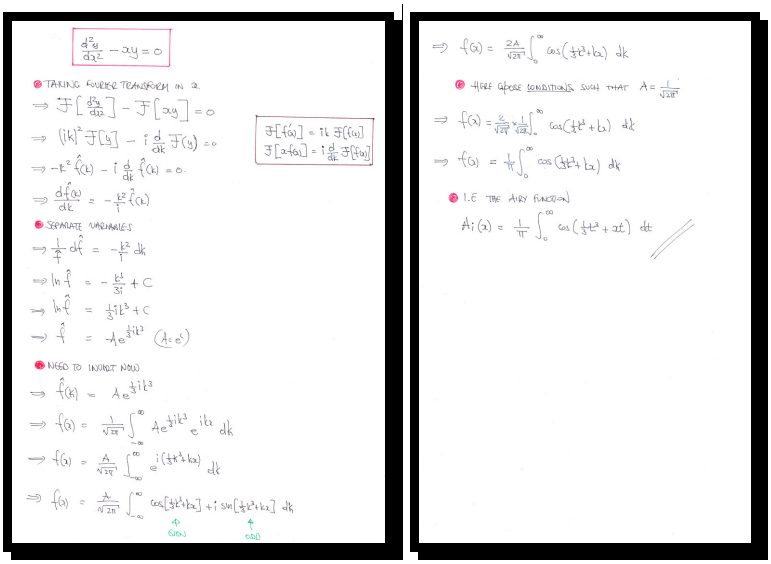
Use Fourier transforms to show that

$$\mathrm{Ai}(x)=\frac{1}{\pi}\int_0^{\infty}\cos\left(\frac{1}{3}t^3+xt\right)\,dt\,,$$

for suitable boundary conditions.

You may assume that $\mathcal{F}\left[x\,f(x)\right]=\mathrm{i}\dfrac{d}{dk}\left\{\mathcal{F}\left[f(x)\right]\right\}$.

proof



Question 4

The function $\psi = \psi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $y \geq 0$.

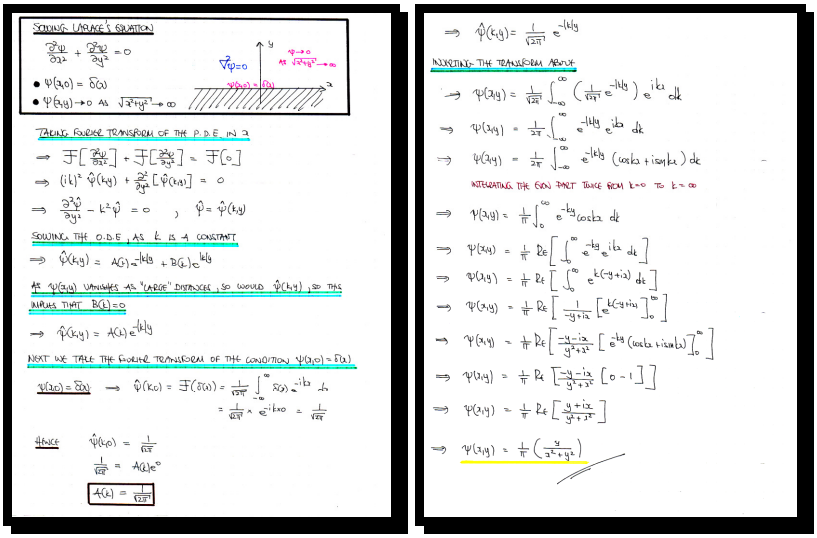
It is further given that

- $\psi(x, 0) = \delta(x)$
- $\psi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\psi(x, y) = \frac{1}{\pi}\left(\frac{y}{x^2 + y^2}\right).$$

, proof



Question 5

The function $u = u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{1}{3}\frac{\partial^3 u}{\partial x^3} = 0 .$$

It is further given that

- $u(x,0) = \delta(x)$
- $u(x,t) \to 0$ as $|x| \to \infty$

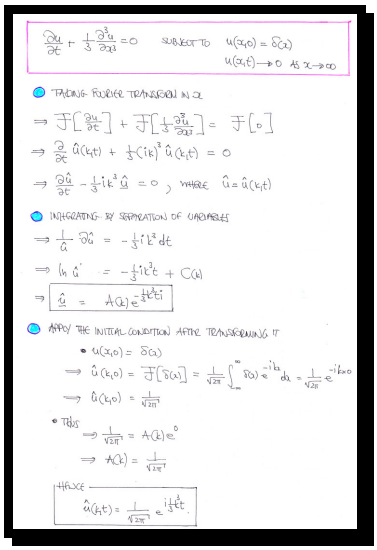
Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

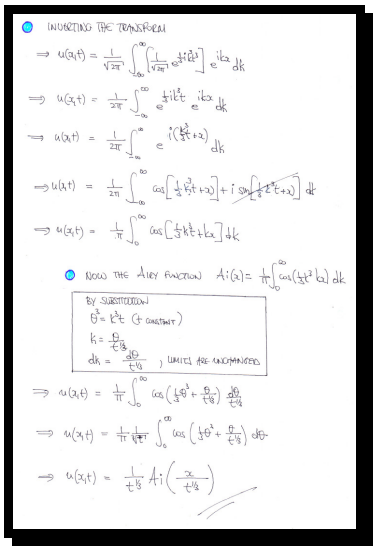
$$u(x,t) = \frac{1}{t^{\frac{1}{3}}}\text{Ai}\left(\frac{x}{t^{\frac{1}{3}}}\right),$$

where the $\text{Ai}(x)$ is the Airy function, defined as

$$\text{Ai}(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left[\frac{1}{3}k^3 + kx\right]\, dk .$$

proof





Question 6

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $x \geq 0$ and $y \geq 0$.

