FOURIER SERIES

The Fourier Theorem

If $f(x)$ is a piecewise continuous function on (α,β), then

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right],$$

where $a_n=\frac{1}{L}\int_{\alpha}^{\beta}f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$

$$b_n=\frac{1}{L}\int_{\alpha}^{\beta}f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

$$L=\frac{\beta-\alpha}{2}=\text{half period}$$

Parseval's Identity

$$\frac{1}{L}\int_{\alpha}^{\beta}\left[f(x)\right]^2\,dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left[a_n^2+b_n^2\right]$$

FOURIER SERIES EXPANSIONS

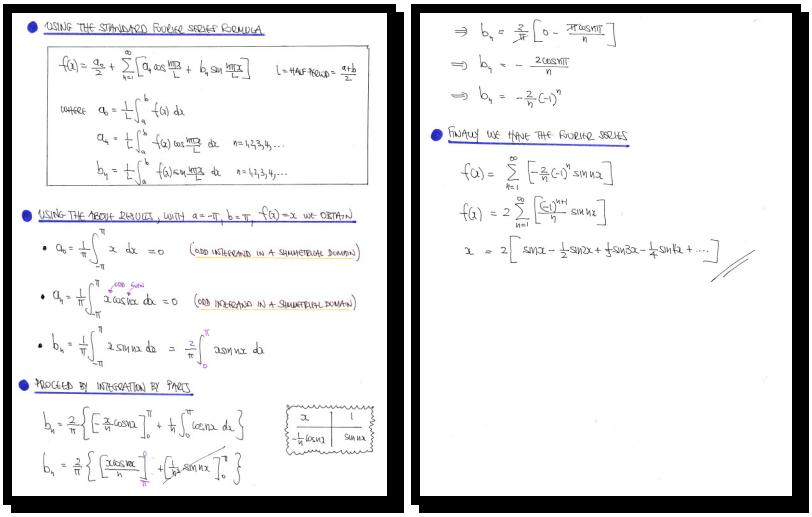
Question 1

$$f(x)=x,\ x\in\mathbb{R},\ -\pi\le x\le\pi.$$

$$f(x)=f(x+2\pi).$$

Determine the Fourier series expansion of $f(x)$.

$$\boxed{},\quad \boxed{f(x)=2\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}\sin nx}{n}\right]}$$



Question 2

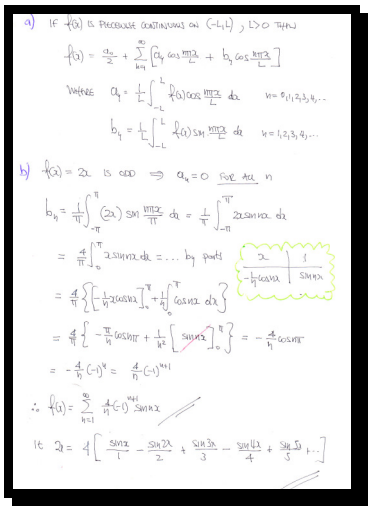
A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Find the Fourier series of

$$f(x) = 2x, \ -\pi \leq x \leq \pi .$$

$$2x = \sum_{n=1}^{\infty} \left[\frac{4(-1)^{n+1}}{n} \sin(nx) \right]$$



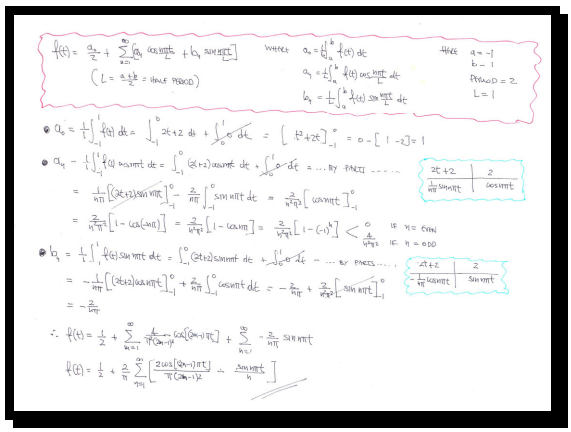
Question 3

$$f(t)=\begin{cases}2t+2 & -1\le t\le 0\\ 0 & 0\le t\le 1\end{cases}.$$

$$f(t)=f(t+2).$$

Determine the Fourier series expansion of $f(t)$.

$$\boxed{f(t)=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{2\cos\left[(2n-1)\pi t\right]}{\pi(2n-1)^2}-\frac{\sin(n\pi t)}{n}\right]}$$



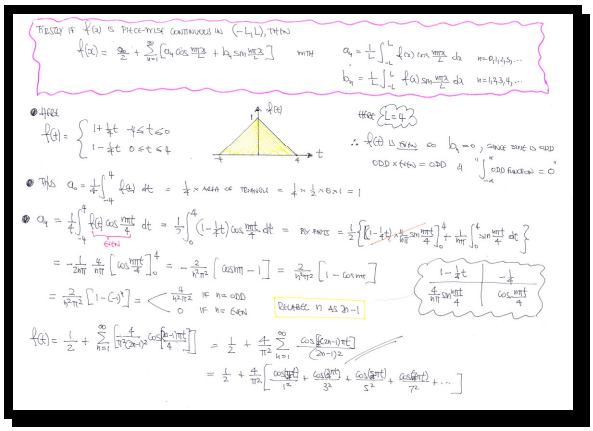
Question 4

$$f(t)=\begin{cases}1+\frac{1}{4}t & -4\le t\le 0\\ 1-\frac{1}{4}t & 0\le t\le 4\end{cases}.$$

$$f(t)=f(t+8).$$

Determine the Fourier series expansion of $f(t)$.

$$\boxed{f(t)=\frac{1}{2}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{\cos\left[\frac{1}{4}(2n-1)\pi t\right]}{(2n-1)^2}\right]}$$



Question 5

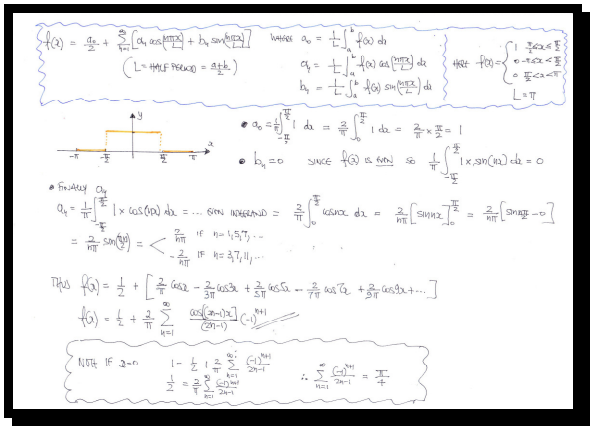
The "Top Hat" function is defined as

$$f(x)=\begin{cases}1 & |x|\le\dfrac{\pi}{2}\\ 0 & \dfrac{\pi}{2}<|x|\le\pi\end{cases}$$

for $x\in\mathbb{R}$, $f(x)=f(x+2\pi)$.

Determine the Fourier series expansion of $f(x)$.

$$f(x)=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}\cos\left[(2n-1)x\right]}{2n-1}\right]$$



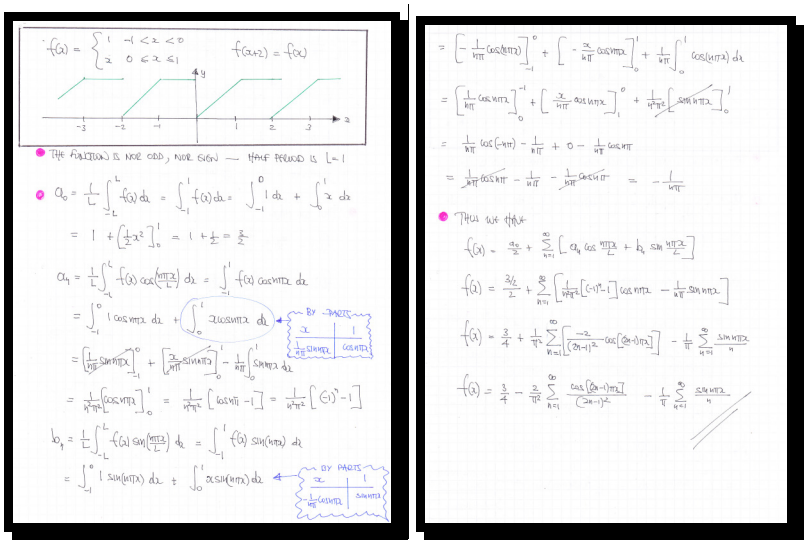
Question 6

$$f(x)=\begin{cases} 1 & -1\le x\le 0 \\ x & 0\le x\le 1 \end{cases}.$$

$$f(x+2)=f(x).$$

Determine the Fourier series expansion of $f(x)$.

$$f(x)=\frac{3}{4}-\frac{2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{\cos\left[(2n-1)\pi x\right]}{(2n-1)^2}\right]-\frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{\sin(n\pi x)}{n}\right]$$



Question 7

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

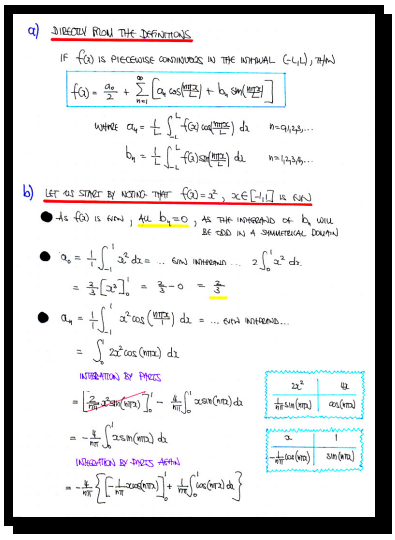
b) Find the Fourier series of

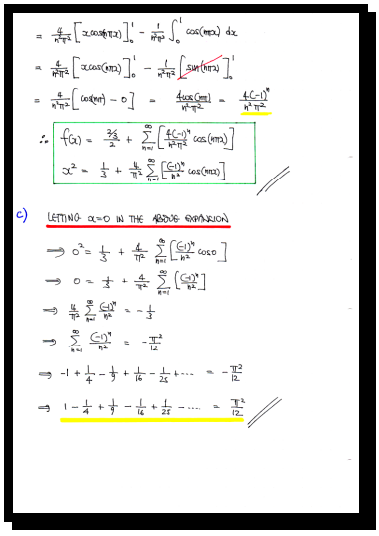
$$f(x) = x^2, \ -1 \le x \le 1.$$

c) Hence determine the exact value of

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \ldots.$$

$\square$, $\boxed{x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos(n\pi x)\right]}$, $\boxed{1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \ldots = \frac{\pi^2}{12}}$





Question 8

A function $f(x)$ is defined in an interval $(-\pi,\pi)$.

a) State the general formula for the Fourier series of $f(x)$ in $(-\pi,\pi)$, giving general expressions for the coefficients of the series.

