DIFFERENTIATION UNDER THE INTEGRAL SIGN

LEIBNIZ INTEGRAL RULE

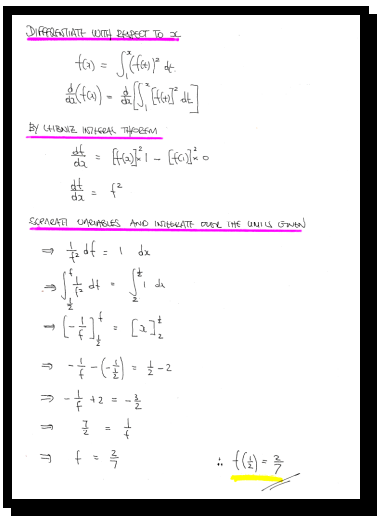
Question 1

The function f satisfies the following relationship.

$$f(x)=\int_1^x \left[f(t)\right]^2\,dt\,, \quad f(2)=\frac{1}{2}\,.$$

Determine the value of $f\left(\frac{1}{2}\right)$.

$\square$, $\boxed{f\left(\frac{1}{2}\right)=\frac{2}{7}}$

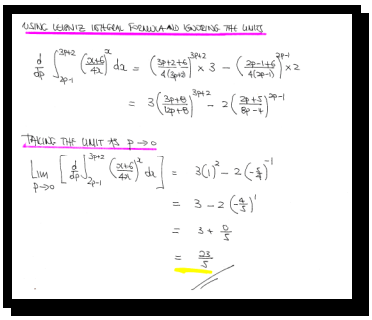


Question 2

Find the value of

$$\lim_{p\to 0}\left[\frac{d}{dp}\left[\int_{2p-1}^{3p+2}\left(\frac{x+6}{4x}\right)^{x}\,dx\right]\right].$$

$\square$, $\boxed{\frac{23}{5}}$

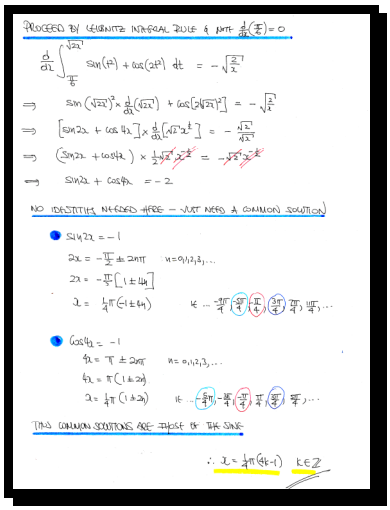


Question 3

Find the general solution of the following equation

$$\frac{d}{dx}\left[\int_{\frac{1}{6}\pi}^{\sqrt{2x}} \sin\left(t^2\right)+\cos\left(2t^2\right)\ dt\right] = -\sqrt{\frac{2}{x}}, \quad x\in\mathbb{R}.$$

$\boxed{}$, $\boxed{x=\frac{1}{4}\pi\left(4k-1\right)\ \ k\in\mathbb{Z}}$



Question 4

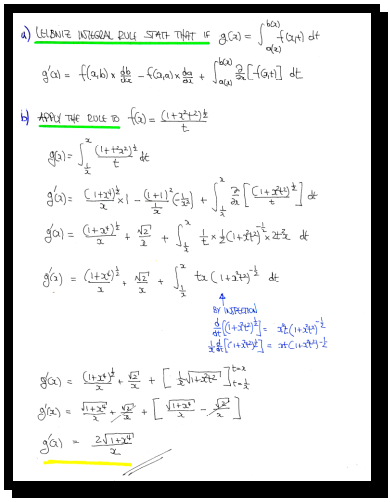
The function g is defined as

$$g(x)=\int_{a(x)}^{b(x)} f(x,t)\ dt\,.$$

a) State Leibniz integral theorem for $g'(x)$.

b) Find a simplified expression for $\dfrac{d}{dx}\left[\displaystyle\int_{x^{-1}}^{x}\frac{\sqrt{1+x^2t^2}}{t}\ dt\right]$.

$\square$, $\dfrac{d}{dx}\left[\displaystyle\int_{x^{-1}}^{x}\frac{\sqrt{1+x^2t^2}}{t}\ dt\right]=\frac{2\sqrt{1+x^4}}{x}$



INTEGRATION APPLICATIONS INTRODUCTION

Question 1

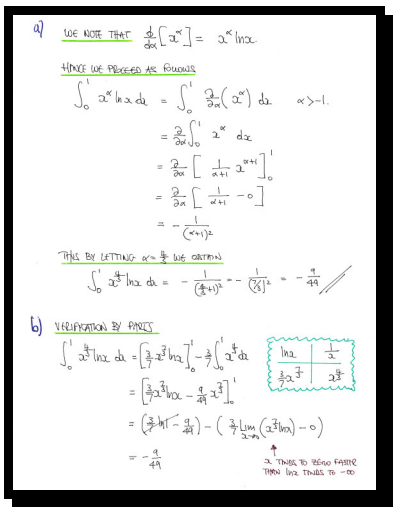
It is given that the following integral converges.

$$\int_0^1 x^{\frac{4}{3}} \ln x \ dx .$$

a) Evaluate the above integral by introducing a parameter and carrying out a suitable differentiation under the integral sign.

b) Verify the answer obtained in part **(a)** by evaluating the integral by standard integration by parts.

V, ☐, $\boxed{-\frac{9}{49}}$



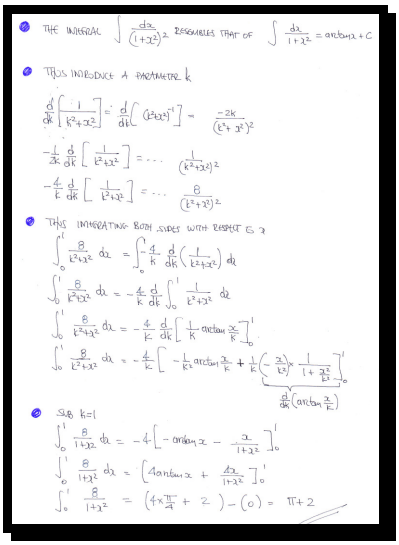
Question 2

$$\int_0^1 \frac{8}{\left(1+x^2\right)^2}\, dx\,.$$

Evaluate the above integral by introducing a parameter k and carrying out a suitable differentiation under the integral sign.

You may not use standard integration techniques in this question.

V, $\boxed{\pi+2}$



Question 3

$$\int \frac{4}{\left(1-4x^2\right)^2}\,dx\,.$$

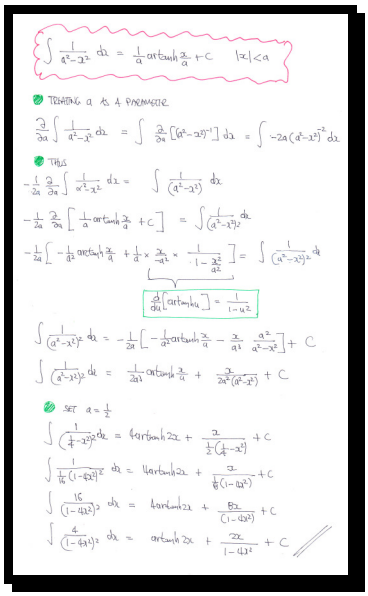
Find a simplified expression for the above integral by introducing a parameter a and carrying out a suitable differentiation under the integral sign.

You may assume

- $\displaystyle\int \frac{1}{a^2-x^2}\,dx=\frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right)+\text{constant}\,,\ |x|<a\,.$
- $\displaystyle\frac{d}{du}\left(\operatorname{artanh} u\right)=\frac{1}{1-u^2}$

You may not use standard integration techniques in this question.

$$\boxed{\operatorname{artanh} 2x+\frac{2x}{1-4x^2}+C}$$



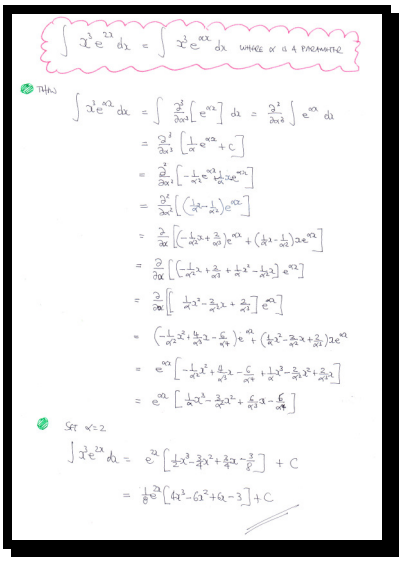
Question 4

$$\int x^3 \mathrm{e}^{2x}\, dx\,.$$

Find a simplified expression for the above integral by introducing a parameter α and carrying out a suitable differentiation under the integral sign.

You may not use integration by parts or a reduction formula in this question.

$$\boxed{\frac{1}{8}\mathrm{e}^{2x}\left[4x^3-6x^2+6x-3\right]+C}$$



Question 5

$$\int \frac{1}{\left(5+4x-x^2\right)^{\frac{3}{2}}}\,dx\,.$$

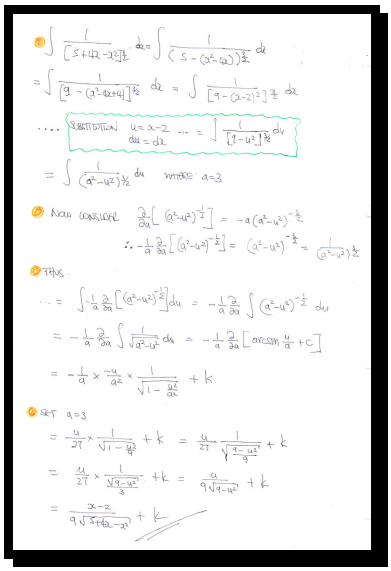
Find a simplified expression for the above integral by introducing a parameter a and carrying out a suitable differentiation under the integral sign.

You may assume

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)+\text{constant}\,,\ |x|\le a\,.$$

You may not use standard integration techniques in this question.

$$\boxed{\frac{x-2}{9\sqrt{5+4x-x^2}}+C}$$



Question 6

It is given that the following integral converges

$$\int_0^{\infty} x^n \mathrm{e}^{-\alpha x} \, dx ,$$

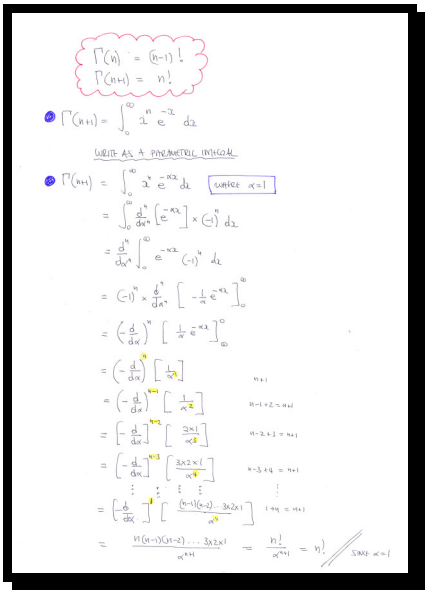
where α is a positive parameter and n is a positive integer.

By carrying out a suitable differentiation under the integral sign, show that

$$\Gamma(n+1) = n! .$$

You may not use integration by parts or a reduction formula in this question.

proof



Question 7

It is given that the following integral converges

$$\int_0^1 x^m [\ln x]^n \ dx ,$$

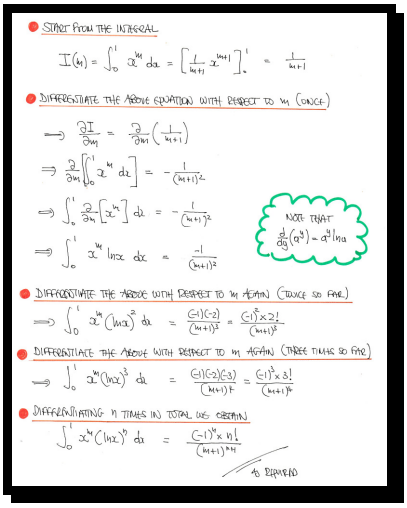
where n is a positive integer and m is a positive constant.

By carrying out a suitable differentiation under the integral sign, show that

$$\int_0^1 x^m [\ln x]^n \ dx = \frac{(-1)^n n!}{(m+1)^{n+1}} \ .$$

You may not use standard integration techniques in this question.