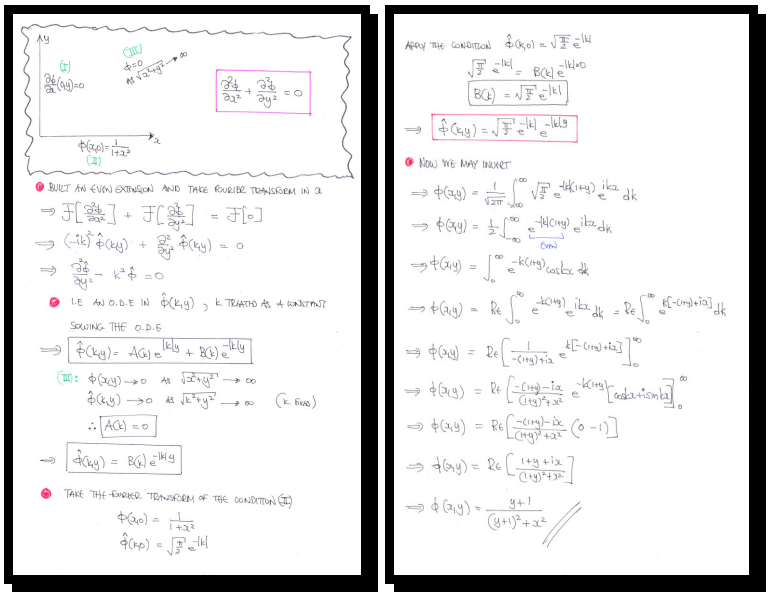
It is further given that

- $\varphi(x,0) = \dfrac{1}{1+x^2}$
- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$
- $\dfrac{\partial}{\partial x}\left[\varphi(x,0)\right] = 0$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\varphi(x, y) = \frac{y+1}{x^2 + (y+1)^2}.$$

proof



Question 7

The function $\Phi = \Phi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0,$$

in the part of the x-y plane for which $y \geq 0$.

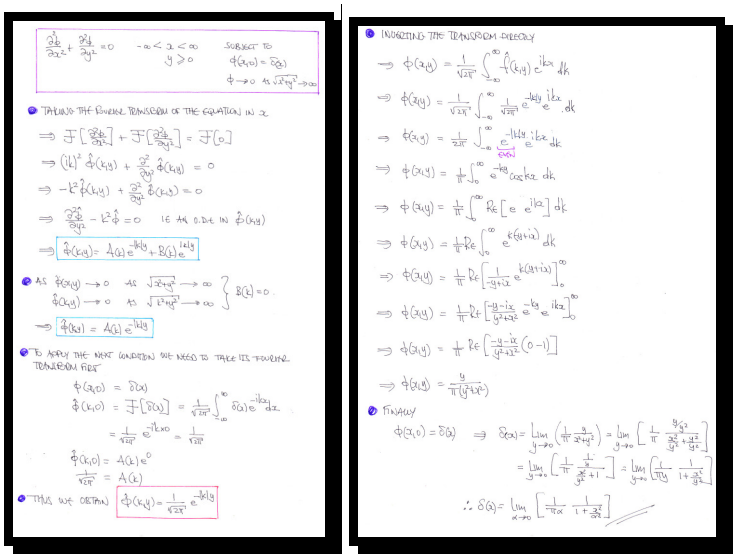
It is further given that

- $\Phi(x, 0) = \delta(x)$
- $\Phi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

Use Fourier transforms to find the solution of the above partial differential equation and hence show that

$$\delta(x) = \lim_{\alpha \to 0} \left[\frac{1}{\pi\alpha} \left(1 + \frac{y^2}{\alpha^2} \right)^{-1} \right].$$