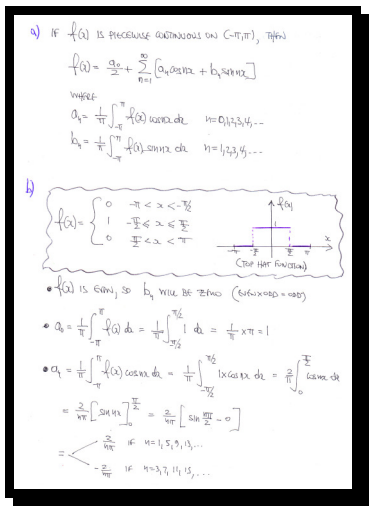
b) Find the Fourier series of

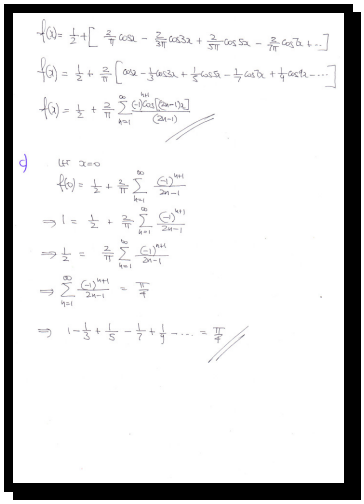
$$f(x)=\begin{cases}0 & -\pi\le x\le -\frac{1}{2}\pi\\ 1 & -\frac{1}{2}\pi< x\le \frac{1}{2}\pi\\ 0 & \frac{1}{2}\pi\le x\le \pi\end{cases}$$

c) Hence determine the exact value of

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots.$$

$$\boxed{f(x)=\frac{1}{2}+\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}\cos(nx)}{2n-1}\right]},\qquad \boxed{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots=\frac{\pi}{4}}$$





Question 9

A function $f(x)$ is defined in an interval $(\alpha,\alpha+2L)$, $L>0$.

a) State the general formula for the Fourier series of $f(x)$ in $(\alpha,\alpha+2L)$, giving general expressions for the coefficients of the series.

$$f(x)=x,\ 0\le x\le 4.$$

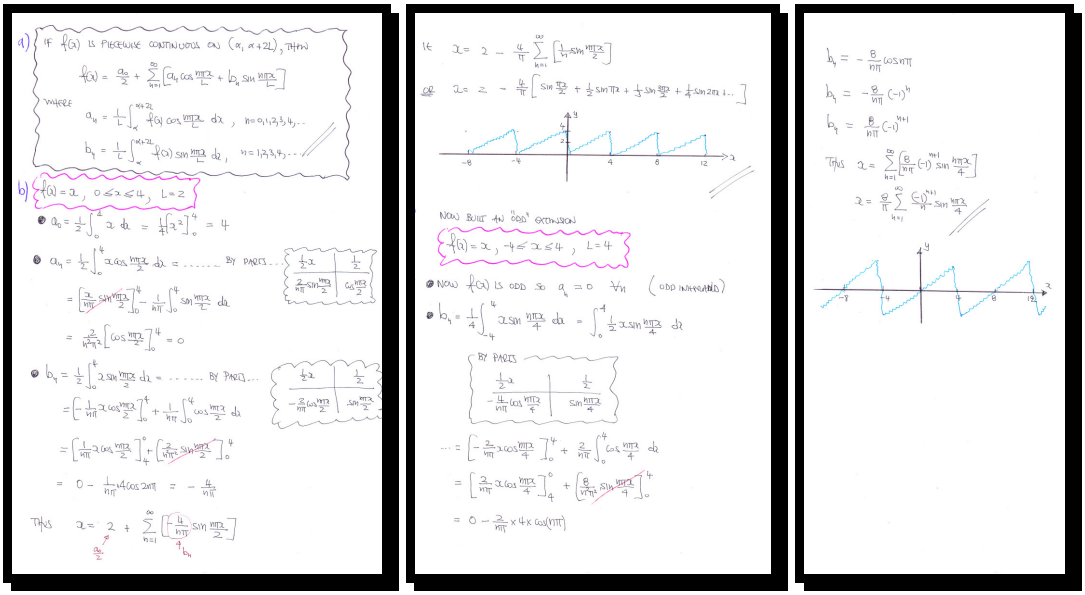
b) Find the Fourier series of $f(x)$…

i. … in the interval $0\le x\le 4$, with period 4.

ii. … in the interval $0\le x\le 4$, with period 8, by building a suitable "extension" to $f(x)$.

Illustrate the solution in each case with a sketch.

$$\boxed{x=2-\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\sin\left(\frac{1}{2}n\pi x\right)\right]},\qquad \boxed{x=\frac{8}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n}\sin\left(\frac{1}{4}n\pi x\right)\right]}$$



Question 10

A function $f(x)$ is defined in an interval $(\alpha,\alpha+2L)$, $L>0$.

a) State the general formula for the Fourier series of $f(x)$ in $(\alpha,\alpha+2L)$, giving general expressions for the coefficients of the series.

$$f(x)=x^2,\ 0\le x\le 1.$$

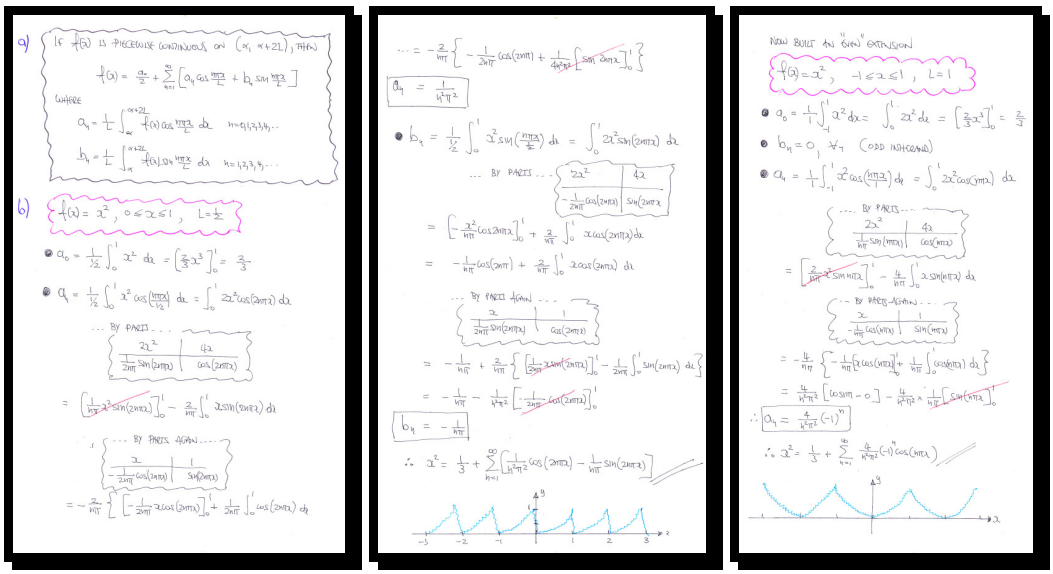
b) Find the Fourier series of $f(x)$…

i. … in the interval $0\le x\le 1$, with period 1.

ii. … in the interval $0\le x\le 1$, with period 2, by building a suitable "extension" to $f(x)$.

Illustrate the solution in each case with a sketch.

$$x^2=\frac{1}{3}+\sum_{n=1}^{\infty}\left[\frac{\cos(2n\pi x)}{n^2\pi^2}-\frac{\sin(2n\pi x)}{n\pi}\right],\qquad x^2=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos(n\pi x)\right]$$



Question 11

$$f(x)=\begin{cases}\pi-x & 0\le x\le\pi\\ \pi+x & -\pi<x\le 0\end{cases}$$

for $x\in\mathbb{R}$, $f(x)=f(x+2\pi)$.

a) Determine the Fourier series expansion of $f(x)$.