, proof



Question 8

$$I(\alpha)=\int_0^{\pi}\frac{1}{\alpha-\cos x}\,dx\,,\ |\alpha|>1.$$

a) Use an appropriate method to show that

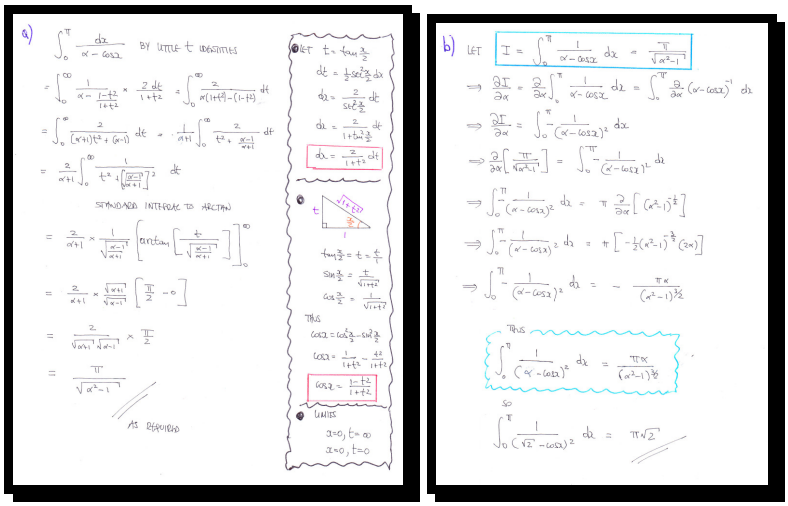
$$I(\alpha)=\frac{\pi}{\sqrt{\alpha^2-1}}.$$

b) By carrying out a suitable differentiation under the integral sign, evaluate

$$\int_0^{\pi}\frac{1}{\left(\sqrt{2}-\cos x\right)^2}\,dx\,.$$

You may not use standard integration techniques in this part of the question.

$\boxed{\pi\sqrt{2}}$



FURTHER INTEGRATION APPLICATIONS

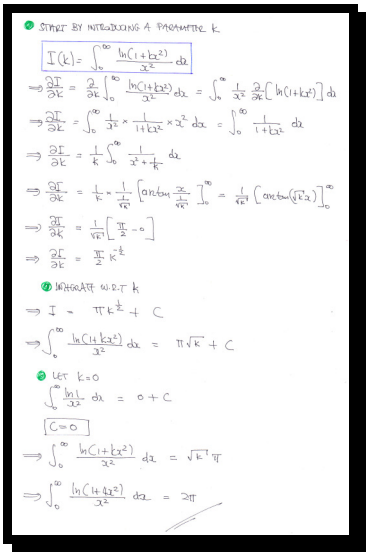
Question 1

$$I = \int_0^{\infty} \frac{\ln\left(1+4x^2\right)}{x^2}\, dx \ .$$

By introducing a parameter in the integrand and carrying a suitable differentiation under the integral sign show that

$$I = 2\pi \, .$$

V, proof



Question 2

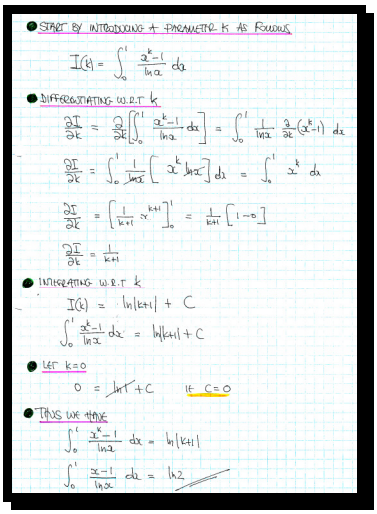
It is given that the following integral converges.

$$I = \int_0^1 \frac{x-1}{\ln x}\, dx .$$

Evaluate I by carrying out a suitable differentiation under the integral sign.

You may not use standard integration techniques in this question.

V, [ ], [ln 2]



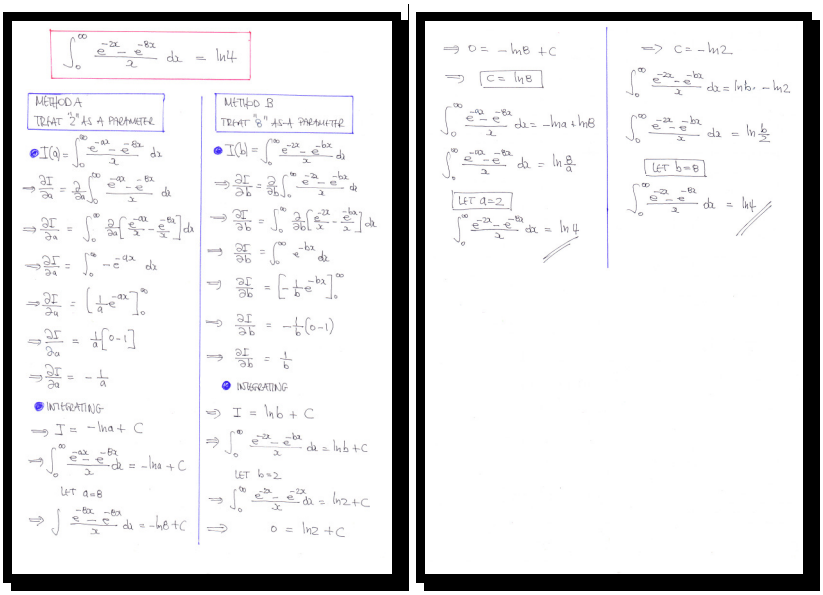
Question 3

$$I = \int_0^{\infty} \frac{\mathrm{e}^{-2x} - \mathrm{e}^{-8x}}{x} \, dx \, .$$

By introducing a parameter in the integrand and carrying a suitable differentiation under the integral sign show that

$$I = \ln 4 \, .$$

V, proof



Question 4

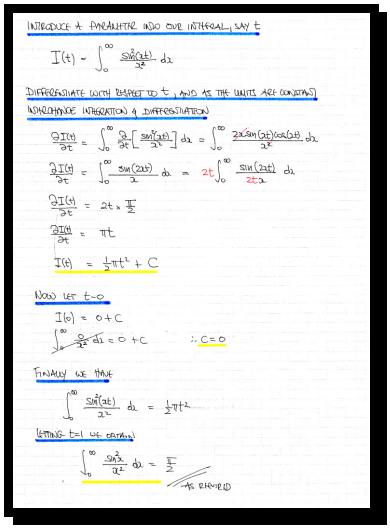
It is given that

$$\int_0^{\infty} \frac{\sin(kx)}{kx}\, dx = \frac{\pi}{2}.$$

Use Leibniz's integral rule to show that

$$\int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}.$$

[ ], [proof]



Question 5

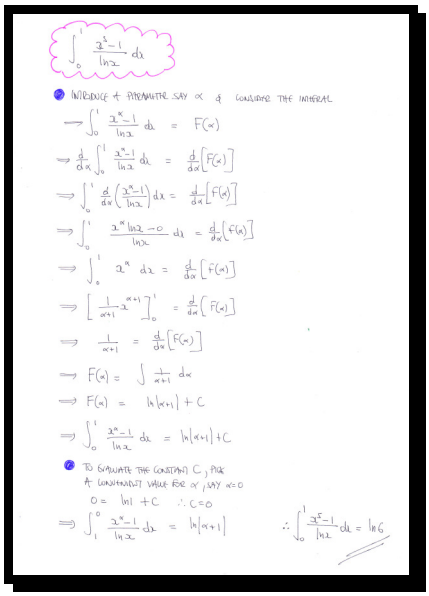
It is given that the following integral converges.

$$\int_0^1 \frac{x^5-1}{\ln x}\,dx\,.$$

Evaluate the above integral by introducing a parameter and carrying out a suitable differentiation under the integral sign.

You may not use standard integration techniques in this question.

$\ln 6$



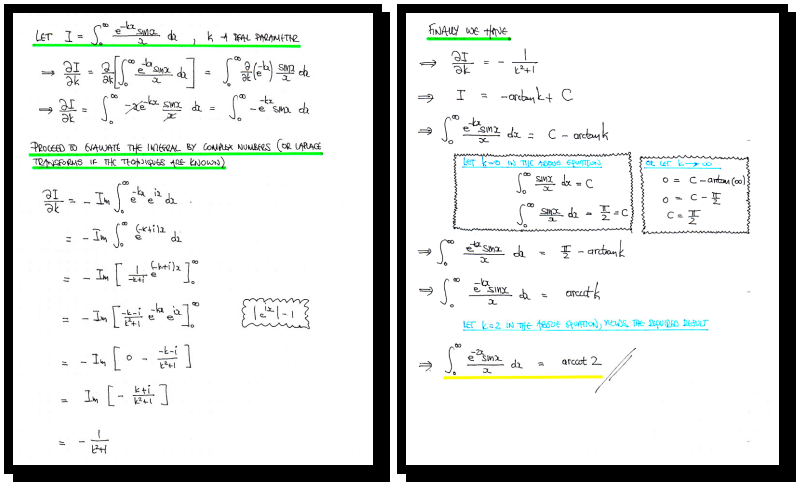
Question 6

$$I = \int_0^{\infty} \frac{\mathrm{e}^{-2x} \sin x}{x} \, dx \, .$$

By introducing in the integrand a parameter k and carrying a suitable differentiation under the integral sign show that

$$I = \operatorname{arccot} 2 \, .$$

V, [ ], proof



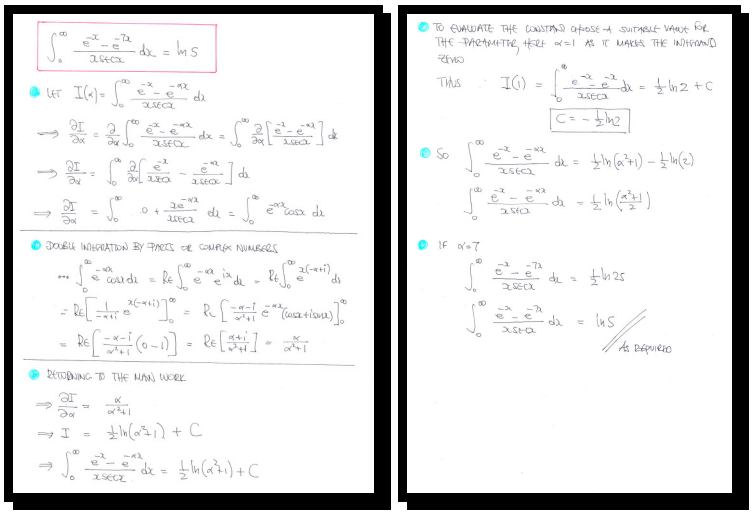
Question 7

$$I=\int_0^{\infty}\frac{\mathrm{e}^{-x}-\mathrm{e}^{-7x}}{x\sec x}\,dx\ .$$

By introducing in the integrand a parameter α and carrying a suitable differentiation under the integral sign show that

$$I=\ln 5\,.$$

V, proof



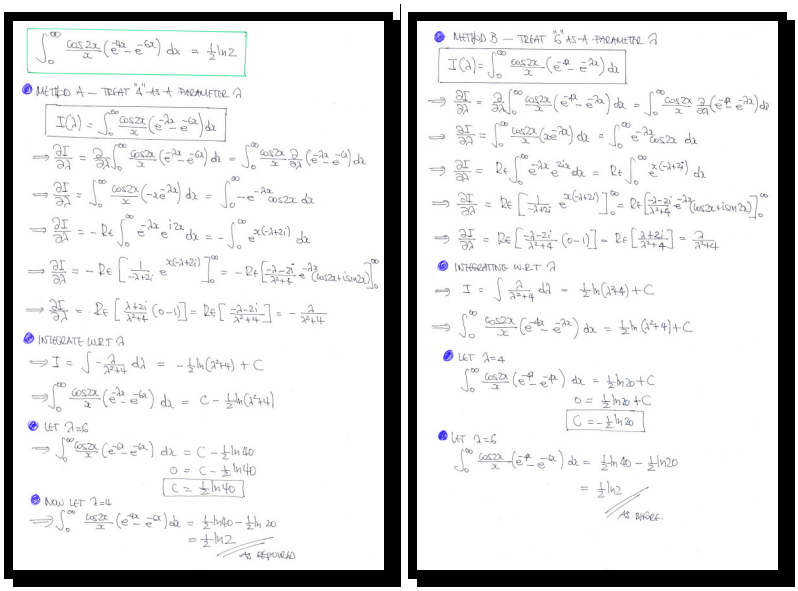
Question 8

$$I=\int_0^{\infty}\frac{\cos x}{x}\left[\mathrm{e}^{-4x}-\mathrm{e}^{-6x}\right]\,dx\ .$$