proof



Question 8

The function $y = y(x)$ satisfies the differential equation

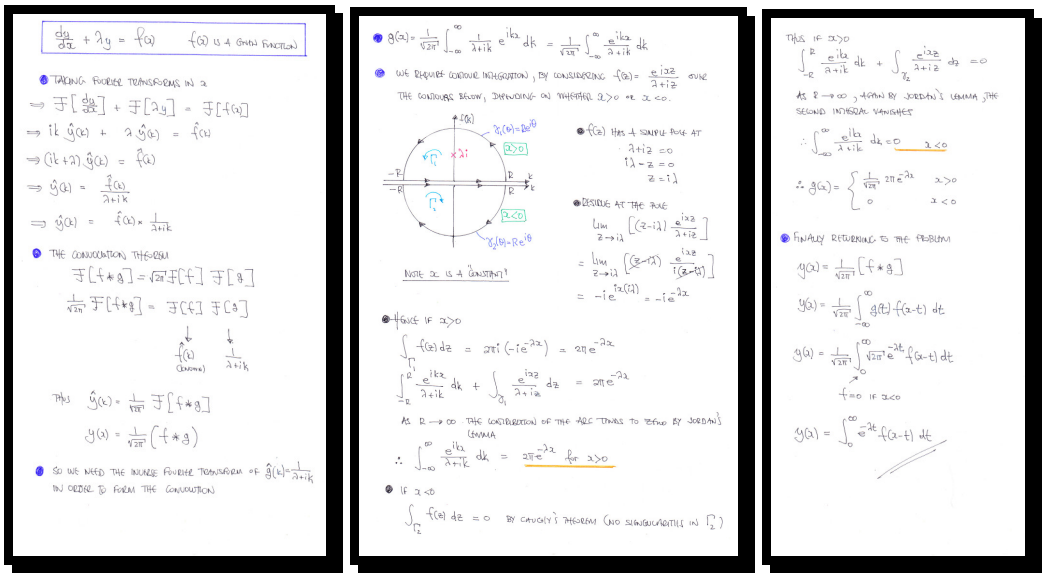
$$\frac{dy}{dx} + \lambda y = f(x),$$

where $f(x)$ is a given function and λ is a real constant.

Use Fourier transforms to show that

$$y(x) = \int_0^{\infty} \mathrm{e}^{\lambda t} f(x-t)\ dt .$$

proof



Question 9

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the semi-infinite region of the x-y plane for which $y \geq 0$.

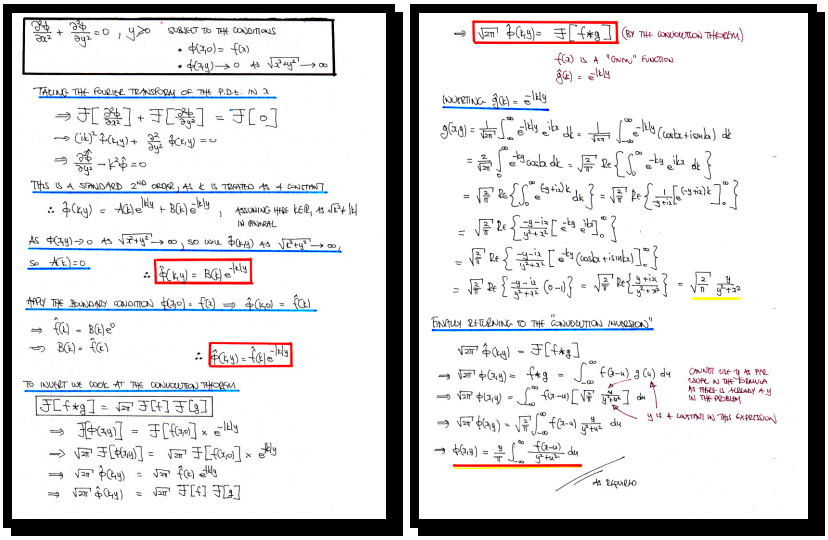
It is further given that

- $\varphi(x,0) = f(x)$
- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\varphi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x-u)}{u^2 + y^2} \, du \, .$$

, proof



Question 10

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in the semi-infinite region of the x-y plane for which $y \geq 0$.

It is further given that for a given function $f = f(x)$

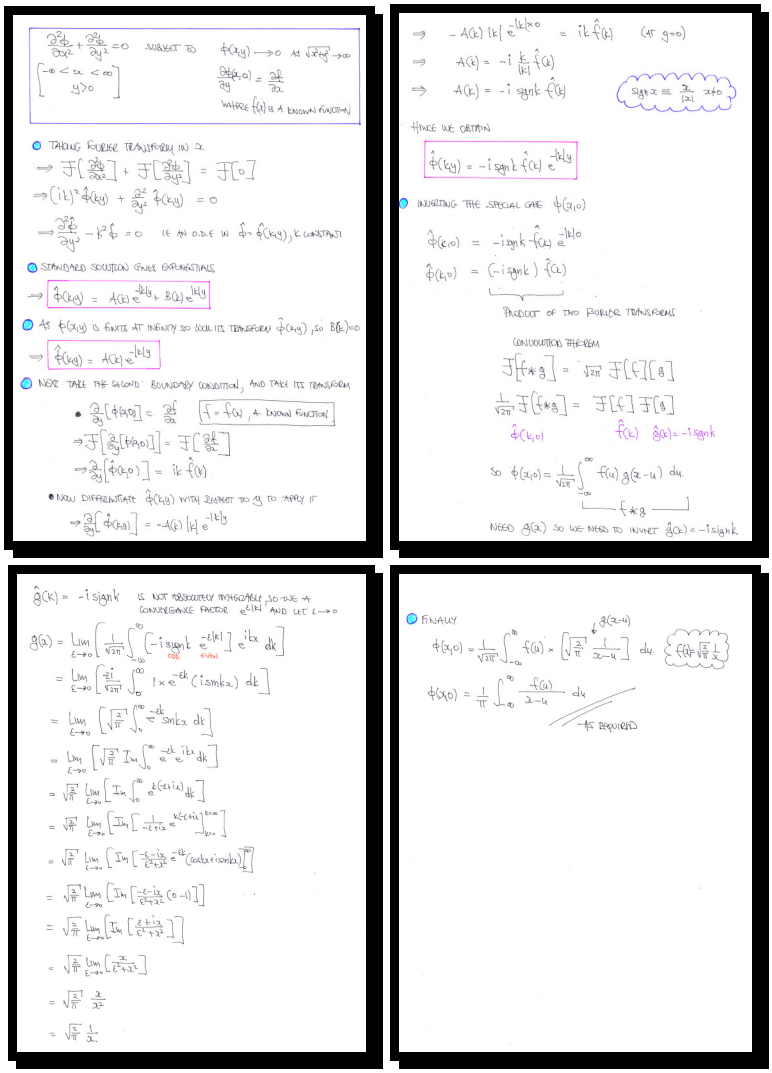
- $\dfrac{\partial}{\partial y}\left[\varphi(x,0)\right] = \dfrac{\partial}{\partial x}\left[f(x)\right]$
- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\varphi(x,0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(u)}{x-u}\, du .$$

proof

[**solution overleaf**]



Question 11

The function $\varphi = \varphi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0, \quad -\infty < x < \infty, \ y \geq 0.$$

It is further given that

- $\varphi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$
- $\varphi(x, 0) = \mathrm{H}(x)$, the Heaviside function.