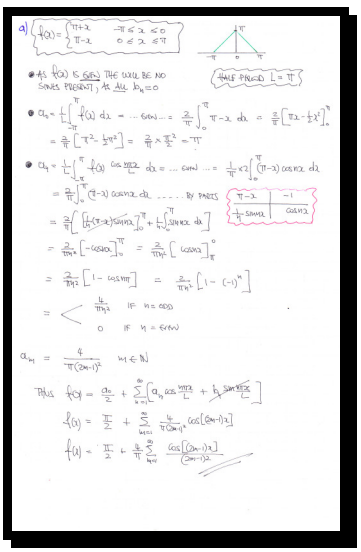
b) Hence determine the exact value of

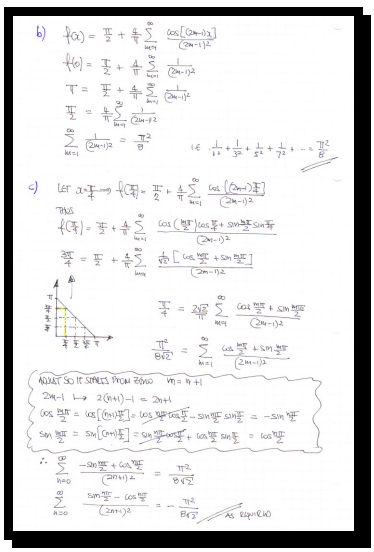
$$\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}.$$

c) Show that

$$\sum_{n=0}^{\infty}\left[\frac{\sin\frac{n\pi}{2}-\cos\frac{n\pi}{2}}{(2n+1)^2}\right]=-\frac{\pi^2}{8\sqrt{2}}$$

$$\boxed{f(x)=\frac{\pi}{2}+\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{\cos\left[(2n-1)x\right]}{(2n-1)^2}\right]},\qquad \boxed{\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}=\frac{\pi^2}{8}}$$





Question 12

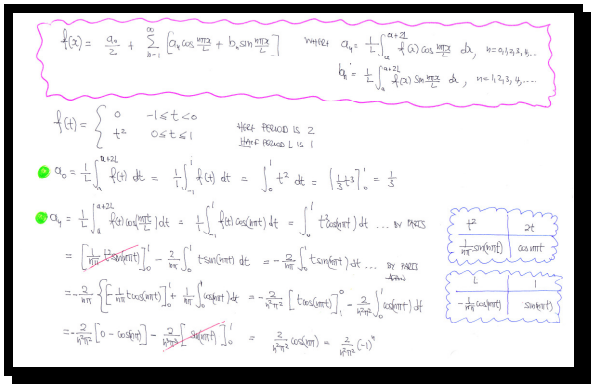
The periodic function f is defined as

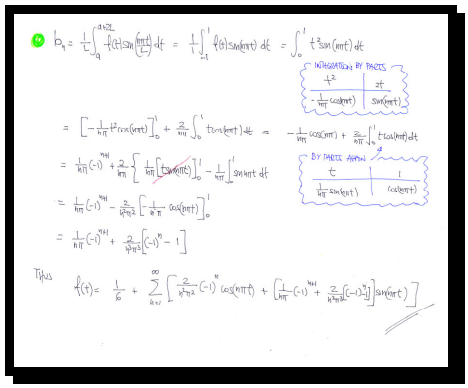
$$f(t)=\begin{cases} 0 & -1\le t<0 \\ t^2 & 0\le t\le 1 \end{cases}$$

for $t\in\mathbb{R}$, $f(t)=f(t+2)$.

Determine the Fourier series expansion of $f(t)$.

$$f(t)=\frac{1}{6}+\sum_{n=1}^{\infty}\left\{\frac{(-1)^n\times 2\cos(n\pi t)}{n^2\pi^2}+\left[\frac{(-1)^{n+1}}{n\pi}+\frac{2}{n^3\pi^3}\left[(-1)^n-1\right]\right]\sin(n\pi t)\right\}$$





Question 13

$$f(x)=x,\ x\in\mathbb{R},\ 0\le x\le 2\pi.$$

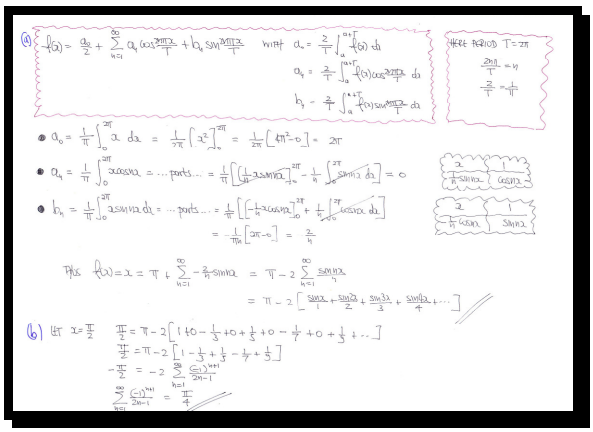
$$f(x)=f(x+2\pi).$$

a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}.$$

$$\boxed{f(x)=\pi-2\sum_{n=1}^{\infty}\frac{\sin nx}{n}},\quad \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=\frac{\pi}{4}}$$



Question 14

A function $f(x)$ is defined in an interval $(-\pi,\pi)$.

a) State the general formula for the Fourier series of $f(x)$ in $(-\pi,\pi)$, giving general expressions for the coefficients of the series.

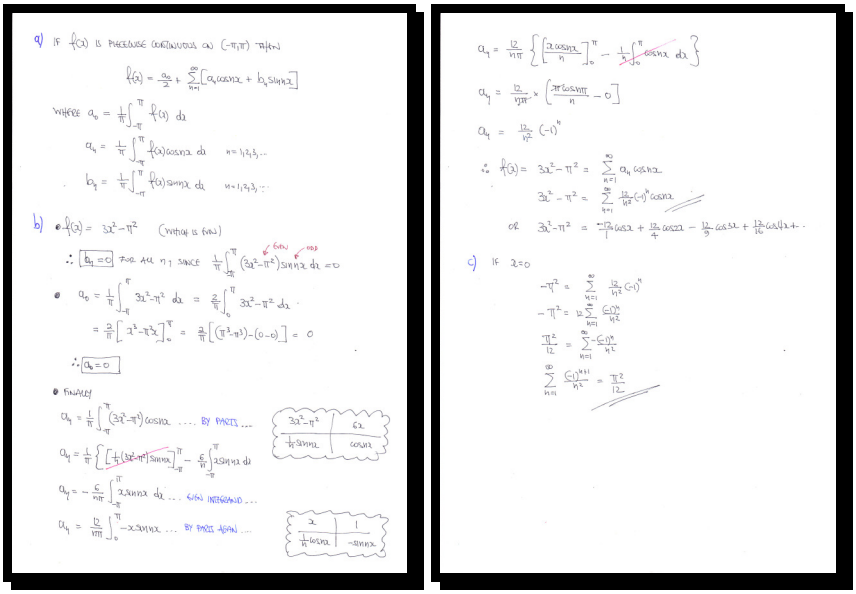
b) Find the Fourier series of

$$f(x)=3x^2-\pi^2,\ -\pi\le x\le\pi\,.$$

c) Hence determine the exact value of

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\,.$$

$$\boxed{3x^2-\pi^2=12\sum_{n=1}^{\infty}\left[\frac{(-1)^n\cos nx}{n^2}\right]},\quad \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}}$$



Question 15

$$f(x)=|x|,\ x\in\mathbb{R},\ -\pi\le x\le\pi.$$

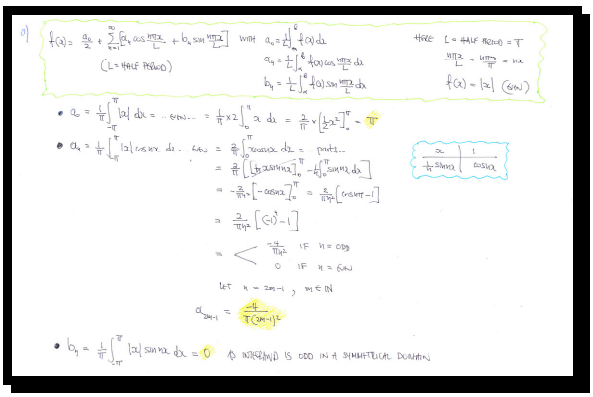
$$f(x)=f(x+2\pi).$$

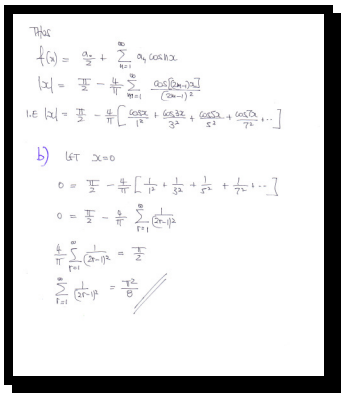
a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}.$$

$$\boxed{f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos\left[(2n-1)x\right]}{(2n-1)^2}},\quad \boxed{\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}=\frac{\pi^2}{8}}$$





Question 16

$$f(x)=x,\ x\in\mathbb{R},\ -1\le x\le 1.$$

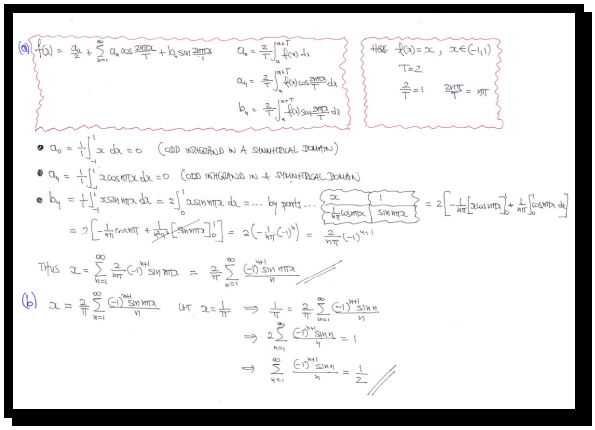
$$f(x)=f(x+2).$$

a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n}{n}.$$

$$\boxed{f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n\pi x}{n}},\qquad \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n=\frac{1}{2}}}$$