By introducing in the integrand a parameter λ and carrying a suitable differentiation under the integral sign show that

$$I=\tfrac{1}{2}\ln 2.$$

proof



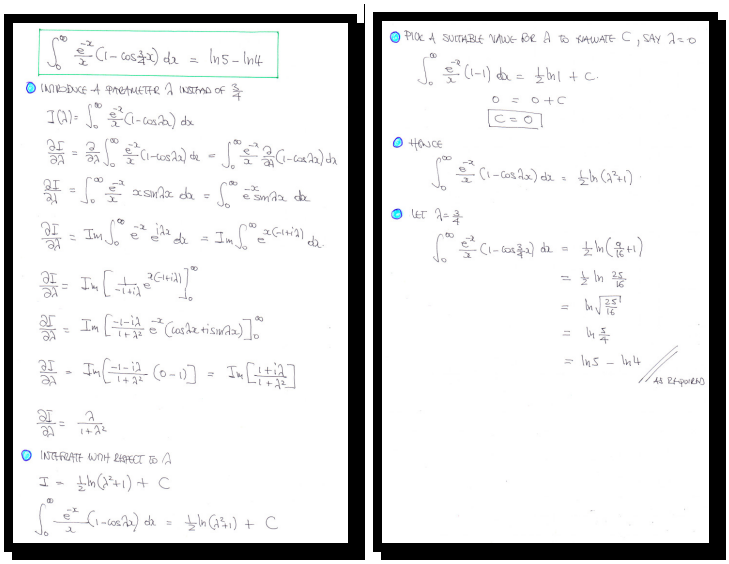
Question 9

$$I=\int_0^{\infty}\frac{\mathrm{e}^{-x}}{x}\left[1-\cos\left(\frac{3}{4}x\right)\right]\,dx\ .$$

By introducing in the integrand a parameter λ and carrying a suitable differentiation under the integral sign show that

$$I\ =\ \ln 5-\ln 4\,.$$

proof



Question 10

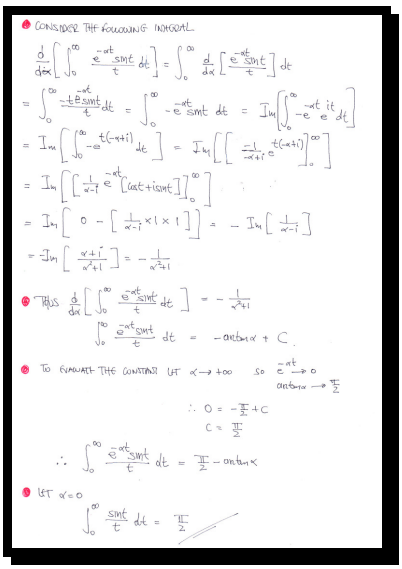
It is given that the following integral converges

$$\int_0^{\infty} \frac{\sin t}{t}\, dt\ .$$

Evaluate the above integral by introducing the term $e^{-\alpha t}$, where α is a positive parameter and carrying out a suitable differentiation under the integral sign.

You may not use contour integration techniques in this question.

V, $\boxed{\frac{\pi}{2}}$



Question 11

Show, by carrying out a suitable differentiation under the integral sign, that

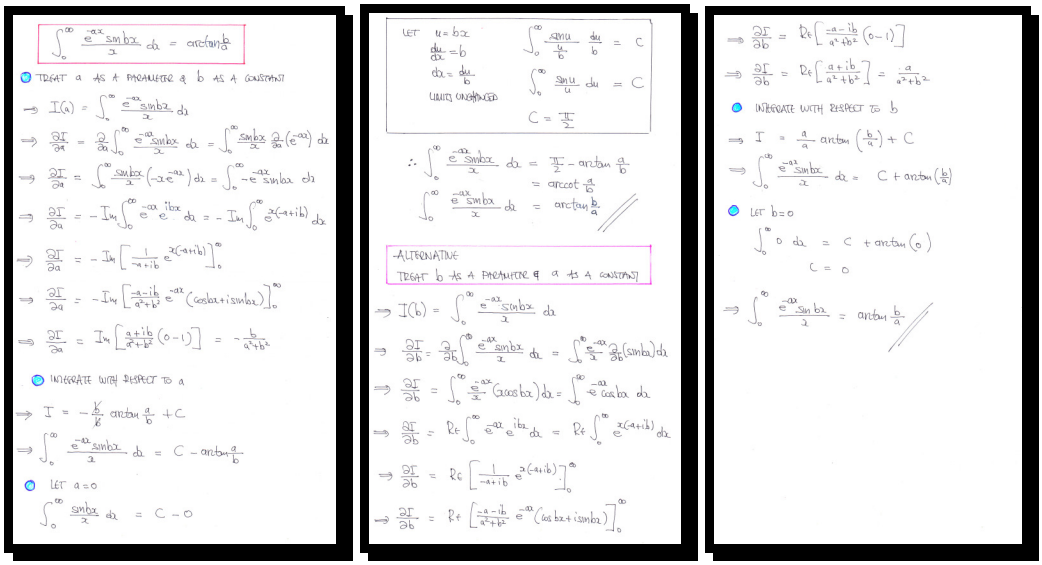
$$\int_0^{\infty} \frac{\mathrm{e}^{-ax}\sin bx}{x}\,dx = \arctan\left(\frac{b}{a}\right),$$

where a and b are positive constants.

You may assume

$$\int_0^{\infty} \frac{\sin t}{t}\,dt = \frac{\pi}{2}.$$

V, proof



Question 12

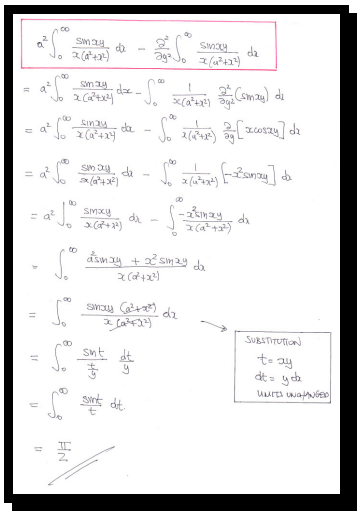
Given that a is a positive constant, find an exact simplified value for

$$a^2\int_0^{\infty}\frac{\sin xy}{x\left(a^2+x^2\right)}\,dx\;-\;\frac{\partial^2}{\partial y^2}\left[\int_0^{\infty}\frac{\sin xy}{x\left(a^2+x^2\right)}\,dx\right].$$

You may assume

$$\int_0^{\infty}\frac{\sin t}{t}\,dt=\frac{\pi}{2}.$$

$$\boxed{\frac{\pi}{2}}$$



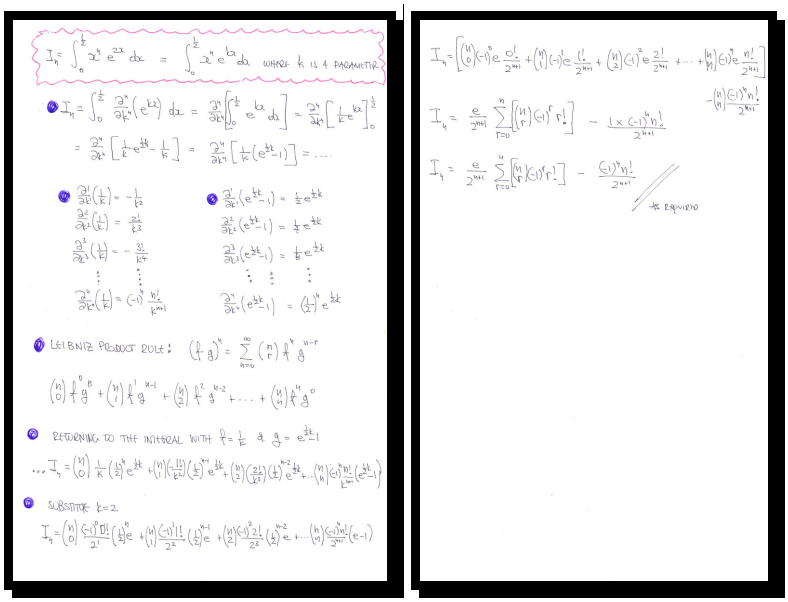
Question 13

$$I_n = \int_0^{\frac{1}{2}} x^n \, \mathrm{e}^{2x} \, dx \, , \quad n = 0, 1, 2, 3, \ldots$$

By introducing in the integrand a parameter k and carrying a suitable differentiation under the integral sign show that

$$I_n = \frac{\mathrm{e}}{2^{n+1}} \sum_{r=0}^{n} \left[\binom{n}{r} (-1)^n \, r! \right] - \frac{(-1)^n \, n!}{2^{n+1}} \, .$$

proof



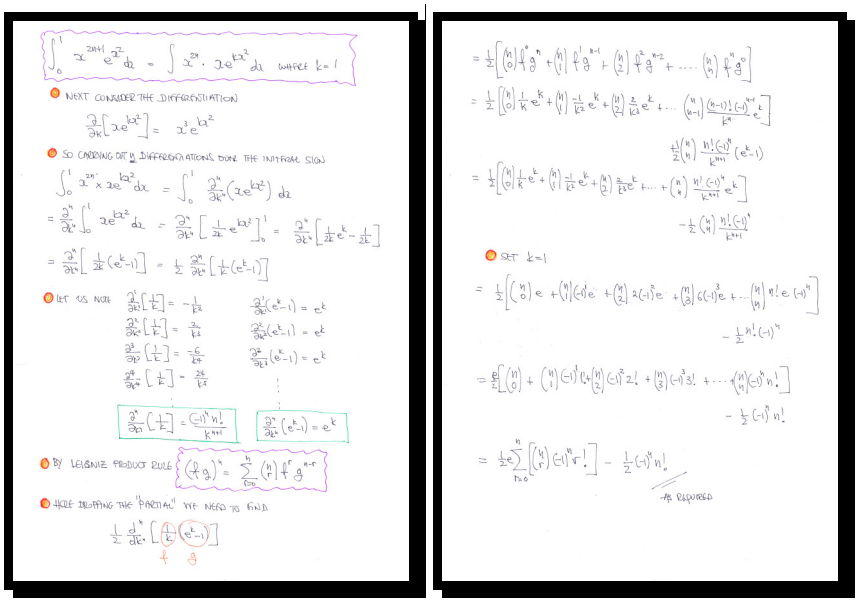
Question 14

$$I_n = \int_0^1 x^{2n+1} \mathrm{e}^{x^2} \, dx \, , \quad n = 0, 1, 2, 3, \ldots$$

By introducing in the integrand a parameter k and carrying a suitable differentiation under the integral sign show that

$$I_n = \frac{\mathrm{e}}{2} \sum_{r=0}^{n} \left[\binom{n}{r} (-1)^n \, r! \right] \; - \frac{1}{2} (-1)^n \, n! \, .$$

proof



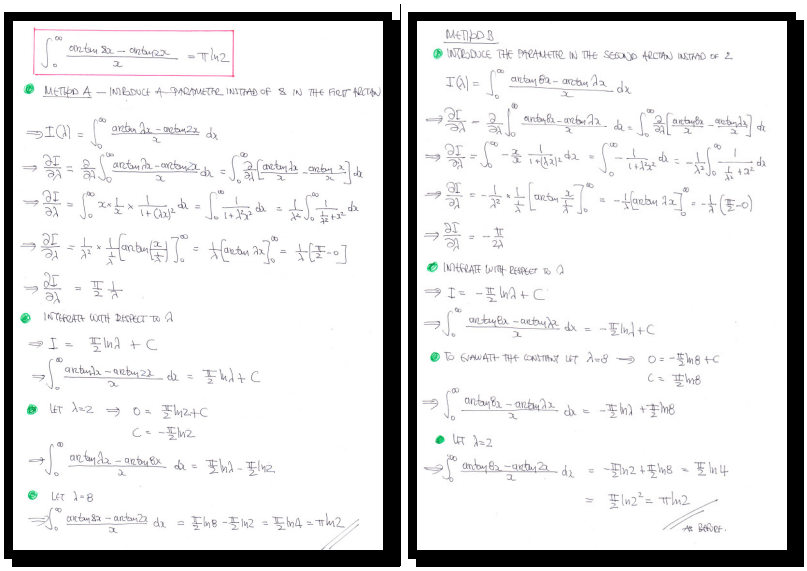
Question 15

$$I=\int_0^{\infty}\frac{\arctan 8x-\arctan 2x}{x}\,dx\ .$$

By introducing a parameter in the integrand and carrying a suitable differentiation under the integral sign show that

$$I=\pi\ln 2\,.$$

V, proof



Question 16

It is given that the following integral converges

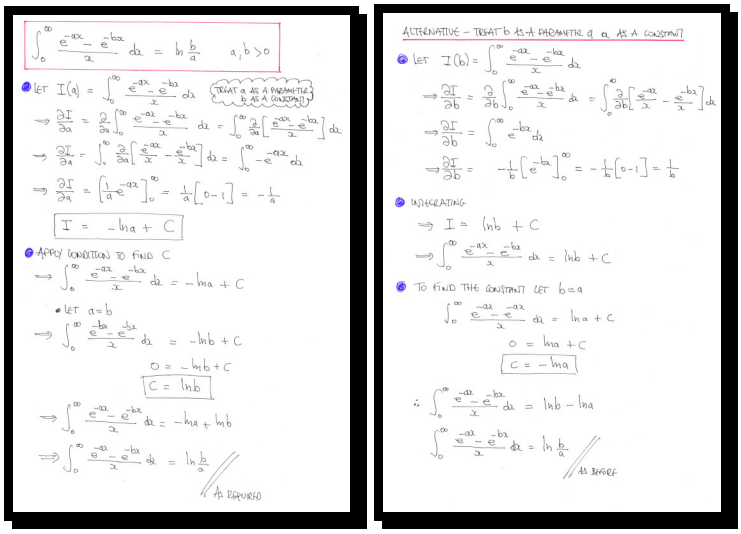
$$\int_0^{\infty} \frac{\mathrm{e}^{-ax}-\mathrm{e}^{-bx}}{x}\,dx\,,$$

where a and b are positive constants.