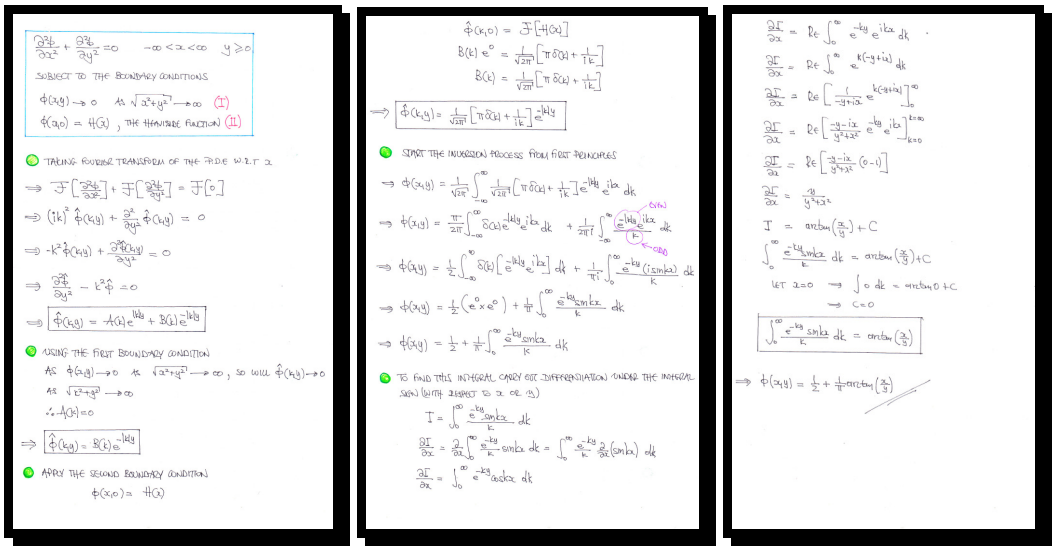
Use Fourier transforms to show that

$$\varphi(x, y) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x}{y}\right).$$

You may assume that

$$\mathcal{F}[\mathrm{H}(x)] = \frac{1}{\sqrt{2\pi}}\left[\pi\delta(k) + \frac{1}{\mathrm{i}k}\right].$$

proof



Question 12

The function $u = u(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad 0 < y < 1.$$

It is further given that

- $u(x,0) = 0$
- $u(x,1) = f(x)$

  where $f(-x) = f(x)$ and $f(x) \to 0$ as $x \to \infty$

a) Use Fourier transforms to show that

$$u(x, y) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}(k) \cos kx \sinh ky}{\sinh k} \, dk, \quad \hat{f}(k) = \mathcal{F}\left[f(x)\right].$$

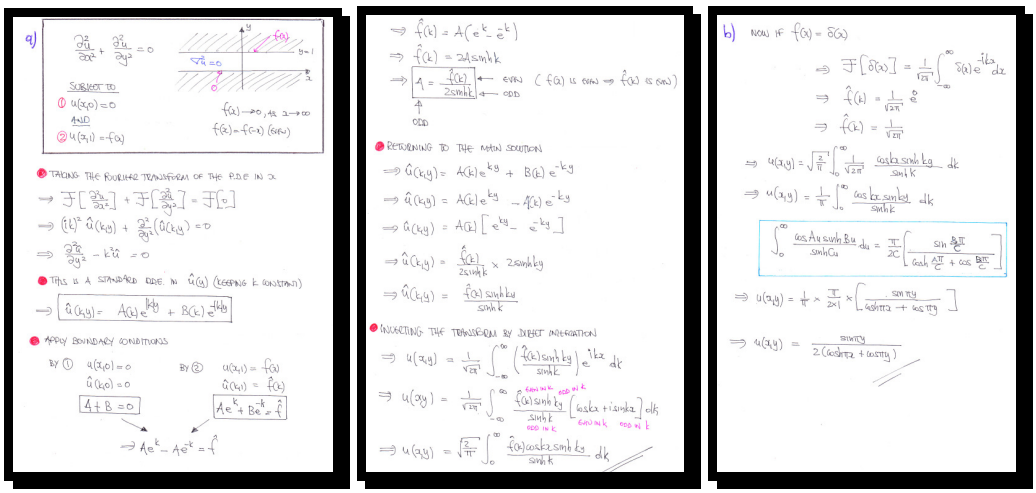
b) Given that $f(x) = \delta(x)$ show further that

$$u(x, y) = \frac{\sin \pi y}{2\left[\cosh \pi x + \cos \pi y\right]}.$$

You may assume without proof

$$\int_0^{\infty} \frac{\cos Au \sinh Bu}{\sinh Cu} \, du = \frac{\pi}{2C}\left[\frac{\sin(B\pi/C)}{\cosh(A\pi/C) + \cos(B\pi/C)}\right], \quad 0 \le B < C.$$

proof



Question 13

The function $\psi = \psi(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 ,$$

in the part of the x-y plane for which $y \geq 0$.