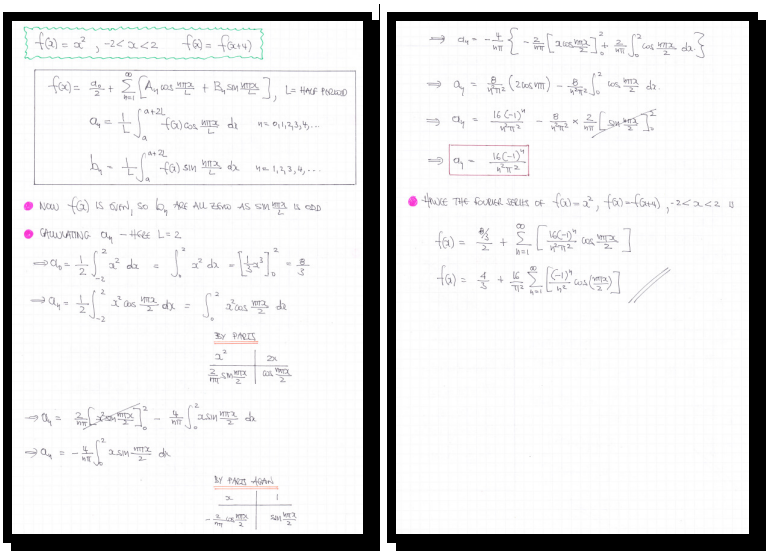
Question 17

$$f(x)=x^2,\ x\in\mathbb{R},\ -2\le x\le 2.$$

$$f(x)=f(x+4).$$

Determine the Fourier series expansion of $f(x)$.

$$f(x)=\frac{4}{3}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{1}{2}n\pi x\right)$$



Question 18

A function $f(x)$ is defined in the interval $(-\pi,\pi)$.

a) State the general formula for the Fourier series of $f(x)$ in $(-\pi,\pi)$, giving general expressions for the coefficients of the series.

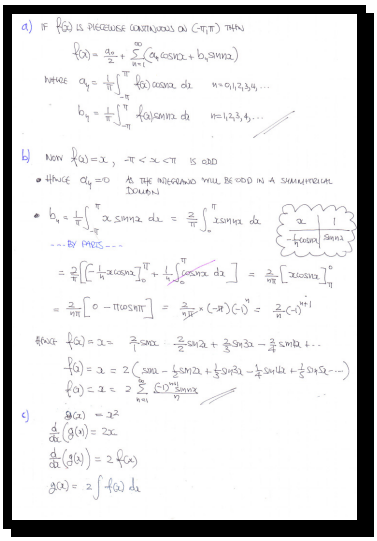
b) Find the Fourier series of

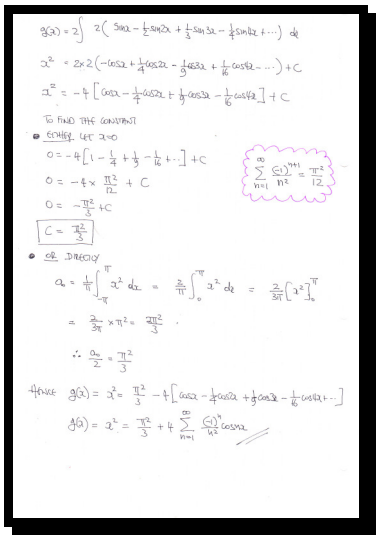
$$f(x)=x\,,\ -\pi\le x\le\pi\,.$$

c) Hence determine the exact value of

$$g(x)=x^2\,,\ -\pi\le x\le\pi\,.$$

$$\boxed{f(x)=2\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}\sin nx}{n}\right]},\qquad \boxed{g(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\left[\frac{(-1)^{n}\cos nx}{n^2}\right]}$$





Question 19

$$f(x)=x^2,\ x\in\mathbb{R},\ 0\le x\le 1.$$

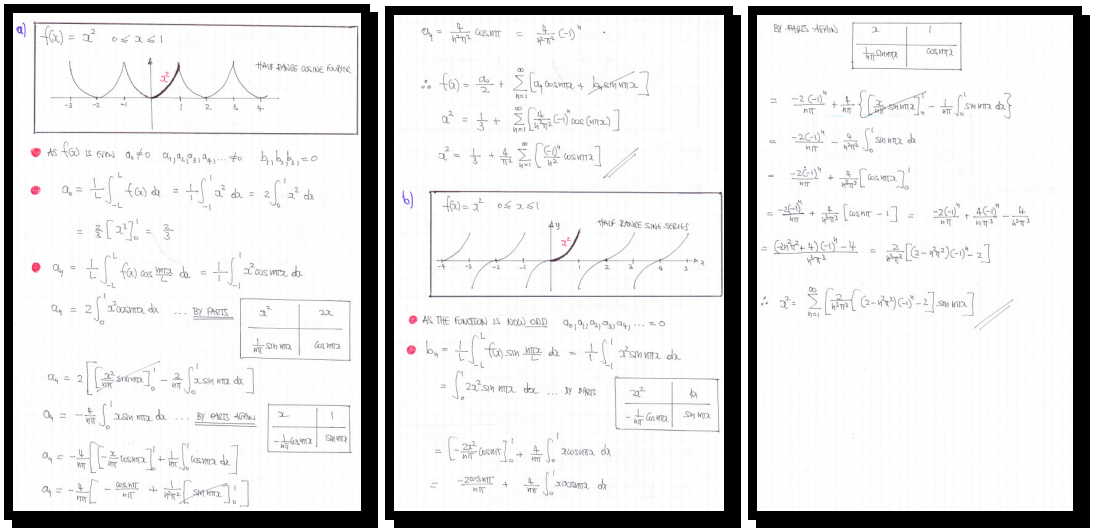
Determine the Fourier series of $f(x)$ as

a) … as half range cosine expansion.

b) … as half range sine expansion.

$$\boxed{f(x)=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos(n\pi x)\right]},$$

$$\boxed{f(x)=\frac{2}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{2}{n^3}\left[\left(2-n^2\pi^2\right)(-1)^n-2\right]\sin(n\pi x)\right]}$$



Question 20

$$f(x)=\begin{cases} \pi-x & 0\le x\le \pi \\ 0 & \pi\le x\le 2\pi \end{cases}, \quad x\in\mathbb{R}.$$

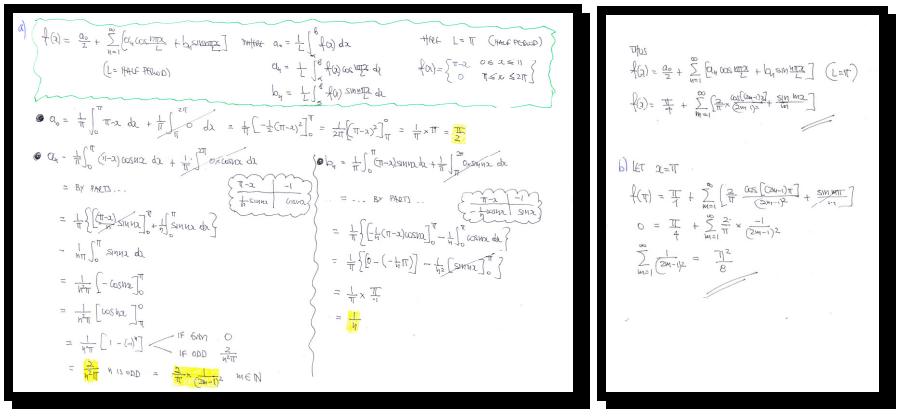
$$f(x)=f(x+2\pi).$$

a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}.$$

$$\boxed{f(x)=\frac{\pi}{4}+\sum_{m=1}^{\infty}\left[\frac{2\cos\left[(2m-1)x\right]}{\pi(2m-1)^2}+\frac{\sin mx}{m}\right]}, \quad \boxed{\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}=\frac{\pi^2}{8}}$$



Question 21

$$f(x)=x^2,\ x\in\mathbb{R},\ -\pi\le x\le\pi.$$

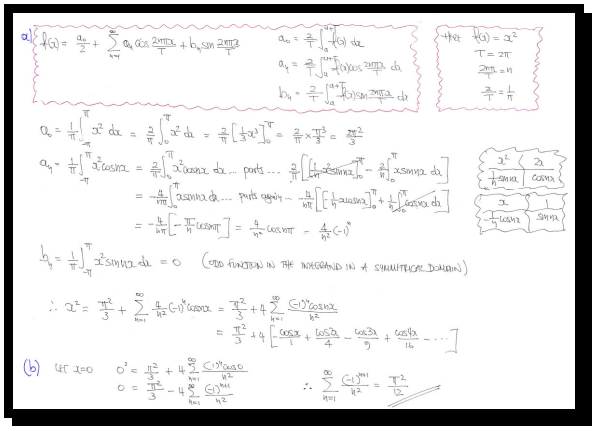
$$f(x)=f(x+2\pi).$$

a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}.$$

$$\boxed{f(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}},\qquad \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}}$$



Question 22

$$f(x)=x^2,\ x\in\mathbb{R},\ 0\le x\le 2\pi .$$

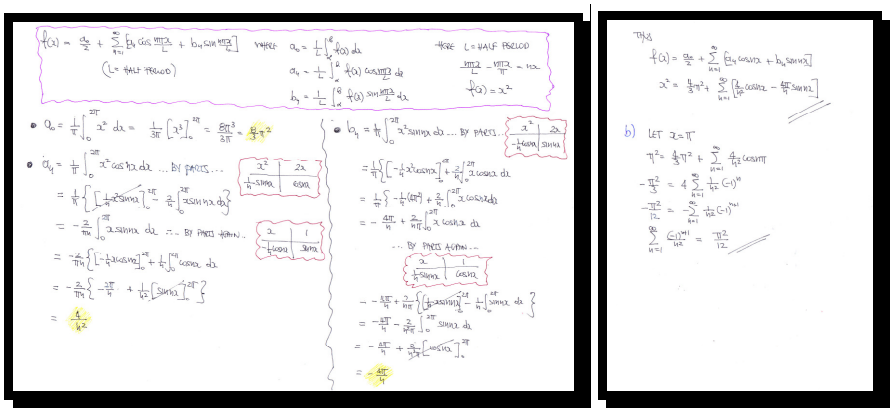
$$f(x)=f(x+2\pi).$$

a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}.$$

$$\boxed{f(x)=\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\left[\frac{\cos nx}{n^2}-\frac{\pi\sin nx}{n}\right]},\qquad \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}}$$