By carrying out a suitable differentiation under the integral sign, show that

$$\int_0^{\infty} \frac{\mathrm{e}^{-ax}-\mathrm{e}^{-bx}}{x}\,dx \;=\; \ln\left[\frac{b}{a}\right].$$

V, proof



Question 17

It is given that the following integral converges

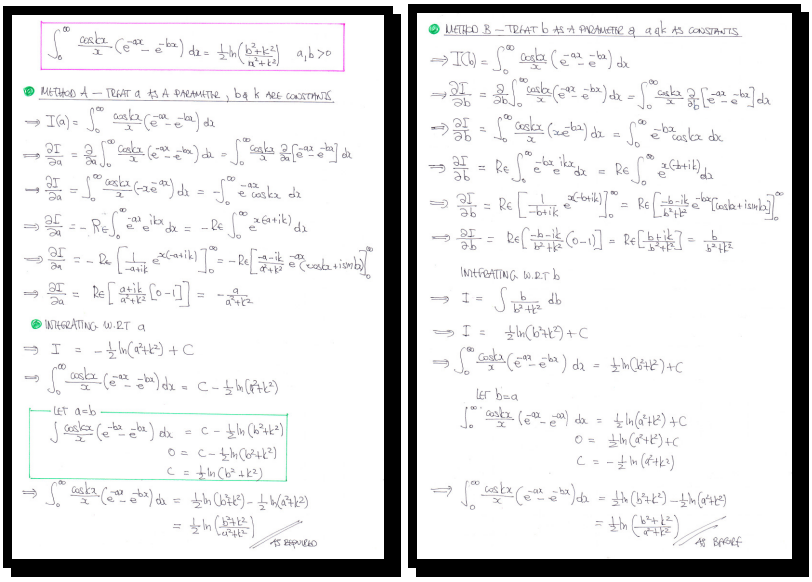
$$\int_0^{\infty} \frac{\cos kx}{x}\left[\mathrm{e}^{-ax}-\mathrm{e}^{-bx}\right]\,dx\,,$$

where k, a and b are constants with $a>0$ and $b>0$.

By carrying out a suitable differentiation under the integral sign, show that

$$\int_0^{\infty} \frac{\cos kx}{x}\left[\mathrm{e}^{-ax}-\mathrm{e}^{-bx}\right]\,dx \;=\; \frac{1}{2}\ln\left[\frac{b^2+k^2}{a^2+k^2}\right].$$

proof



Question 18

It is given that the following integral converges

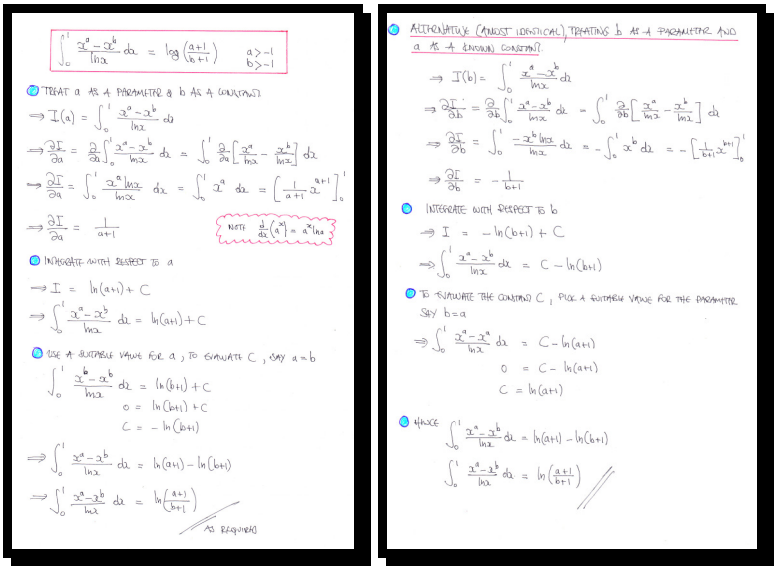
$$\int_0^1 \frac{x^a - x^b}{\ln x}\, dx \ ,$$

where a and b are constants greater than -1.

By carrying out a suitable differentiation under the integral sign, show that

$$\int_0^1 \frac{x^a - x^b}{\ln x}\, dx \ = \ \ln\left[\frac{a+1}{b+1}\right] .$$

proof



Question 19

It is given that the following integral converges

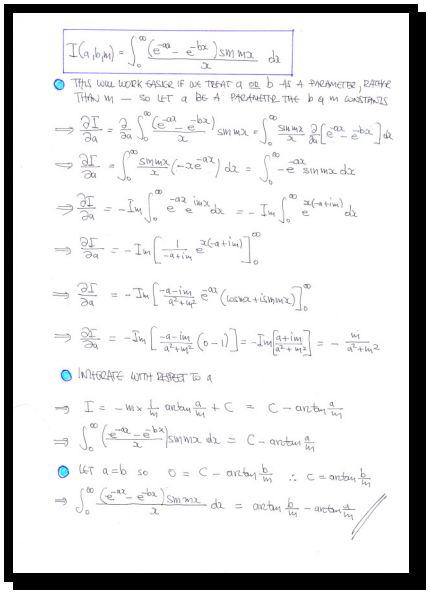
$$\int_0^{\infty} \frac{\sin mx}{x}\left[\mathrm{e}^{-ax}-\mathrm{e}^{-bx}\right]\,dx\ ,$$

where a, b and m are constants, with $m\neq 0$, $a>0$, $b>0$.

By carrying out a suitable differentiation under the integral sign, show that

$$\int_0^{\infty} \frac{\sin mx}{x}\left[\mathrm{e}^{-ax}-\mathrm{e}^{-bx}\right]\,dx=\arctan\left(\frac{b}{m}\right)-\arctan\left(\frac{a}{m}\right)\ .$$

proof



Question 20

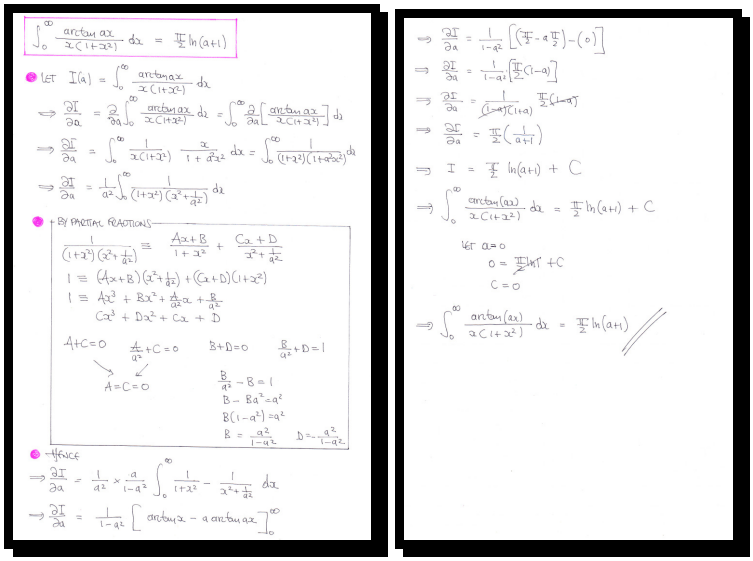
It is given that the following integral converges

$$\int_0^{\infty} \frac{\arctan ax}{x\left(1+x^2\right)}\, dx\,,\ a>-1\,.$$

By carrying out a suitable differentiation under the integral sign, show that

$$\int_0^{\infty} \frac{\arctan ax}{x\left(1+x^2\right)}\, dx=\frac{\pi}{2}\ln\left(a+1\right)\,.$$

proof



Question 21

It is given that the following integral converges

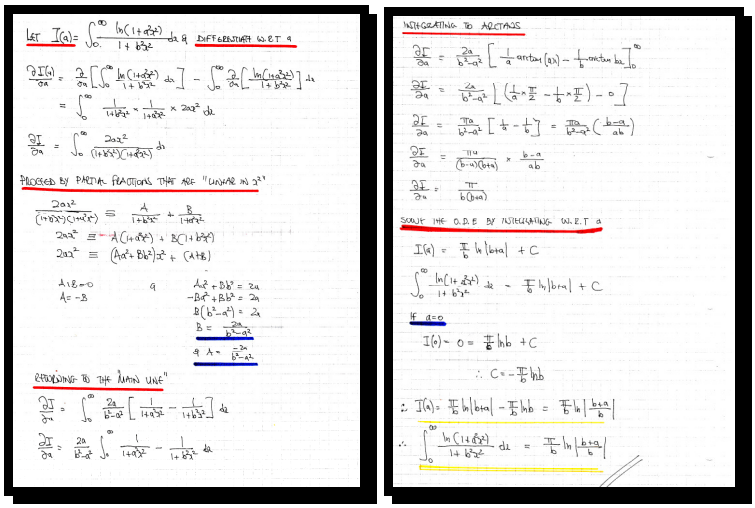
$$\int_0^{\infty} \frac{\ln\left(1+a^2x^2\right)}{1+b^2x^2}\,dx\,,$$

where a and b are constants.

By carrying out a suitable differentiation under the integral sign, show that the exact value of the above integral is

$$\frac{\pi}{b}\ln\left|\frac{a+b}{b}\right|\,.$$

[ ], [proof]



Question 22

It is given that the following integral converges

$$\int_0^{\infty} \frac{\mathrm{e}^{-x}-\mathrm{e}^{-2x}}{x}\,dx\,.$$

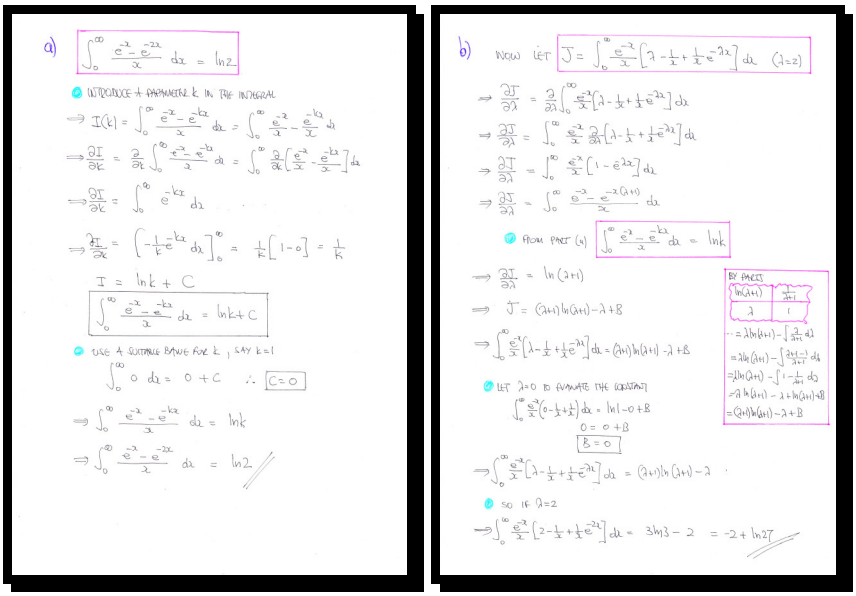
a) By introducing a parameter k and carrying out a suitable differentiation under the integral sign, show that

$$\int_0^{\infty} \frac{\mathrm{e}^{-x}-\mathrm{e}^{-2x}}{x}\,dx=\ln 2\,.$$

b) Use the result of part **(a)** and differentiation under the integral sign to show further that

$$\int_0^{\infty} \frac{\mathrm{e}^{-x}}{x}\left[2-\frac{1}{x}+\frac{1}{x}\mathrm{e}^{-2x}\right]\,dx=-2+\ln 27\,.$$

proof



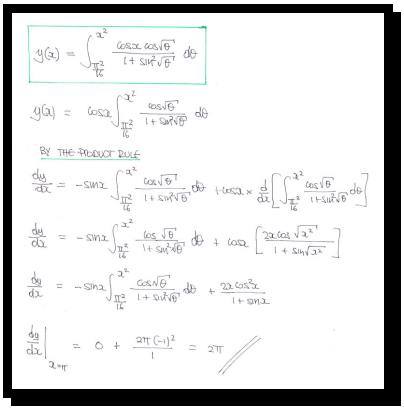
Question 23

The integral function $y = y(x)$ is defined as

$$y(x) = \int_{\frac{1}{16}\pi^2}^{x^2} \frac{\cos x \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\, d\theta .$$

Evaluate $y'(\pi)$.