It is further given that

- $\psi(x, 0) = f(x)$
- $\psi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$

c) Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\psi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(u)}{(x-u)^2 + y^2} \, du .$$

d) Evaluate the above integral for …

i. … $f(x) = 1$.

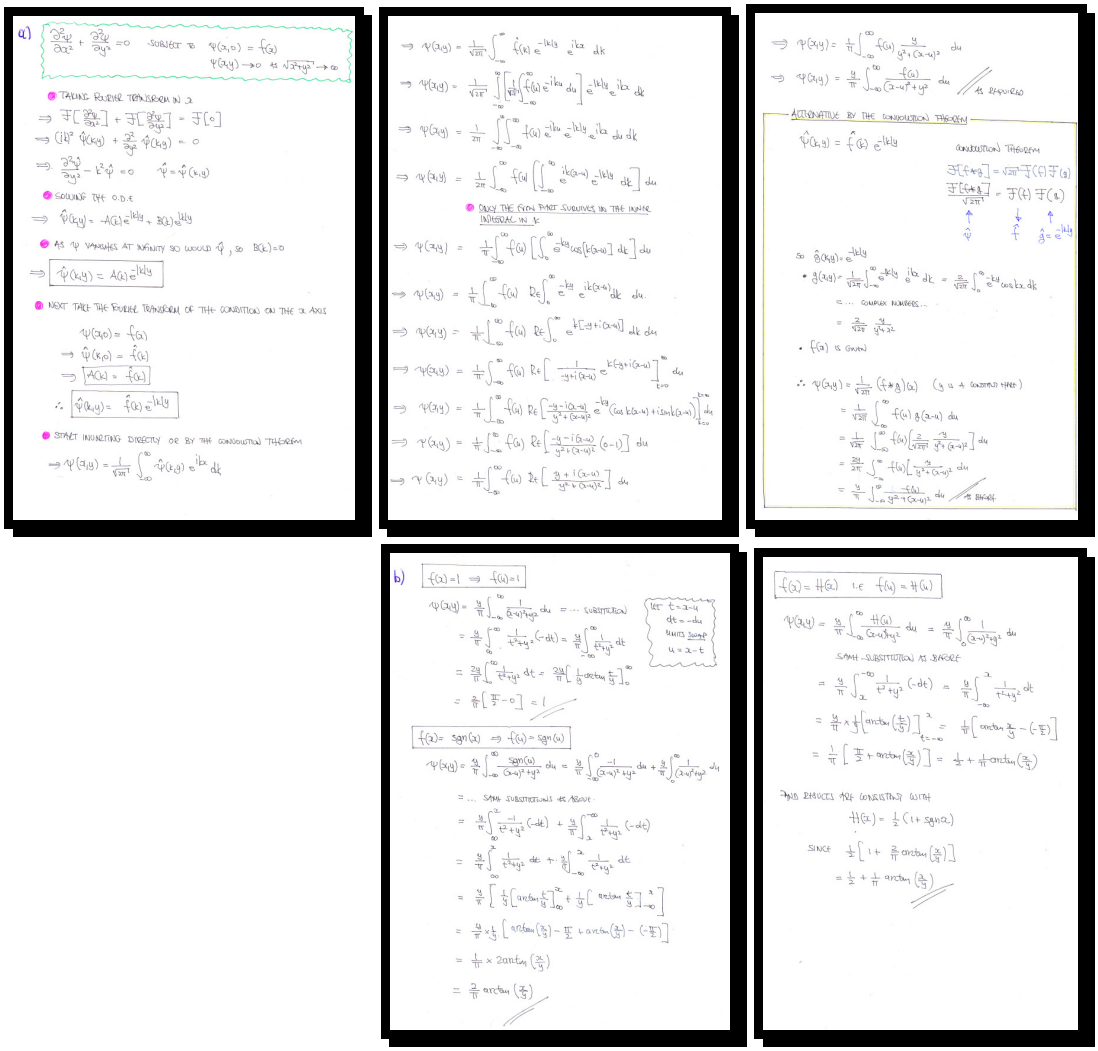
ii. … $f(x) = \operatorname{sgn} x$

iii. … $f(x) = \mathrm{H}(x)$

commenting further whether these answers are consistent.

$$\boxed{\psi(x, y) = 1} , \quad \boxed{\psi(x, y) = \frac{2}{\pi} \arctan\left(\frac{x}{y}\right)} , \quad \boxed{\psi(x, y) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x}{y}\right)}$$

[solution overleaf]



Question 14

The function $\theta = \theta(x,t)$ satisfies the heat equation in one spatial dimension,