Question 23

It is given that for $x \in \mathbb{R}$, $-\pi \leq x \leq \pi$,

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos\left[(2n-1)x\right]}{(2n-1)^2}, \qquad |x| = |x+2\pi|.$$

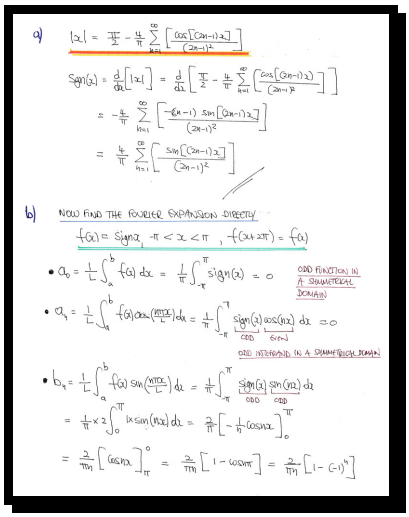
a) Use the above Fourier series expansion to deduce the Fourier series expansion of $\operatorname{sgn}(x)$.

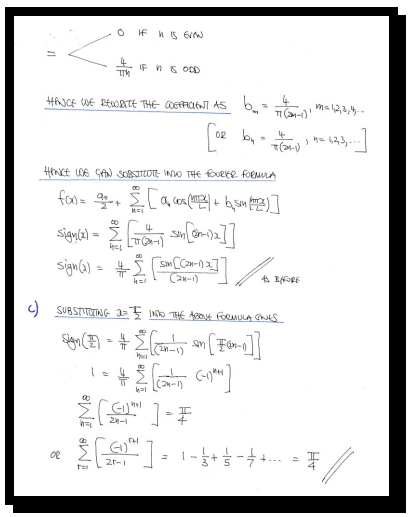
b) Verify the answer of part **(a)** by obtaining directly the Fourier series expansion of $\operatorname{sgn}(x)$.

c) Hence determine the exact value of

$$\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{2r-1}.$$

$\square$, $\boxed{\operatorname{sgn}(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left[(2n-1)x\right]}{(2n-1)^2}}$, $\boxed{\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{2r-1} = \frac{\pi}{4}}$





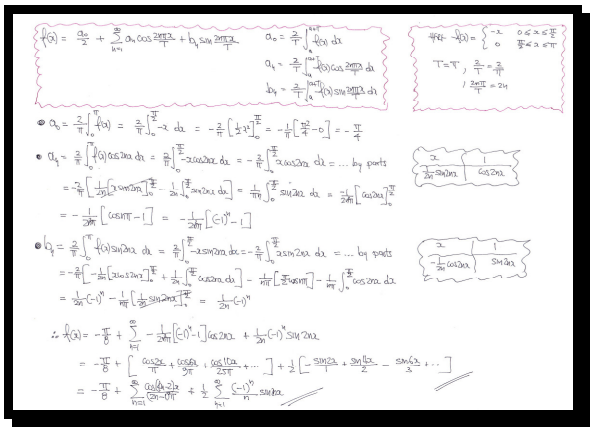
Question 24

$$f(x)=\begin{cases} -x & 0\le x\le \dfrac{\pi}{2} \\ 0 & \dfrac{\pi}{2}<x<\pi \end{cases}$$

$$f(x)=f(x+\pi).$$

Determine the Fourier series expansion of $f(x)$.

$$\boxed{f(x)=-\frac{\pi}{8}+\sum_{n=1}^{\infty}\frac{\cos\left[(4n-2)x\right]}{(2n-1)^2\pi}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n\sin 2nx}{n}}$$



Question 25

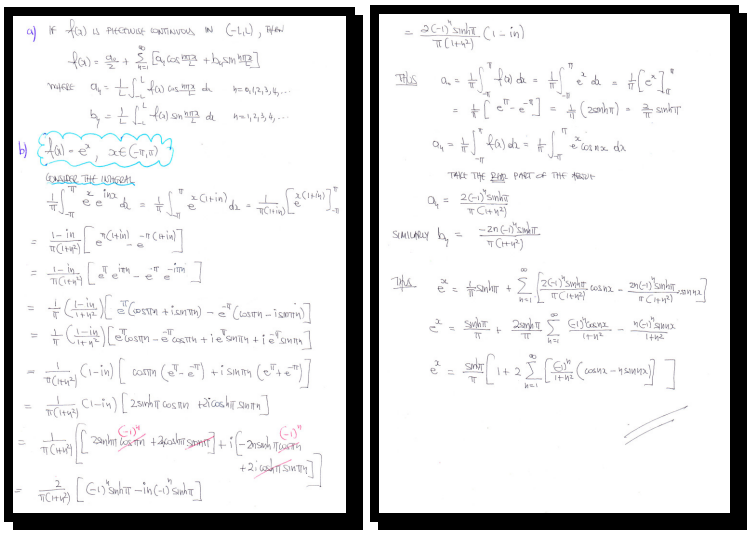
A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Determine the Fourier series of

$$f(x) = \mathrm{e}^x, \ -\pi \le x \le \pi .$$

$$\mathrm{e}^x = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \left[\cos(nx) - n\sin(nx) \right]}{1+n^2} \right]$$



Question 26

A function $f(x)$ is defined in an interval $(-L,L)$, $L>0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L,L)$, giving general expressions for the coefficients of the series.

b) Show that

$$\int_{-\pi}^{\pi} \mathrm{e}^{ax}\,\mathrm{e}^{\mathrm{i}nx}\,dx = \frac{2(a-n\mathrm{i})(-1)^n}{\mathrm{a}^2+n^2}\sinh(a\pi)$$

c) Determine the Fourier series of

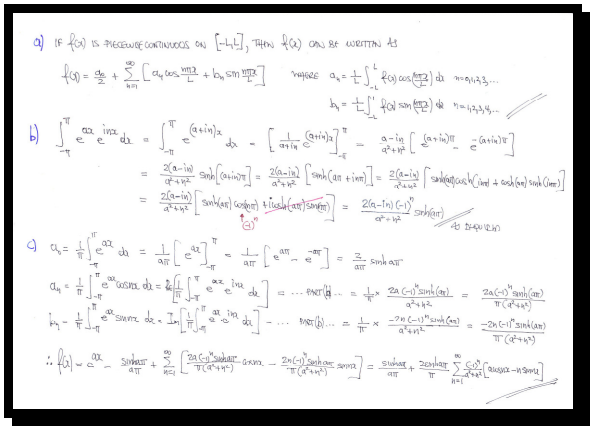
$$f(x)=\mathrm{e}^{ax},\ a>0,\ -\pi\le x\le\pi.$$

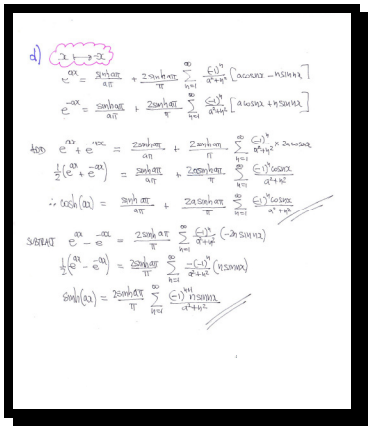
d) Hence find the Fourier series of $\cosh(ax)$ and $\sinh(ax)$, for $-\pi\le x\le\pi$.

$$\mathrm{e}^{ax}=\frac{\sinh(a\pi)}{a\pi}+\frac{2\sinh(a\pi)}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n\left[a\cos(nx)-n\sin(nx)\right]}{a^2+n^2}\right],$$

$$\cosh(ax)=\frac{\sinh(a\pi)}{a\pi}+\frac{2a\sinh(a\pi)}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n\cos(nx)}{a^2+n^2}\right],$$

$$\sinh(ax)=\frac{2\sinh(a\pi)}{\pi}\sum_{n=1}^{\infty}\left[\frac{n(-1)^{n+1}\sin(nx)}{a^2+n^2}\right]$$





Question 27

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Determine the Fourier series of

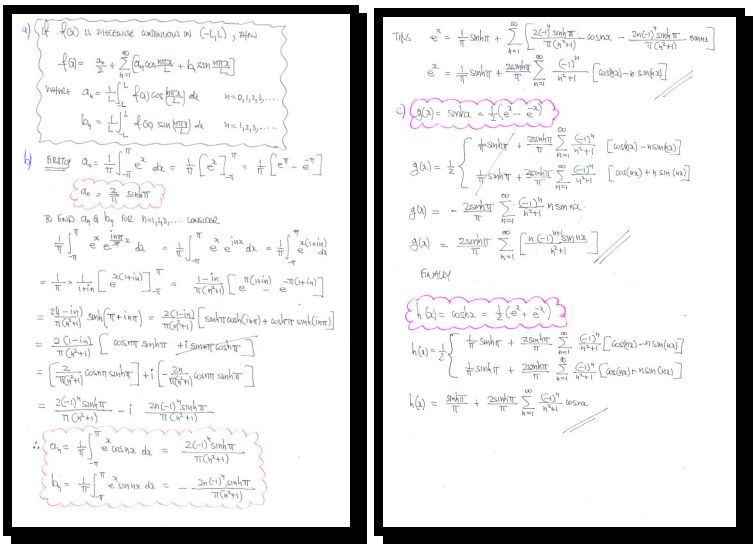
$$f(x) = \mathrm{e}^x, \ -\pi \le x \le \pi .$$

c) Hence find the Fourier series of $\sinh x$ and $\cosh x$, for $-\pi \le x \le \pi$.