$\boxed{2\pi}$



$$y(x) = \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos x \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\, d\theta$$

$$y(x) = \cos x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\, d\theta$$

BY THE PRODUCT RULE

$$\frac{dy}{dx} = -\sin x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\, d\theta + \cos x \times \frac{d}{dx}\left[\int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\, d\theta\right]$$

$$\frac{dy}{dx} = -\sin x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\, d\theta + \cos x \left[\frac{2x\cos\sqrt{x^2}}{1+\sin\sqrt{x^2}}\right]$$

$$\frac{dy}{dx} = -\sin x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\, d\theta + \frac{2x\cos^2 x}{1+\sin x}$$

$$\left.\frac{dy}{dx}\right|_{x=\pi} = 0 + \frac{2\pi(-1)^2}{1} = 2\pi$$

Question 24

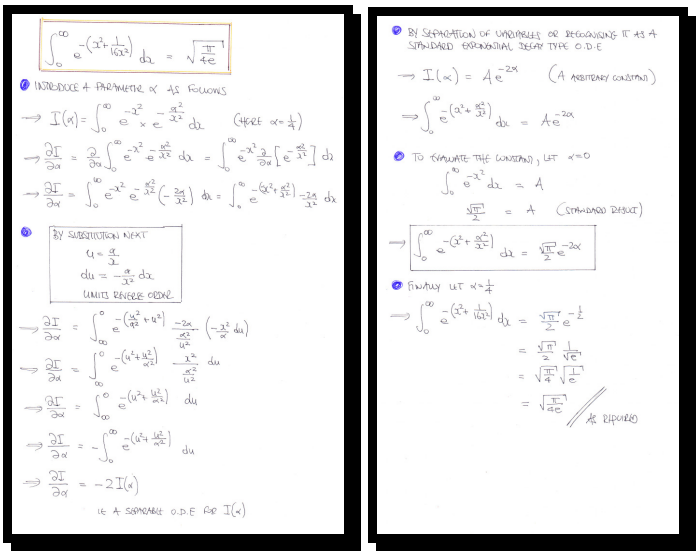
An integral I is defined in terms of a parameter α as

$$I(\alpha) = \int_0^{\infty} \exp\left[-x^2 - \frac{\alpha^2}{x^2}\right] dx\,.$$

By carrying out a suitable differentiation on I under the integral sign, show that

$$\int_0^{\infty} \exp\left[-x^2 - \frac{1}{16x^2}\right] dx = \sqrt{\frac{\pi}{4\mathrm{e}}}\,.$$

proof



Question 25

An integral I with variable limits is defined as

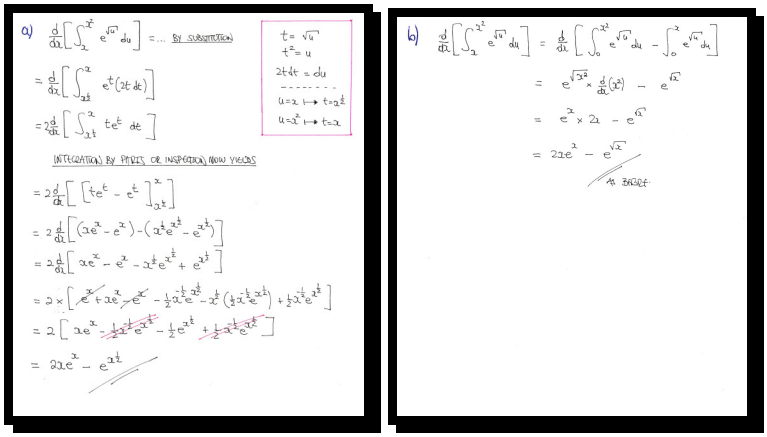
$$I(x) = \int_{x}^{x^2} \mathrm{e}^{\sqrt{u}}\, du\,.$$

a) Use a suitable substitution followed by integration by parts to find a simplified expression for

$$\frac{d}{dx}\Big[I(x)\Big].$$

b) Verify the answer obtained in part **(a)** by carrying the differentiation over the integral sign.

$\square$, $\boxed{\dfrac{d}{dx}\Big[I(x)\Big] = 2x\mathrm{e}^{x} - \mathrm{e}^{\sqrt{x}}}$



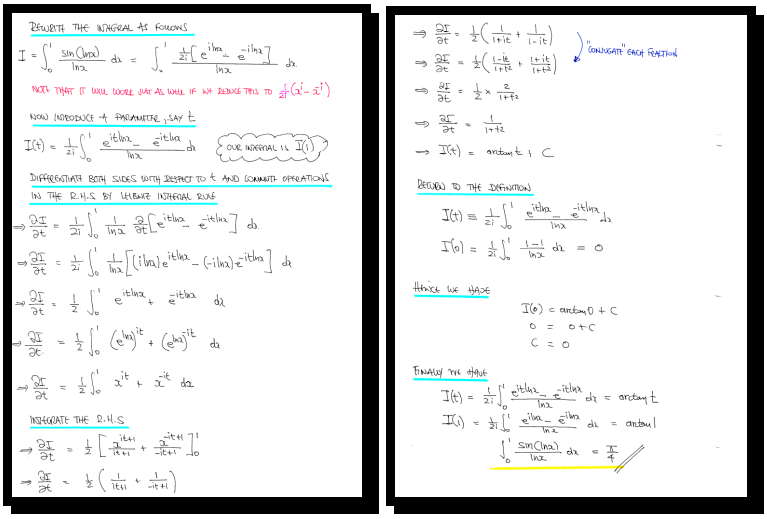
Question 26

Use complex variables and the Leibniz integral rule to evaluate

$$\int_0^1 \frac{\sin(\ln x)}{\ln x}\, dx\,.$$

You may assume that the integral converges.

$\square$, $\boxed{\frac{1}{4}\pi}$



Question 27

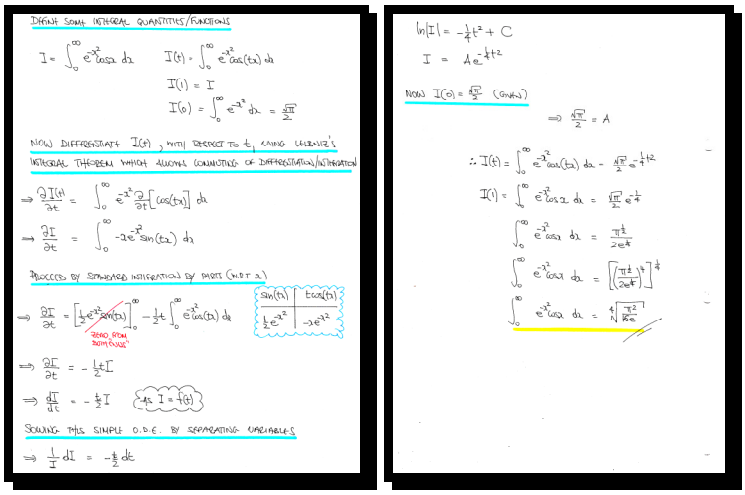
$$I=\int_0^{\infty} \mathrm{e}^{-x^2}\cos x \ \ dx$$

Assuming that the above integral converges, use the Leibniz integral rule to evaluate it.

Give the answer in the form $\sqrt[4]{k}$, where k is an exact constant.

You may use without proof $\displaystyle\int_0^{\infty} \mathrm{e}^{-x^2} \ \ dx=\frac{1}{2}\sqrt{\pi}$.

$\square$, $\boxed{I=\sqrt[4]{\dfrac{\pi^2}{16\mathrm{e}}}}$



Question 28

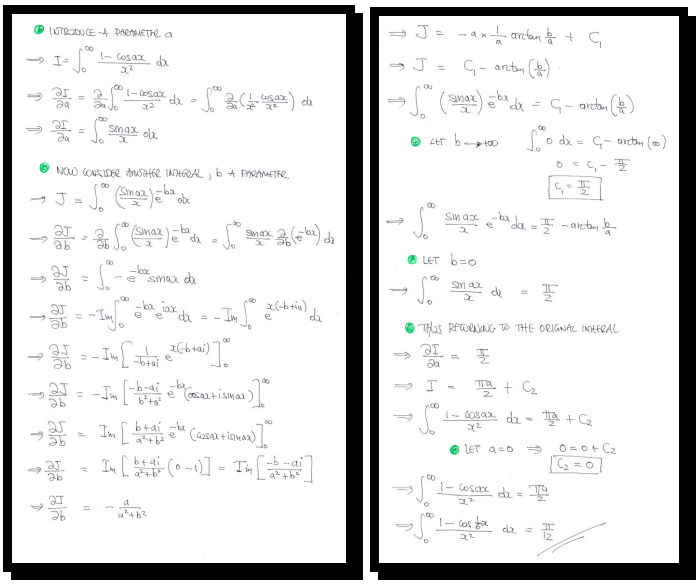
It is given that the following integral converges

$$\int_0^{\infty} \frac{1-\cos\left(\frac{1}{6}x\right)}{x^2}\, dx\,.$$

By introducing a parameter in the integrand and carrying out a suitable differentiation under the integral sign, show that

$$\int_0^{\infty} \frac{1-\cos\left(\frac{1}{6}x\right)}{x^2}\, dx \;=\; \frac{\pi}{12}\,.$$

proof



Question 29

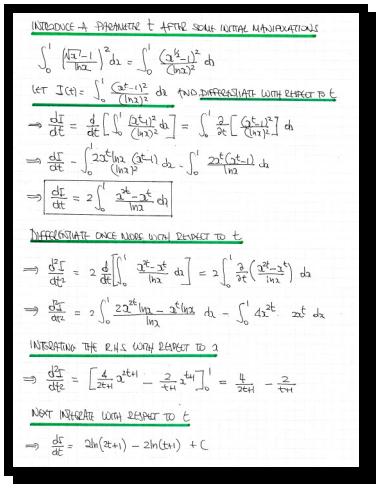
It is given that the following integral converges

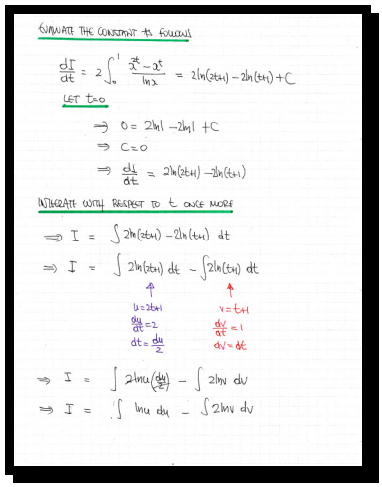
$$I = \int_0^1 \left[\frac{\sqrt{x} - 1}{\ln x} \right]^2 dx \ .$$

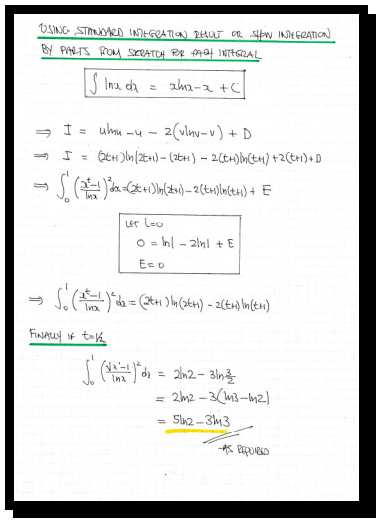
By carrying out a suitable differentiation under the integral sign, show that

$$I = 5\ln 3 - 3\ln 3 \ .$$

V , [ ] , proof







Question 30

It is given that the following integral converges

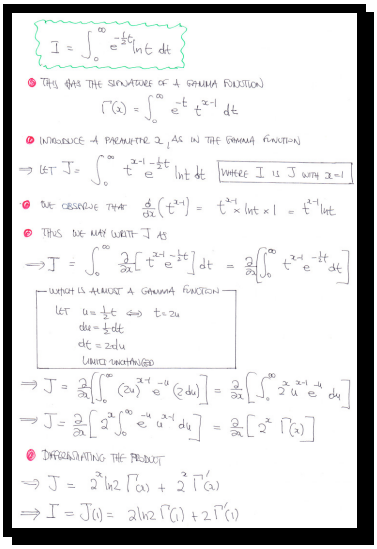
$$\int_0^{\infty} \mathrm{e}^{-\frac{1}{2}t} \ln t \ \ dt .$$

Evaluate the above integral by introducing a new parametric term in the integrand and carrying out a suitable differentiation under the integral sign.