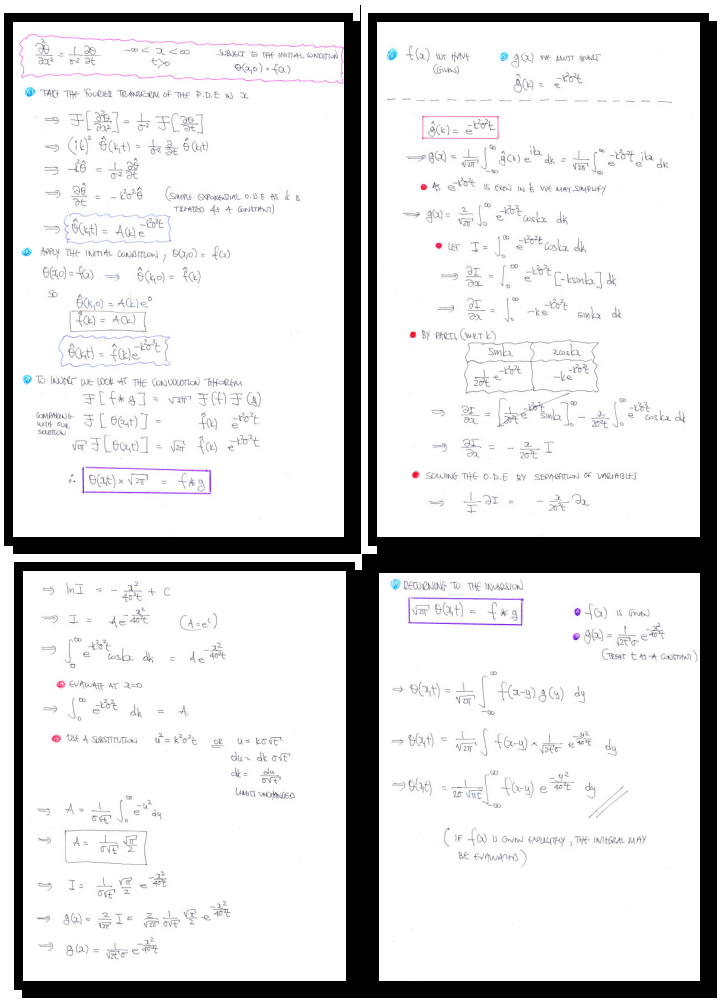
$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\sigma^2}\frac{\partial \theta}{\partial t}, \quad -\infty < x < \infty, \ t \geq 0,$$

where σ is a positive constant.

Given further that $\theta(x,0) = f(x)$, use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$\theta(x,t) = \frac{1}{2\sigma\sqrt{\pi t}} \int_{-\infty}^{\infty} f(x-u) \exp\left(\frac{u^2}{4t\sigma^2}\right) du .$$

proof



Question 15

The function $u = u(x, y)$ satisfies Laplace's equation in Cartesian coordinates,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

in the part of the x-y plane for which $x \geq 0$ and $y \geq 0$.

It is further given that

- $u(0, y) = 0$
- $u(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$
- $u(x, 0) = f(x)$, $f(0) = 0$, $f(x) \to 0$ as $x \to \infty$

Use Fourier transforms to show that

$$u(x, y) = \frac{y}{\pi} \int_0^{\infty} f(w) \left[\frac{1}{y^2 + (x - w)^2} - \frac{1}{y^2 + (x + w)^2} \right] dw .$$

proof

[solution overleaf]



$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

SUBJECT TO

① $u(x,y)\to 0$ AS $\sqrt{x^2+y^2}\to\infty$

② $u(0,y)=0$

③ $u(x,0)=f(x)$, $f(0)=0$, $f(x)\to 0$ AS $x\to\infty$

$\nabla u=0$

A) ALTHOUGH THE REGION IS NOT SYMMETRICAL IN x (OR IN y), EXTEND $u(x,y)$ & $f(x)$ IN THE NEGATIVE OF INFLECTION, SO BOTH ARE ODD

B) TAKE FOURIER TRANSFORM OF THE P.D.E IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right]+\mathcal{F}\left[\frac{\partial^2 u}{\partial y^2}\right]=\mathcal{F}[0]$$

$$\Rightarrow (ik)^2\hat{u}(k,y)+\frac{\partial^2}{\partial y^2}\hat{u}(k,y)=0$$

$$\Rightarrow \frac{\partial^2 \hat{u}}{\partial y^2}-k^2\hat{u}=0$$

$$\Rightarrow \boxed{\hat{u}(k,y)=A(k)e^{|k|y}+B(k)e^{-|k|y}}$$

C) APPLY BOUNDARY CONDITION ①

IF $u(x,y)\to 0$ AS $\sqrt{x^2+y^2}\to\infty$, THEN $\hat{u}(k,y)\to 0$ AS $\sqrt{k^2+y^2}\to\infty$

$\therefore A(k)=0$

$$\Rightarrow \boxed{\hat{u}(k,y)=B(k)e^{-|k|y}}$$

D) APPLY BOUNDARY CONDITION ③

$\Rightarrow u(x,0)=f(x)$

$\Rightarrow \hat{u}(k,0)=\hat{f}(k)$

$\Rightarrow \hat{f}(k)=B(k)e^{0}$

$\Rightarrow B(k)=\hat{f}(k)$

$\therefore \boxed{\hat{u}(k,y)=\hat{f}(k)e^{-|k|y}}$

E) AS WE DO NOT KNOW $f(x)$ EXPLICITLY, WE START THE INVERSION FROM FIRST PRINCIPLES

$$\Rightarrow u(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}(k,y)e^{ikx}\,dk$$