$$\mathrm{e}^x = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \left[\cos(nx) - n\sin(nx) \right]}{1+n^2} \right],$$

$$\sinh x = \frac{2\sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{n(-1)^{n+1} \sin(nx)}{1+n^2} \right],$$

$$\cosh x = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \cos(nx)}{1+n^2} \right],$$



Question 28

A function f is defined by

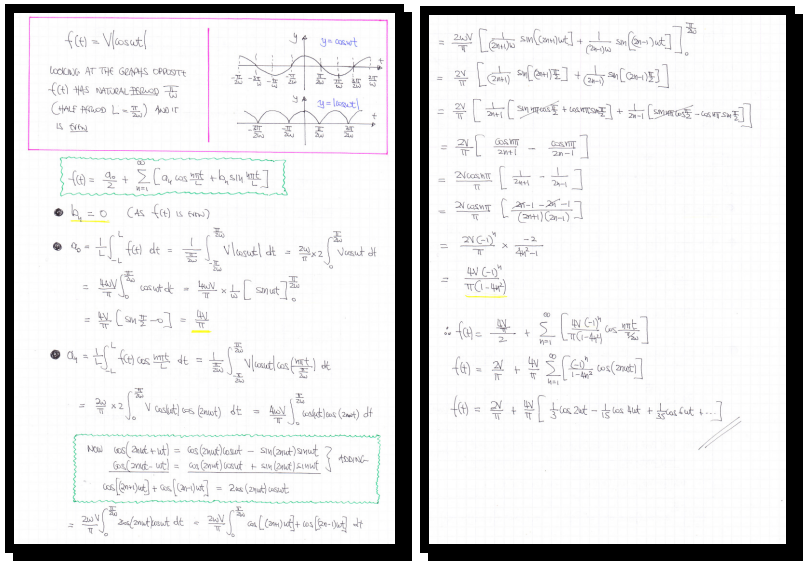
$$f(t)=V\left|\cos\omega t\right|,\ t\in\mathbb{R},$$

where V and ω are positive constants.

Show that the Fourier series of f is given by

$$f(t)=\frac{2V}{\pi}+\frac{4V}{\pi}\left[\frac{1}{3}\cos(2\omega t)-\frac{1}{15}\cos(4\omega t)+\frac{1}{35}\cos(6\omega t)+...\right]$$

proof



PARSEVAL'S IDENTITY

Question 1

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Find the Fourier series of

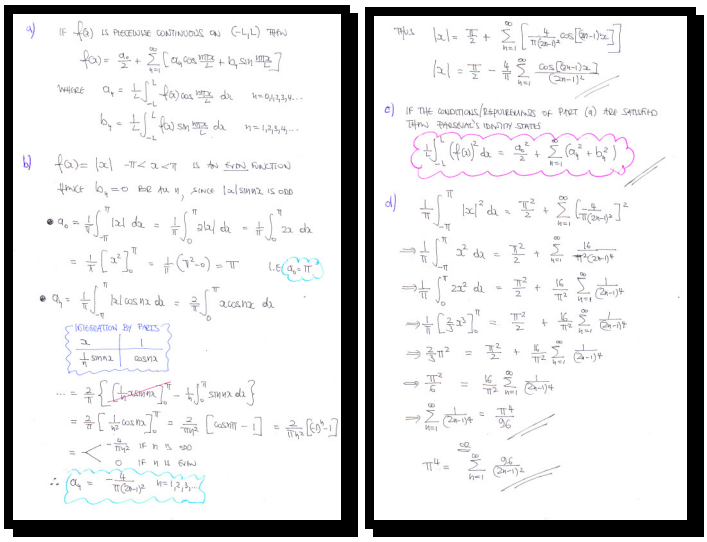
$$f(x) = |x|, \quad -\pi \le x \le \pi .$$

c) State Parseval's identity for the Fourier series of $f(x)$ from part **(a)**.

d) Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96} .$$

$$\boxed{|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{\cos\left[(2n-1)x\right]}{(2n-1)^2} \right]}$$



Question 2

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Find the Fourier series of

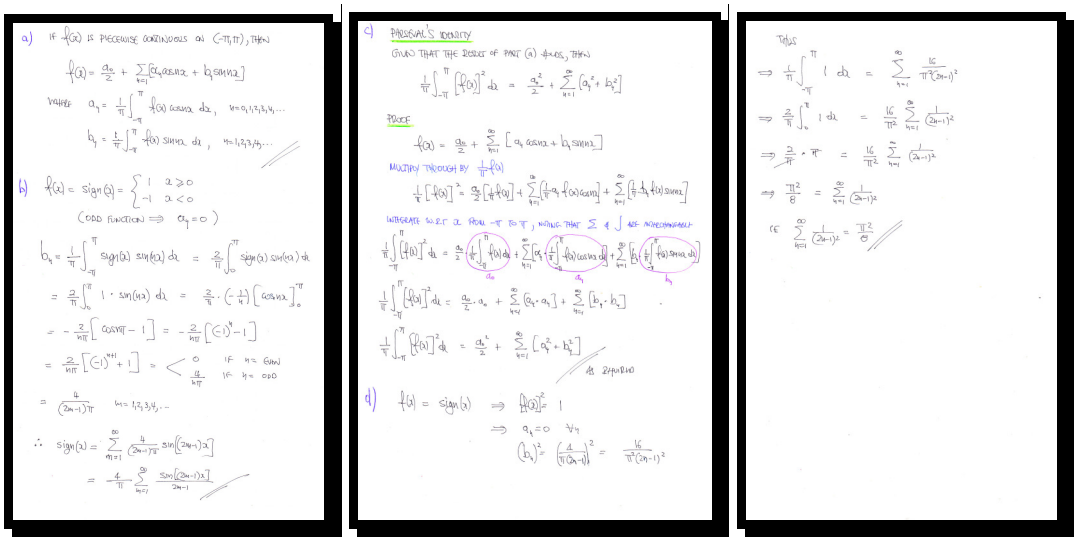
$$f(x) = \text{sign}(x), \quad -\pi \le x \le \pi .$$

c) Prove Parseval's identity for the Fourier series of $f(x)$ in $(-\pi, \pi)$.

d) Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} .$$

$$\boxed{\text{sign}(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{\sin\left[(2n-1)x\right]}{(2n-1)} \right]}$$



Question 3

A function $f(x)$ is defined in an interval $(-L,L)$, $L>0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L,L)$, giving general expressions for the coefficients of the series.

b) Find the Fourier series of

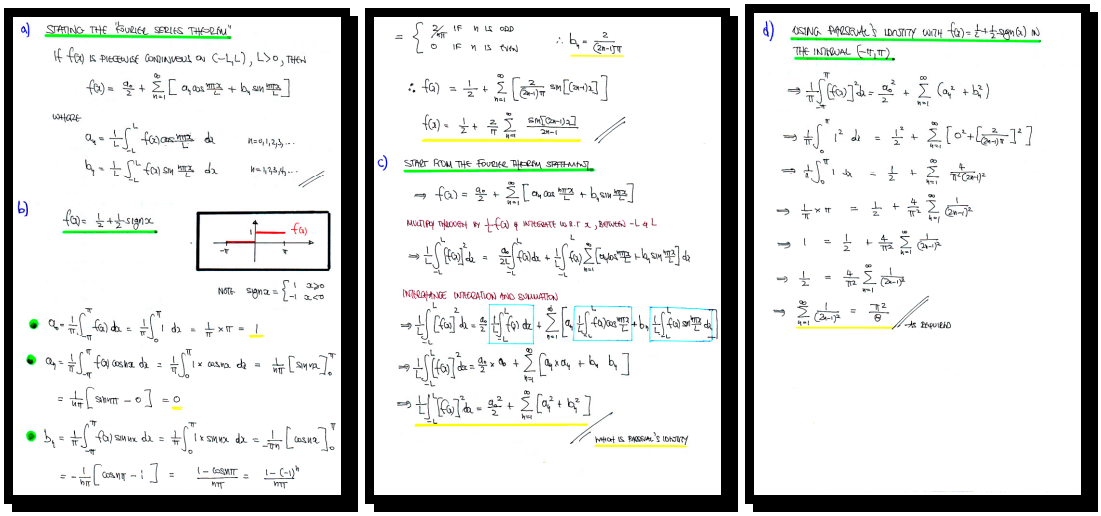
$$f(x)=\frac{1}{2}+\frac{1}{2}\text{sign}(x), \quad -\pi\le x\le\pi.$$

c) Prove the validity of Parseval's identity for the Fourier series of $f(x)$ in the interval $(-L,L)$.

d) Hence show that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}.$$

$\square$, $\dfrac{1}{2}+\dfrac{1}{2}\text{sign}(x)=\dfrac{1}{2}+\dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{\sin\left[(2n-1)x\right]}{(2n-1)}\right]$