You may assume that

$$\Gamma'(x)=\Gamma(x)\left[-\gamma+\frac{1}{x}+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{x+k}\right)\right].$$

$$\boxed{2(-\gamma+\ln 2)}$$



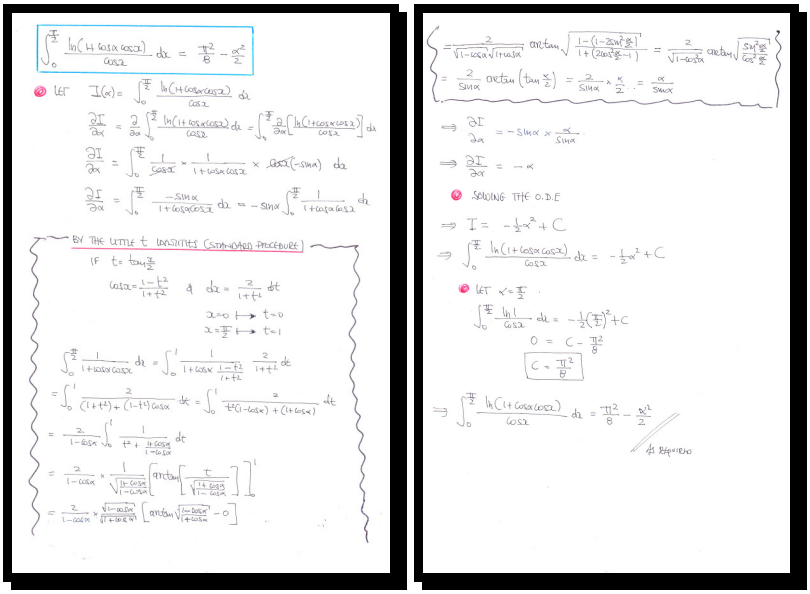
Question 31

$$I=\int_0^{\frac{\pi}{2}} \frac{\ln\left(1+\cos\alpha\cos x\right)}{\cos x}\,dx\,.$$

By carrying out a suitable differentiation on I under the integral sign, show that

$$I=\frac{1}{8}\pi^2-\frac{1}{2}\alpha^2\,.$$

proof



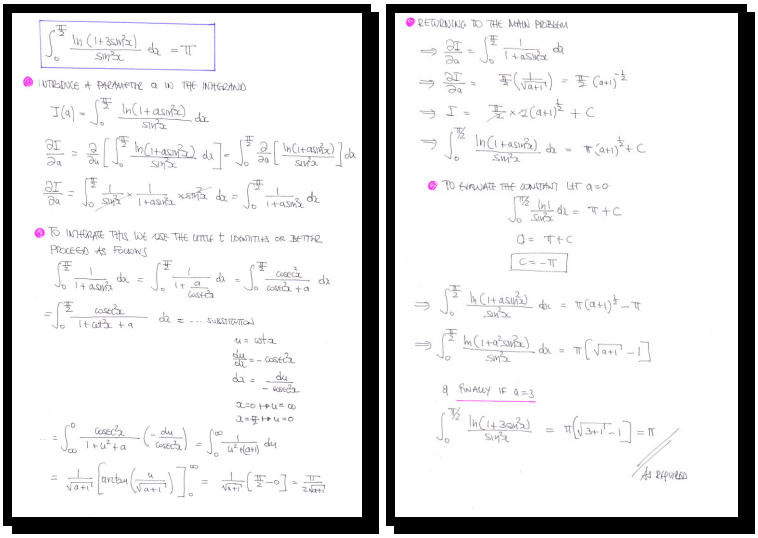
Question 32

$$I=\int_0^{\frac{\pi}{2}} \frac{\ln\left(1+3\sin^2 x\right)}{\sin^2 x}\,dx\,.$$

By introducing a parameter a in the integrand and carrying out differentiation on I under the integral sign, show that

$$I=\pi\,.$$

proof



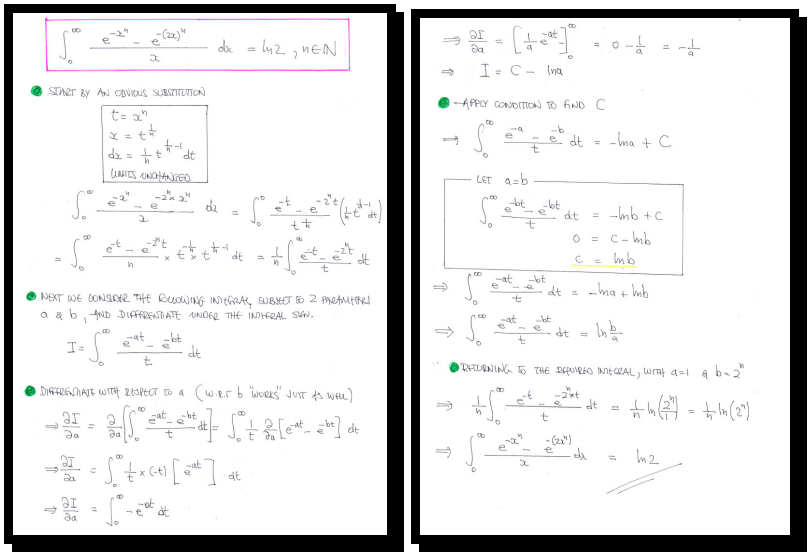
Question 33

$$I=\int_0^{\infty}\frac{\mathrm{e}^{-x^n}-\mathrm{e}^{-(2x)^n}}{x}\,dx\,,\ n\in\mathbb{N}\,.$$

By carrying out a suitable differentiation on I under the integral sign, show that for all $n\in\mathbb{N}$,

$$I=\ln 2\,.$$

proof



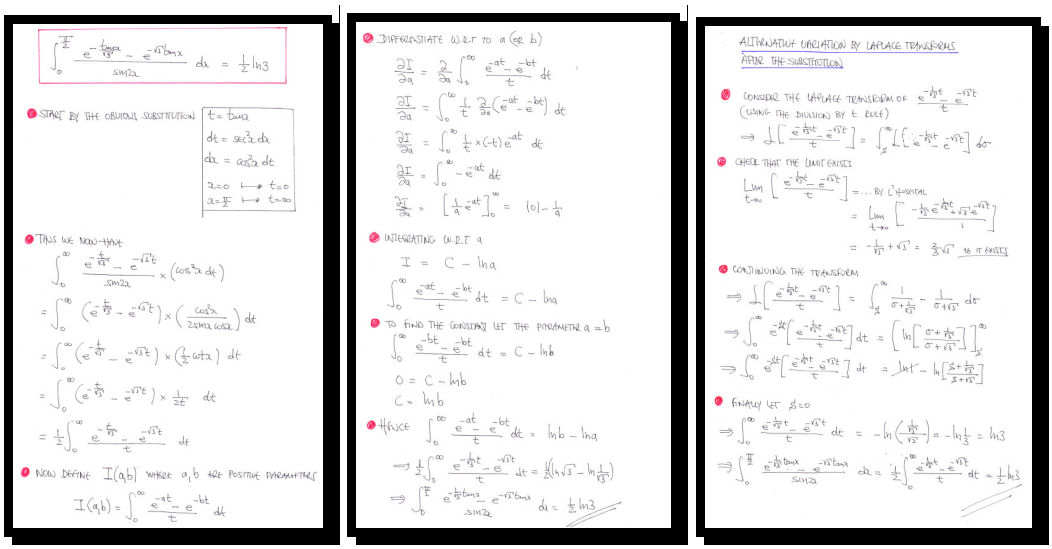
Question 34

$$I=\int_0^{\frac{1}{2}\pi}\frac{\exp\left(-\frac{1}{\sqrt{3}}\tan x\right)-\exp\left(-\sqrt{3}\tan x\right)}{\sin 2x}\,dx\,.$$

By carrying out a suitable differentiation on I under the integral sign, show that

$$I=\frac{1}{2}\ln 3\,.$$

proof



Question 35

$$J=\int_0^{\frac{\pi}{2}}\frac{1}{1+k^2\tan^2 x}\,dx,\quad |k|\neq 1.$$

a) Use appropriate methods to find, in terms of k, a simplified expression for J.

$$I(k)=\int_0^{\frac{\pi}{2}}\frac{\arctan(k\tan x)}{\tan x}\,dx,\quad |k|\neq 1.$$

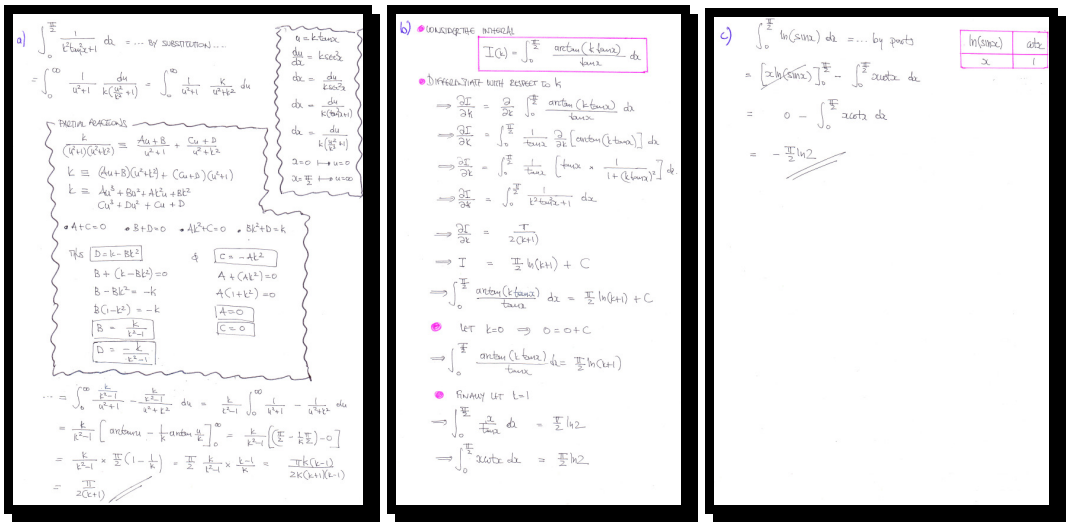
b) By carrying out a suitable differentiation on I under the integral sign, show that

$$\int_0^{\frac{\pi}{2}} x\cot x\,dx=\frac{1}{2}\pi\ln 2.$$

c) Deduce the value of

$$\int_0^{\frac{\pi}{2}}\ln(\sin x)\,dx.$$

$$\boxed{J=\frac{\pi}{2(k+1)}},\quad \boxed{-\frac{1}{2}\pi\ln 2}$$



Question 36

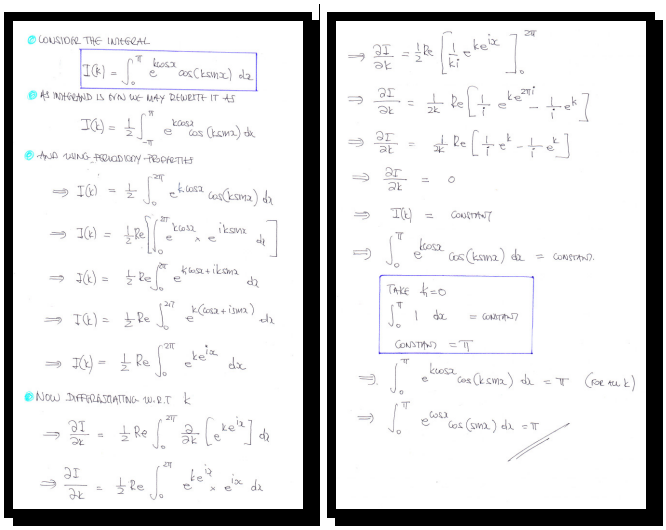
The integral function $I(k)$ is defined as

$$I(k)=\int_0^{\pi} \mathrm{e}^{k\cos x}\cos(k\sin x)\ dx, \quad k\in\mathbb{R}.$$

By carrying out a suitable differentiation on I under the integral sign, show that

$$\int_0^{\pi} \mathrm{e}^{\cos x}\cos(\sin x)\ dx \ = \ \pi .$$

proof



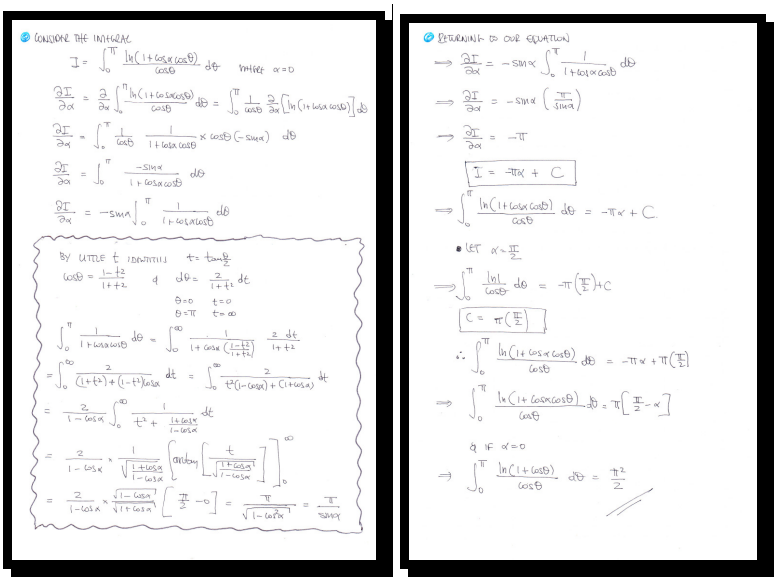
Question 37

$$I=\int_0^{\pi}\frac{\ln\left(1+\cos\alpha\cos\theta\right)}{\cos\theta}\,d\theta\,.$$