- AS $u(x,y)$ IS ODD (WE BUILT ODD EXTENSION), $\hat{u}(k,y)$ WILL ALSO BE ODD

$$\Rightarrow u(x,y)=\frac{2}{\sqrt{2\pi}}\int_{0}^{\infty}\hat{u}(k,y)i\sin kx\,dk$$

$$\Rightarrow u(x,y)=\sqrt{\frac{2}{\pi}}i\int_{0}^{\infty}\hat{f}(k)e^{-|k|y}\sin kx\,dk$$

$$\Rightarrow u(x,y)=\sqrt{\frac{2}{\pi}}i\int_{0}^{\infty}e^{-|k|y}\sin kx\left[\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(w)e^{-ikw}\,dw}_{\hat{f}(k)}\right]dk$$

- AS f IS "BUILT" ODD, ONLY THE ODD (SINE PART) SURVIVES

$$\Rightarrow u(x,y)=\sqrt{\frac{2}{\pi}}i\int_{0}^{\infty}e^{-|k|y}\sin kx\left[-i\sqrt{\frac{2}{\pi}}i\int_{0}^{\infty}f(w)\sin kw\,dw\right]dk$$

$$\Rightarrow u(x,y)=-\frac{2}{\pi}(i)^2\int_{0}^{\infty}e^{-|k|y}\sin kx\left[\int_{0}^{\infty}f(w)\sin kw\,dw\right]dk$$

$$\Rightarrow u(x,y)=\frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}e^{-|k|y}f(w)\sin kx\sin kw\,dw\,dk$$

F) REVERSING THE ORDER OF INTEGRATION NOTING THAT THE LIMITS ARE UNDERSTOOD ("BOX" REGION FROM 0 TO ∞)

$$\Rightarrow u(x,y)=\frac{2}{\pi}\int_{w=0}^{\infty}f(w)\left[\int_{k=0}^{\infty}e^{-|k|y}\sin kx\sin kw\,dk\right]dw$$

NEED TO DERIVE AN IDENTITY

$$\left.\begin{aligned}\cos(kx+kw)&\equiv\cos kx\cos kw-\sin kx\sin kw\\ \cos(kx-kw)&\equiv\cos kx\cos kw+\sin kx\sin kw\end{aligned}\right\}$$

$$\cos(kx-kw)-\cos(kx+kw)\equiv 2\sin kx\sin kw$$

$$\Rightarrow u(x,y)=\frac{1}{\pi}\int_{w=0}^{\infty}f(w)\left[\int_{k=0}^{\infty}e^{-|k|y}\cos[k(x-w)]-e^{-|k|y}\cos[k(x+w)]\,dk\right]dw$$

G) LOOKING AT EACH OF THE "IMAGE" INTEGRALS

$$\int_{k=0}^{\infty}e^{-|k|y}\cos[k(x-w)]\,dk=\mathrm{Re}\int_{k=0}^{\infty}e^{-|k|y}e^{ik(x-w)}\,dk$$

$$=\mathrm{Re}\int_{k=0}^{\infty}e^{k[-y+i(x-w)]}\,dk=\mathrm{Re}\left[\frac{1}{-y+i(x-w)}e^{k(-y+i(x-w))}\right]_{k=0}^{\infty}$$

$$=\mathrm{Re}\left[\frac{-y-i(x-w)}{y^2+(x-w)^2}e^{-|k|y}e^{ik(x-w)}\right]_{k=0}^{\infty}=\mathrm{Re}\left[\frac{-y-i(x-w)}{y^2+(x-w)^2}(0-1)\right]$$

$$=\frac{y}{y^2+(x-w)^2}$$

SIMILARLY THE OTHER INTEGRAL EXCEPT THAT WE HAVE $(x+w)$ INSTEAD OF $(x-w)$

H) $$u(x,y)=\frac{1}{\pi}\int_{0}^{\infty}f(w)\left[\frac{y}{y^2+(x-w)^2}-\frac{y}{y^2+(x+w)^2}\right]dw$$

$$u(x,y)=\frac{y}{\pi}\int_{0}^{\infty}f(w)\left[\frac{1}{y^2+(x-w)^2}-\frac{1}{y^2+(x+w)^2}\right]dw$$

Question 16

The function $T=T\left(x,t\right)$ satisfies the heat equation in one spatial dimension,

$$\frac{\partial^2\theta}{\partial x^2}=\frac{1}{\sigma}\frac{\partial\theta}{\partial t},\quad x\geq 0\,,\ t\geq 0\,,$$

where σ is a positive constant.

It is further given that

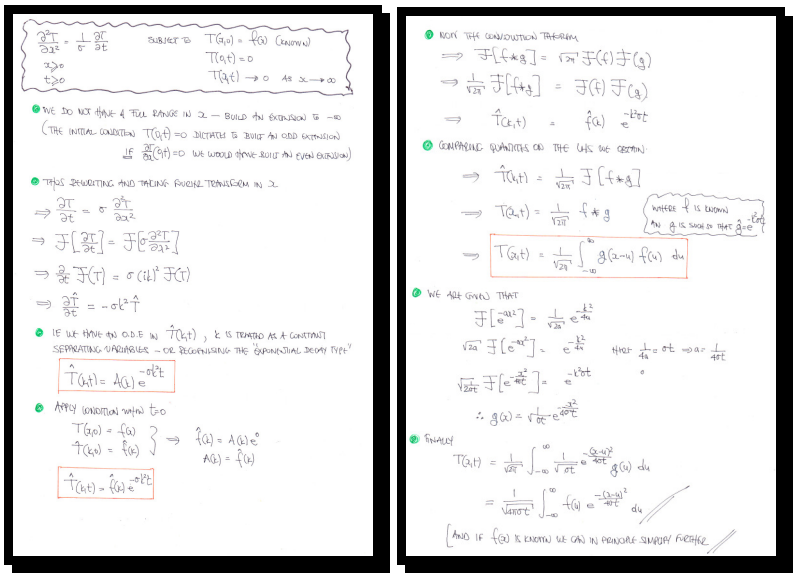
- $T\left(x,0\right)=f\left(x\right)$
- $T\left(0,t\right)=0$
- $T\left(x,t\right)\to 0$ as $x\to\infty$