Question 4

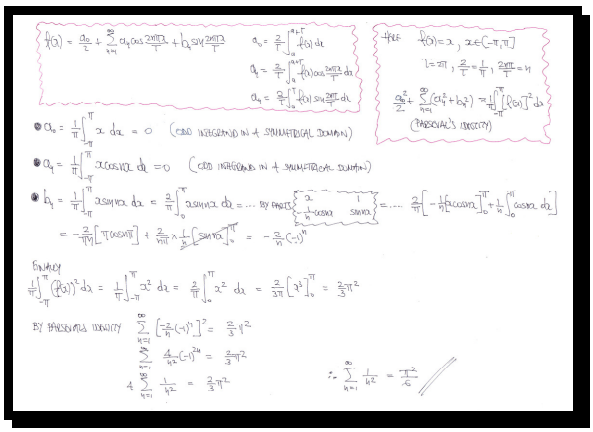
$$f(x)=x,\ x\in\mathbb{R},\ -\pi\le x\le\pi.$$

$$f(x)=f(x+2\pi).$$

Use Parseval's identity for the Fourier coefficients of $f(x)$ to determine the exact value of

$$\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}}$$



Question 5

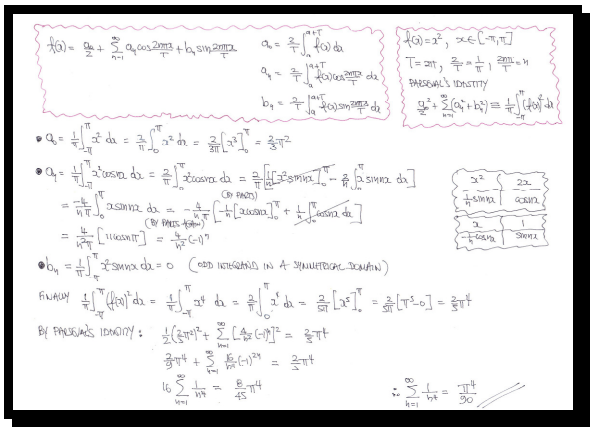
$$f(x)=x^2,\ x\in\mathbb{R},\ -\pi\le x\le\pi.$$

$$f(x)=f(x+2\pi).$$

Use Parseval's identity for the Fourier coefficients of $f(x)$ to determine the exact value of

$$\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}}$$



Question 6

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) Prove the validity of Parseval's identity for the Fourier series of $f(x)$ in the interval $(-L, L)$.

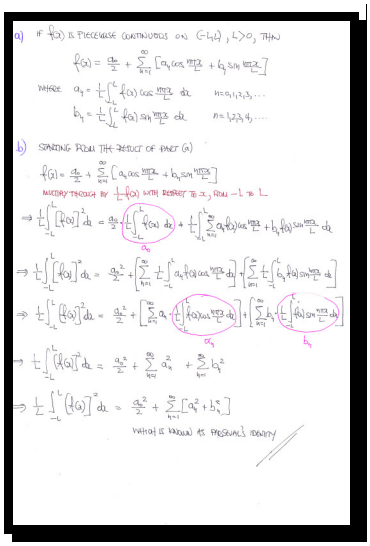
c) Find the Fourier series of

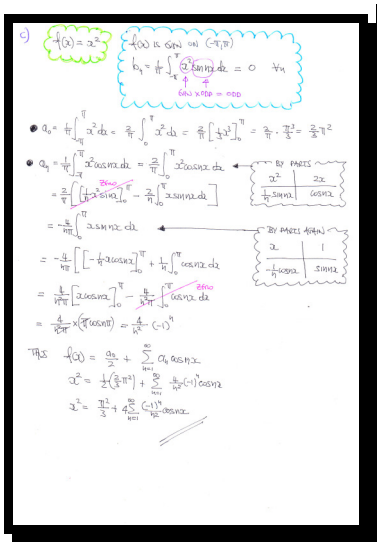
$$f(x) = x^2, \quad -\pi \le x \le \pi .$$

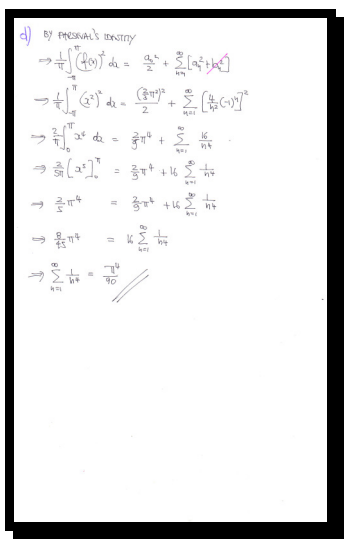
d) Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos nx\right]$$







Question 7

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

b) State and prove Parseval's identity for the Fourier series of $f(x)$ in $(-L, L)$.

c) By considering the Fourier series of

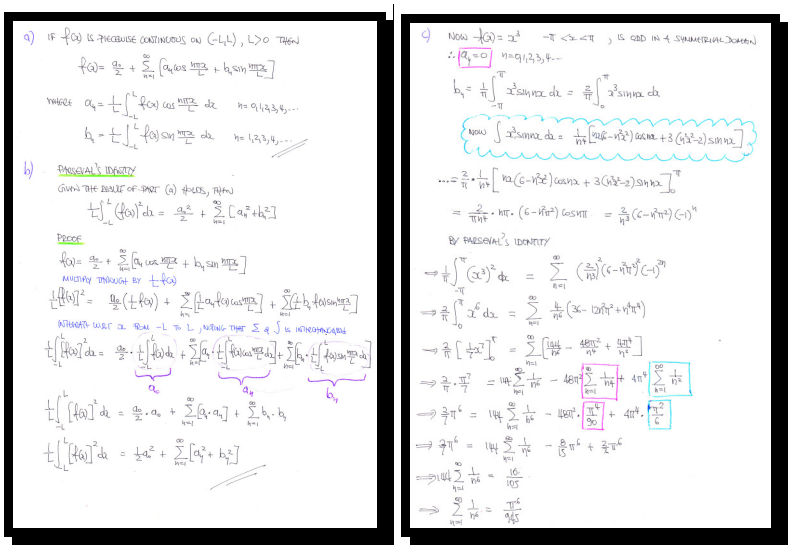
$$f(x) = x^3, \ -\pi \le x \le \pi,$$

show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$.

You may use without proof the following results.

- $\displaystyle\int x^3 \sin nx \, dx = \frac{1}{n^4}\left[nx\left(6 - n^2x^2\right)\cos nx + 3\left(n^2x^2 - 2\right)\sin nx \right] + C$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

proof



FOURIER SERIES
Complex Expansions

Question 1

A periodic function $f(t)$ is defined in the interval $(-L, L)$, $L>0$, $f(t+2L)=f(t)$.

It is further given that $f(t)$ is continuous or piecewise continuous in $(-L, L)$ and has Fourier series

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi t}{L}\right)+b_n\sin\left(\frac{n\pi t}{L}\right)\right],$$

where $a_n=\dfrac{1}{L}\displaystyle\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\,dt$, $n=0,\,1,\,2,\,3,\,...$

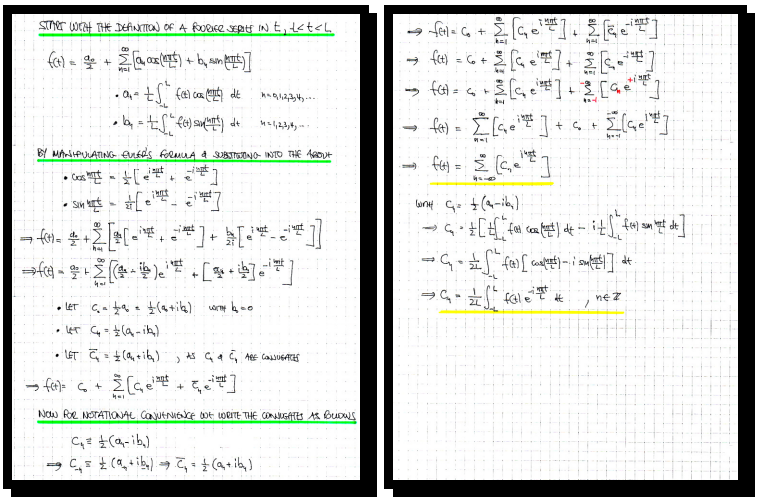
and $b_n=\dfrac{1}{L}\displaystyle\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\,dt$, $n=1,\,2,\,3,\,4,\,...$

Show that the complex Fourier series expansion of $f(t)$ is

$$f(t)=\sum_{n=-\infty}^{\infty}\left[c_n\,\mathrm{e}^{\frac{in\pi t}{L}}\right],$$

where $c_n=\dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(t)\,\mathrm{e}^{-\frac{in\pi t}{L}}\,dt$, $n\in\mathbb{Z}$

, proof

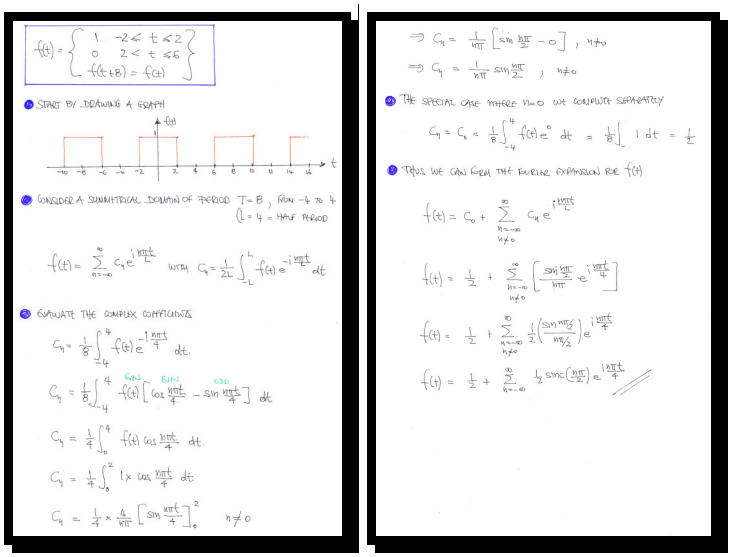


Question 2

$$f(t)=\begin{cases}1 & -2\le t\le 2\\ 0 & 2<t<6\end{cases}, \qquad f(t+8)=f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$f(t)=\frac{1}{2}+\frac{1}{2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left[\mathrm{e}^{\frac{1}{4}n\pi t\mathrm{i}}\ \operatorname{sinc}\left(\frac{1}{2}n\pi\right)\right]$$