By carrying out a suitable differentiation on I under the integral sign, show that

$$\int_0^{\pi}\frac{\ln\left(1+\cos\theta\right)}{\cos\theta}\,d\theta=\frac{\pi^2}{2}\,.$$

proof



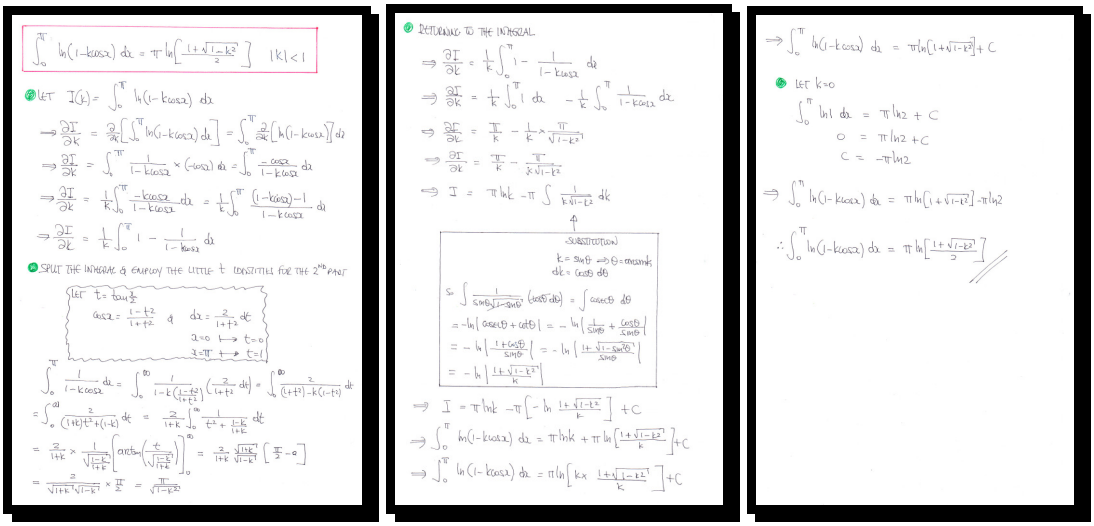
Question 38

$$I(k) \equiv \int_0^{\pi} \ln(1-k\cos x)\ dx\,, \quad |k|<1\,.$$

By differentiating both sides of the above equation with respect to k, show that

$$I(k) = \pi \ln\left[\frac{1}{2}\left(1-\sqrt{1-k^2}\right)\right].$$

proof

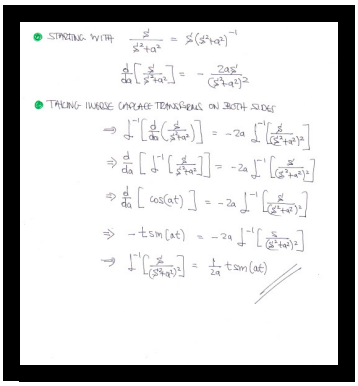


Question 39

Find the following inverse Laplace transform, by using differentiation under the integral sign.

$$\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right],\ a>0\,.$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=\frac{t\sin at}{2a}}$$



Question 40

The integral I is defined in terms of the constants α and k, by

$$I(\alpha,k) \equiv \int_0^{\infty} \mathrm{e}^{-\alpha x^2} \cos(kx)\ dx, \quad \alpha > 0 .$$

By differentiating both sides of the above equation with respect to k, followed by integration by parts, show that

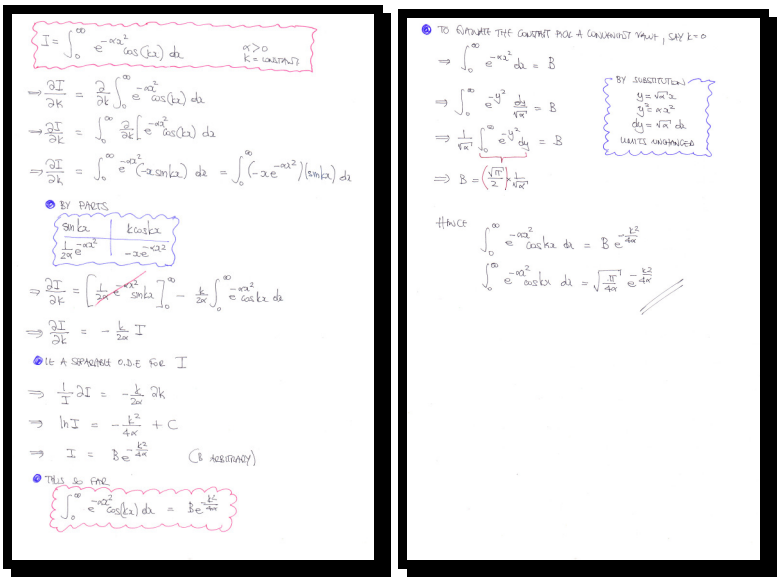
$$\int_0^{\infty} \mathrm{e}^{-\alpha x^2} \cos(kx)\ dx = \sqrt{\frac{\pi}{4\alpha}} \exp\left(-\frac{k^2}{4\alpha}\right).$$

You may assume without proof that

$$\int_0^{\infty} \mathrm{e}^{-x^2}\ dx = \frac{1}{2}\sqrt{\pi} .$$

You may not use contour integration techniques in this question.

proof



Question 41

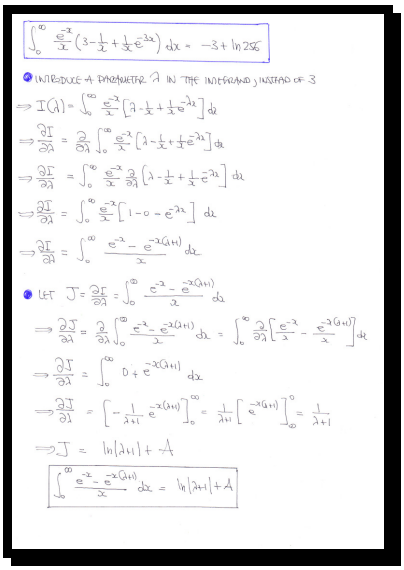
It is given that the following integral converges

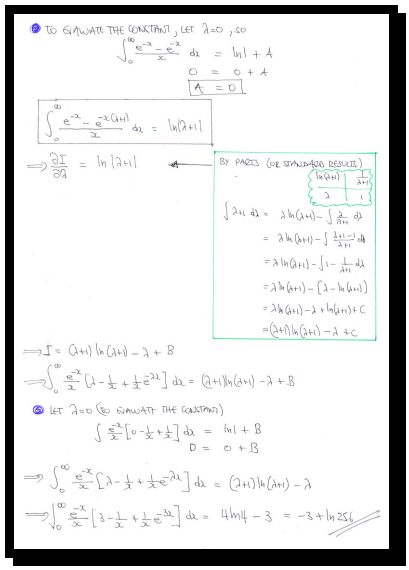
$$\int_0^{\infty} \frac{\mathrm{e}^{-x}}{x}\left[3-\frac{1}{x}+\frac{1}{x}\mathrm{e}^{-3x}\right]\,dx\,.$$

By introducing a parameter λ and carrying out a suitable differentiation under the integral sign, show that

$$\int_0^{\infty} \frac{\mathrm{e}^{-x}}{x}\left[3-\frac{1}{x}+\frac{1}{x}\mathrm{e}^{-3x}\right]\,dx=-3+\ln 256\ .$$

proof





Question 42

The following integral is to be evaluated

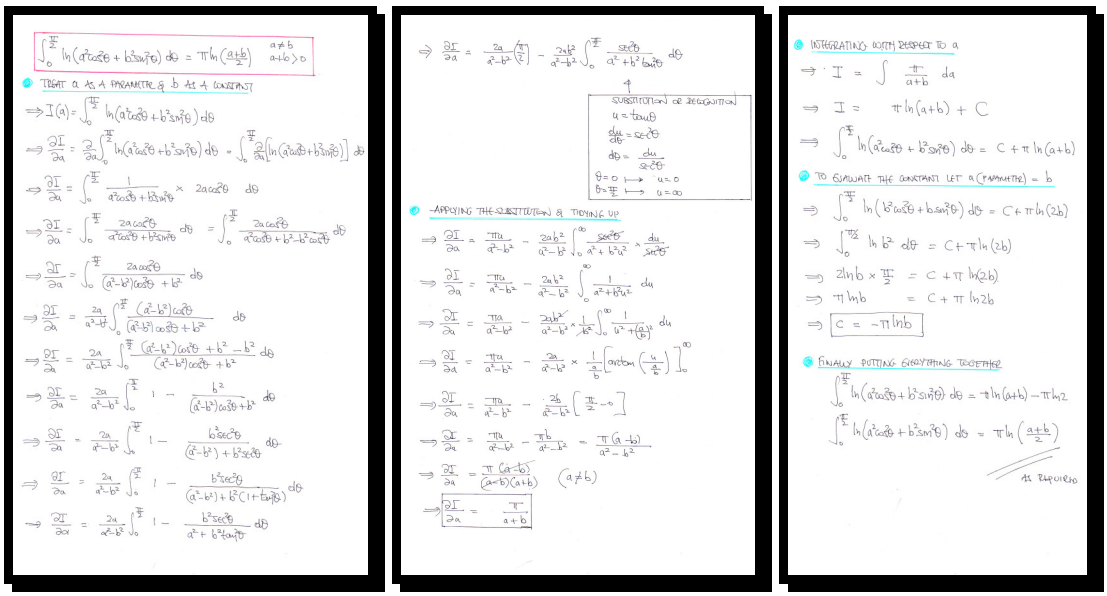
$$\int_0^{\frac{\pi}{2}} \ln\left[a^2\cos^2\theta+b^2\sin^2\theta\right]\,d\theta\,,$$

where a and b are distinct constants such that $a+b>0$.

By carrying out a suitable differentiation under the integral sign, show that

$$\int_0^{\frac{\pi}{2}} \ln\left[a^2\cos^2\theta+b^2\sin^2\theta\right]\,d\theta \;=\; \pi\ln\left[\frac{a+b}{2}\right].$$

proof



Question 43

It is given that

$$y = \arcsin\left[\frac{\alpha + \cos x}{1 + \alpha \cos x}\right],$$

where α is a constant.

a) Show that

$$\frac{dy}{dx} = -\frac{\sqrt{1-\alpha^2}}{1+\alpha\cos x}.$$

The integral function $I(\alpha, x)$ is defined as

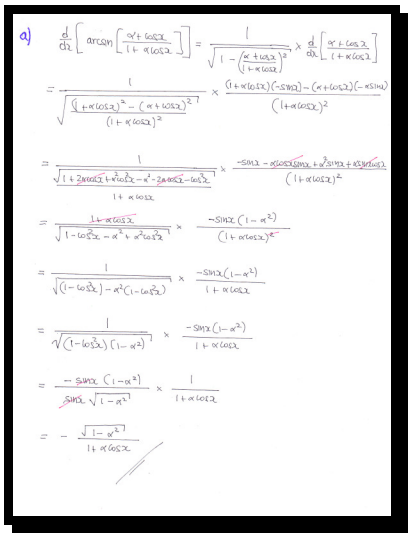
$$I(\alpha) = \int_0^{\pi} \ln(1+\alpha\cos x)\ dx.$$

b) By differentiating both sides of the above relationship with respect to α, show further that

$$I(1) = -\pi \ln 2.$$

proof

[solution overleaf]



a) $\frac{d}{dx}\left[\arcsin\left(\frac{\alpha+\cos x}{1+\alpha\cos x}\right)\right] = \frac{1}{\sqrt{1-\left(\frac{\alpha+\cos x}{1+\alpha\cos x}\right)^2}} \times \frac{d}{dx}\left[\frac{\alpha+\cos x}{1+\alpha\cos x}\right]$

$= \frac{1}{\sqrt{\frac{(1+\alpha\cos x)^2-(\alpha+\cos x)^2}{(1+\alpha\cos x)^2}}} \times \frac{(1+\alpha\cos x)(-\sin x)-(\alpha+\cos x)(-\alpha\sin x)}{(1+\alpha\cos x)^2}$

$= \frac{1}{\sqrt{1+2\alpha\cos x+\alpha^2\cos^2 x-\alpha^2-2\alpha\cos x-\cos^2 x}} \times \frac{-\sin x-\alpha\cos x\sin x+\alpha^2\sin x+\alpha\sin x\cos x}{(1+\alpha\cos x)^2}$