Use Fourier transforms to convert the above partial differential equation into an ordinary differential equation and hence show that

$$T\left(x,t\right)=\frac{1}{\sqrt{4\pi\sigma t}}\int_{-\infty}^{\infty}f\left(u\right)\exp\left[\frac{\left(x-u\right)^2}{4t\sigma}\right]du\,.$$

You may assume that $\mathcal{F}\left[\mathrm{e}^{ax^2}\right]=\dfrac{1}{\sqrt{2a}}\mathrm{e}^{\frac{k^2}{4a}}$.

proof



Question 17

The function $f = f(x)$ satisfies the integral equation

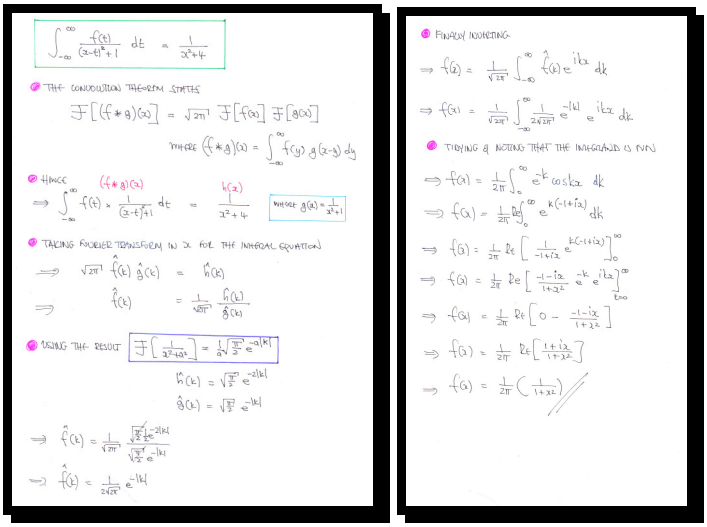
$$\int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2+1}\, dt = \frac{1}{x^2+4},$$

where $f(x) \to 0$ as $x \to \infty$

Use Fourier transforms to find the solution of the above integral equation.

You may assume that $\mathcal{F}\left[\dfrac{1}{x^2+a^2}\right] = \dfrac{1}{a}\sqrt{\dfrac{\pi}{2}}\,\mathrm{e}^{-a|k|}$.

$$\boxed{f(x) = \frac{1}{2\pi\left(1+x^2\right)}}$$



Question 18

The function $f = f(x)$ satisfies the integral equation

$$\int_{-\infty}^{\infty} f(x-u)\, f(u)\ du = \frac{1}{1+x^2},$$

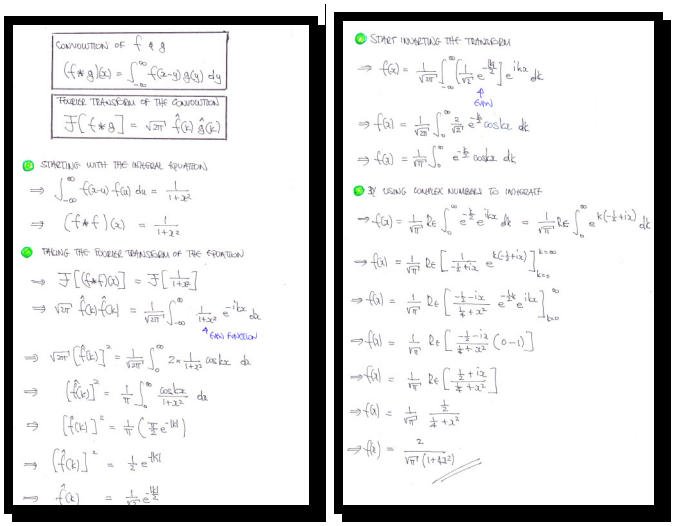
where $f(x) \to 0$ as $x \to \infty$

Use Fourier transforms to find the solution of the above integral equation.

You may assume that

$$\int_0^{\infty} \frac{\cos kx}{x^2+1}\ dx = \frac{1}{2}\pi \mathrm{e}^{|k|}.$$

$$\boxed{f(x) = \frac{2}{\left(1+4x^2\right)\sqrt{\pi}}}$$



Question 19

The function $f = f(x)$ satisfies the integral equation

$$e^{-\frac{1}{2}x^2} = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-u|} f(u)\ du\,,$$

where $f(x) \to 0$ as $x \to \infty$

Use Fourier transforms to find the solution of the above integral equation.

You may assume that

- $\mathcal{F}\left[e^{ax^2}\right] = \dfrac{1}{\sqrt{2a}} e^{\frac{k^2}{4a}}.$
- $\mathcal{F}\left[e^{a|x|}\right] = \sqrt{\dfrac{2}{\pi}}\ \dfrac{a}{a^2+k^2}.$

$$\boxed{f(x) = \left(2-x^2\right)e^{-\frac{1}{2}x^2}}$$