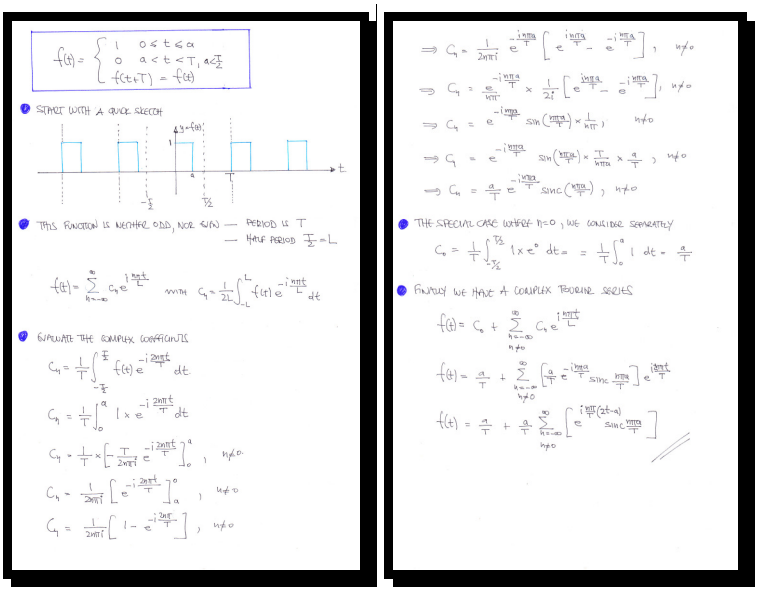


Question 3

$$f(t)=\begin{cases}1 & 0\le t\le a\\ 0 & a<t<T\end{cases}\quad,\quad a<\frac{1}{2}T\,,\quad f(t+T)=f(t)\,.$$

Determine the complex Fourier series expansion of $f(t)$.

$$\boxed{f(t)=\frac{a}{T}+\frac{a}{T}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left[\exp\left(\frac{n\pi(2t-a)}{T}\right)\operatorname{sinc}\left(\frac{n\pi a}{T}\right)\right]}$$

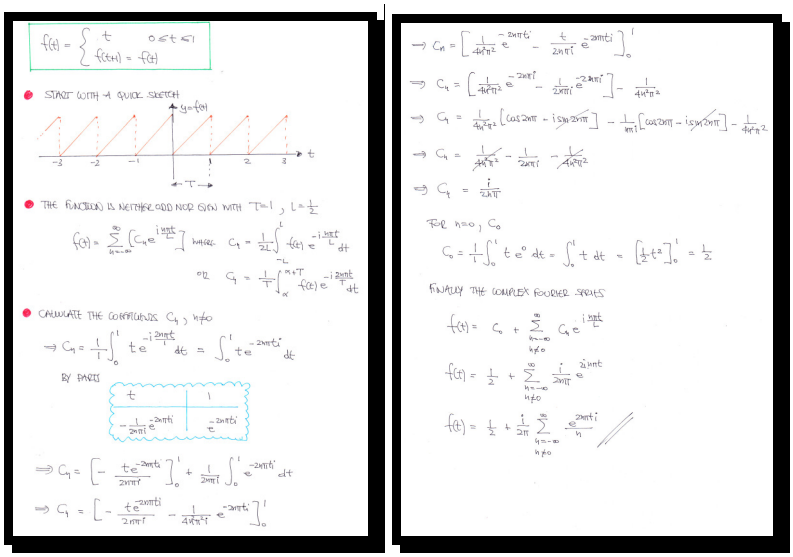


Question 4

$$f(t)=t, \quad 0\le t<1, \quad f(t+1)=f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$f(t)=\frac{1}{2}+\frac{\mathrm{i}}{2\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left[\frac{\mathrm{e}^{2n\pi t\mathrm{i}}}{n}\right]$$

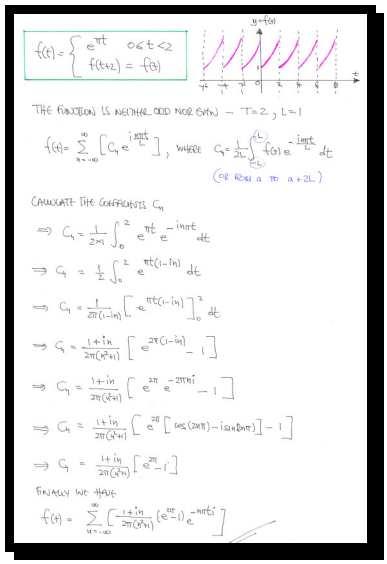


Question 5

$$f(t)=\mathrm{e}^{\pi t} \quad , \quad 0\le t<2, \quad f(t+2)=f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$\boxed{f(t)=\frac{\left(\mathrm{e}^{2\pi}-1\right)}{2\pi}\sum_{n=-\infty}^{\infty}\left[\frac{(1+\mathrm{i}n)\mathrm{e}^{2n\pi t\mathrm{i}}}{\left(n^2+1\right)}\right]}$$

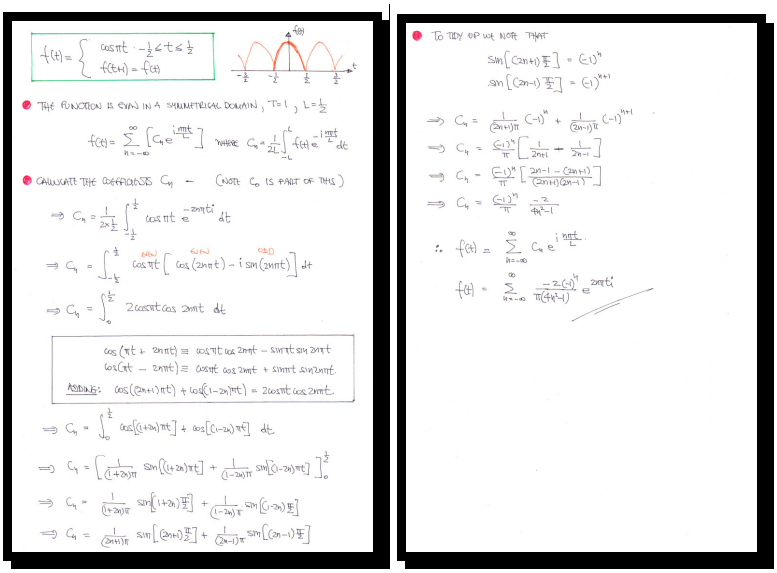


Question 6

$$f(t)=\cos(\pi t)\ \ ,\ \ -\frac{1}{2}\le t<\frac{1}{2},\quad f(t+1)=f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$\boxed{f(t)=\frac{2}{\pi}\sum_{n=-\infty}^{\infty}\left[\frac{(-1)^{n+1}\,e^{2n\pi t\,i}}{4n^2-1}\right]}$$

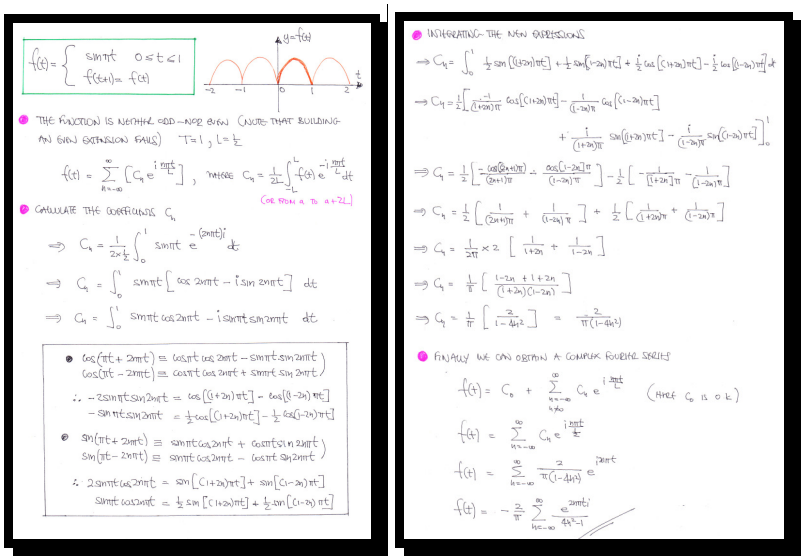


Question 7

$$f(t)=\sin(\pi t)\ ,\ \ 0\le t<1,\quad f(t+1)=f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$\boxed{f(t)=-\frac{2}{\pi}\sum_{n=-\infty}^{\infty}\left[\frac{\mathrm{e}^{2n\pi t\,\mathrm{i}}}{4n^2-1}\right]}$$



Question 8

The function f is defined as

$$f(t)=V\cos\left(\frac{\pi t}{T}\right)\ ,\ -\frac{1}{2}T\le t<\frac{1}{2}T\ ,\quad f(t)=f(t+T),$$

where V and T are positive constants.

Determine the complex Fourier series expansion of $f(t)$.

$$\boxed{f(t)=\frac{2V}{\pi}\sum_{n=-\infty}^{\infty}\left[\frac{\mathrm{e}^{2n\pi t\mathrm{i}}}{\left(1-4n^2\right)}\right]}$$