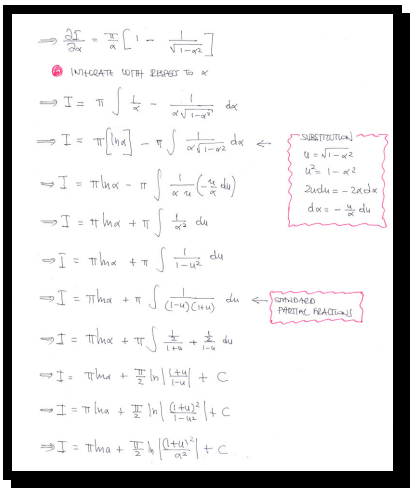
$= \frac{1}{\sqrt{1-\cos^2 x-\alpha^2+\alpha^2\cos^2 x}} \times \frac{-\sin x(1-\alpha^2)}{(1+\alpha\cos x)^2}$

$= \frac{1}{\sqrt{(1-\cos^2 x)-\alpha^2(1-\cos^2 x)}} \times \frac{-\sin x(1-\alpha^2)}{1+\alpha\cos x}$

$= \frac{1}{\sqrt{(1-\cos^2 x)(1-\alpha^2)}} \times \frac{-\sin x(1-\alpha^2)}{1+\alpha\cos x}$

$= \frac{-\sin x\sqrt{1-\alpha^2}}{\sin x\sqrt{1-\alpha^2}} \times \frac{1}{1+\alpha\cos x}$

$= -\frac{\sqrt{1-\alpha^2}}{1+\alpha\cos x}$



b) $I = \int_0^{\pi} \ln(1+\cos x)\,dx$

INTRODUCE A PARAMETER α

$\Rightarrow I(\alpha x) = \int_0^{\pi} \ln(1+\alpha\cos x)\,dx$

$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[\int_0^{\pi} \ln(1+\alpha\cos x)\,dx\right] = \int_0^{\pi} \frac{\partial}{\partial \alpha}\left[\ln(1+\alpha\cos x)\right]dx$

$\Rightarrow \frac{\partial I}{\partial \alpha} = \int_0^{\pi} \frac{\cos x}{1+\alpha\cos x}\,dx$

$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{1}{\alpha}\int_0^{\pi} \frac{\alpha\cos x}{1+\alpha\cos x}\,dx$

$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{1}{\alpha}\int_0^{\pi} \frac{(1+\alpha\cos x)-1}{1+\alpha\cos x}\,dx = \frac{1}{\alpha}\int_0^{\pi} 1-\frac{1}{1+\alpha\cos x}\,dx$

FROM PART (a) $\frac{\partial}{\partial x}\left[\arcsin\left(\frac{\alpha+\cos x}{1+\alpha\cos x}\right)\right] = -\frac{\sqrt{1-\alpha^2}}{1+\alpha\cos x}$

$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{1}{\alpha}\left[x+\frac{1}{\sqrt{1-\alpha^2}}\arcsin\left(\frac{\alpha+\cos x}{1+\alpha\cos x}\right)\right]_0^{\pi}$

$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{1}{\alpha}\left[\pi+\frac{1}{\sqrt{1-\alpha^2}}\arcsin\left(\frac{\alpha-1}{1-\alpha}\right)-\frac{1}{\sqrt{1-\alpha^2}}\arcsin\left(\frac{\alpha+1}{1+\alpha}\right)\right]$

$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{1}{\alpha}\left[\pi+\frac{1}{\sqrt{1-\alpha^2}}\left[\arcsin(-1)-\arcsin 1\right]\right]$

$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{1}{\alpha}\left[\pi+\frac{1}{\sqrt{1-\alpha^2}}\left[-\frac{\pi}{2}-\frac{\pi}{2}\right]\right]$

$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{1}{\alpha}\left[\pi-\frac{\pi}{\sqrt{1-\alpha^2}}\right]$

$\Rightarrow \frac{\partial I}{\partial \alpha} = \frac{\pi}{\alpha}\left[1-\frac{1}{\sqrt{1-\alpha^2}}\right]$

INTEGRATE WITH RESPECT TO α

$\Rightarrow I = \pi\int \frac{1}{\alpha} - \frac{1}{\alpha\sqrt{1-\alpha^2}}\,d\alpha$

$\Rightarrow I = \pi\left[\ln\alpha\right] - \pi\int \frac{1}{\alpha\sqrt{1-\alpha^2}}\,d\alpha$ ←

SUBSTITUTION
$u = \sqrt{1-\alpha^2}$
$u^2 = 1-\alpha^2$
$2u\,du = -2\alpha\,d\alpha$
$d\alpha = -\frac{u}{\alpha}\,du$

$\Rightarrow I = \pi\ln\alpha - \pi\int \frac{1}{\alpha u}\left(-\frac{u}{\alpha}\,du\right)$

$\Rightarrow I = \pi\ln\alpha + \pi\int \frac{1}{\alpha^2}\,du$

$\Rightarrow I = \pi\ln\alpha + \pi\int \frac{1}{1-u^2}\,du$

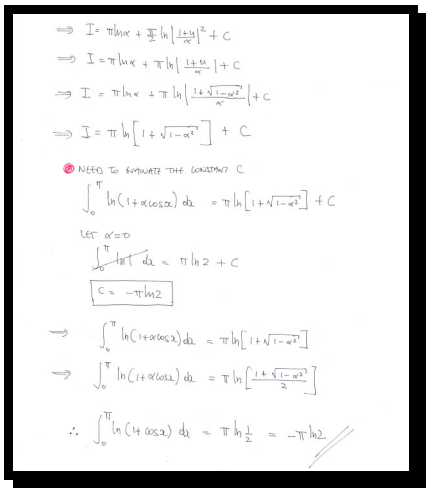
$\Rightarrow I = \pi\ln\alpha + \pi\int \frac{1}{(1-u)(1+u)}\,du$ ← SIMILAR PARTIAL FRACTIONS

$\Rightarrow I = \pi\ln\alpha + \pi\int \frac{\frac{1}{2}}{1+u}+\frac{\frac{1}{2}}{1-u}\,du$

$\Rightarrow I = \pi\ln\alpha + \frac{\pi}{2}\ln\left|\frac{1+u}{1-u}\right| + C$

$\Rightarrow I = \pi\ln\alpha + \frac{\pi}{2}\ln\left|\frac{(1+u)^2}{1-u^2}\right| + C$

$\Rightarrow I = \pi\ln\alpha + \frac{\pi}{2}\ln\left|\frac{(1+u)^2}{\alpha^2}\right| + C$



$\Rightarrow I = \pi\ln\alpha + \frac{\pi}{2}\ln\left|\frac{1+u}{\alpha^2}\right|^2 + C$

$\Rightarrow I = \pi\ln\alpha + \pi\ln\left|\frac{1+u}{\alpha}\right| + C$

$\Rightarrow I = \pi\ln\alpha + \pi\ln\left|\frac{1+\sqrt{1-\alpha^2}}{\alpha}\right| + C$

$\Rightarrow I = \pi\ln\left[1+\sqrt{1-\alpha^2}\right] + C$

NEED TO EVALUATE THE CONSTANT C

$\int_0^{\pi} \ln(1+\alpha\cos x)\,dx = \pi\ln\left[1+\sqrt{1-\alpha^2}\right] + C$

LET $\alpha = 0$

$\int_0^{\pi} \ln 1\,dx = \pi\ln 2 + C$

$C = -\pi\ln 2$

$\Rightarrow \int_0^{\pi} \ln(1+\alpha\cos x)\,dx = \pi\ln\left[1+\sqrt{1-\alpha^2}\right]$

$\Rightarrow \int_0^{\pi} \ln(1+\alpha\cos x)\,dx = \pi\ln\left[\frac{1+\sqrt{1-\alpha^2}}{2}\right]$

$\therefore \int_0^{\pi} \ln(1+\cos x)\,dx = \pi\ln\frac{1}{2} = -\pi\ln 2$

Question 44

By carrying out suitable differentiations on I under the integral sign, show that

$$I = \int_0^{\infty} \operatorname{arccot}(2x)\operatorname{arccot}(4x)\ dx = \frac{1}{8}\pi \ln\left(\frac{27}{4}\right).$$

V, proof



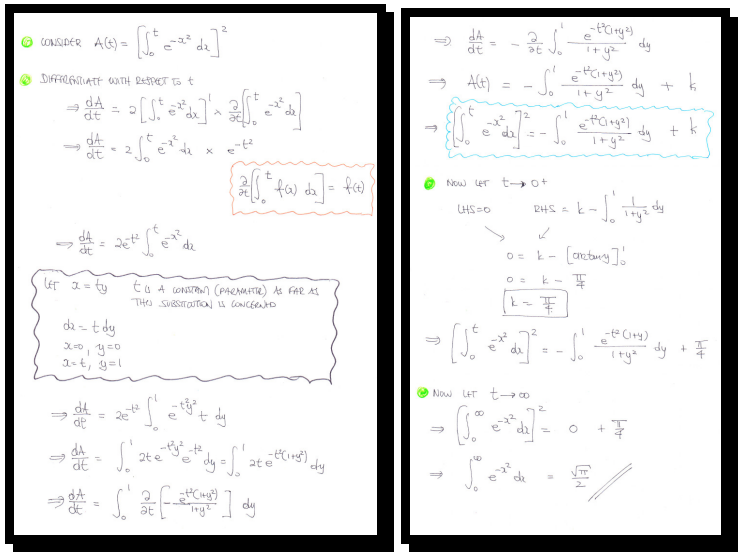
Question 45

$$A(t) \equiv \left[\int_0^t \mathrm{e}^{-x^2} \, dx \right]^2 .$$

By differentiating both sides of the above equation with respect to t, followed by the substitution $x = ty$, show that

$$\int_0^{\infty} \mathrm{e}^{-x^2} \, dx = \frac{1}{2}\sqrt{\pi} .$$

proof



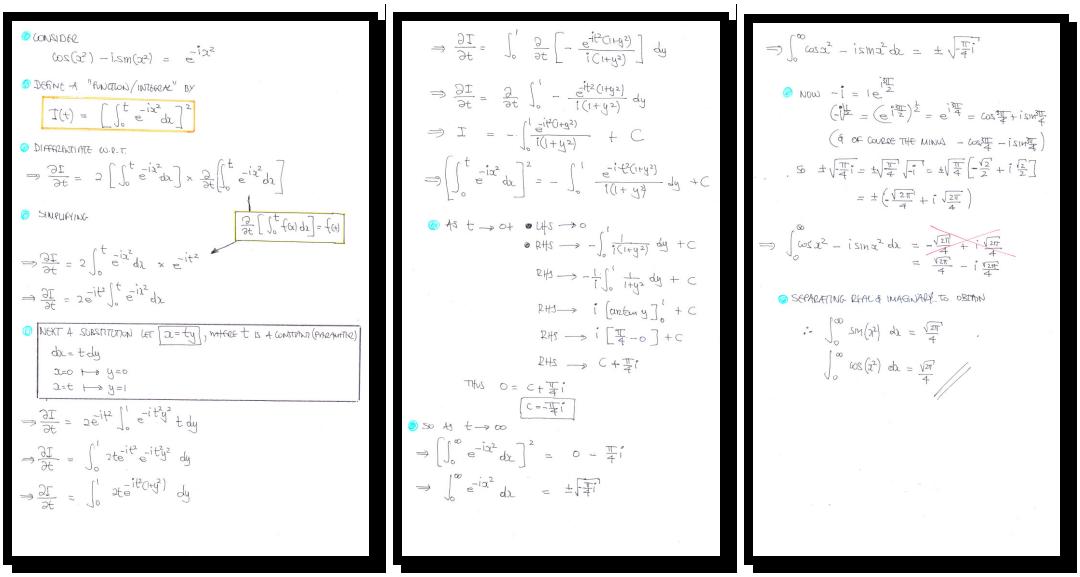
Question 46

$$I(t)\equiv\left[\int_0^t \mathrm{e}^{-\mathrm{i}x^2}\ dx\right]^2 .$$

By differentiating both sides of the above equation with respect to t, followed by the substitution $x=ty$, show that

$$\int_0^\infty \cos\left(x^2\right)\ dx\ =\ \int_0^\infty \sin\left(x^2\right)\ dx = \frac{1}{4}\sqrt{2\pi}\ .$$

proof



INTEGRATION UNDER THE INTEGRAL SIGN

Question 1

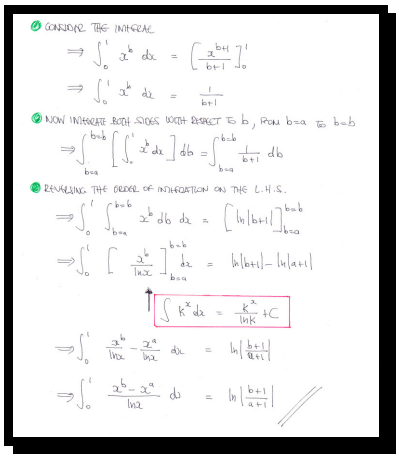
By integrating both sides of an appropriate integral relationship, with suitable limits, show that

$$\int_0^1 \frac{x^b - x^a}{\ln x}\, dx = \ln\left[\frac{b+1}{a+1}\right],$$

where $b > a > 0$.

You may assume that for $k > 0$, $\displaystyle\int k^x\, dx = \frac{k^x}{\ln k} + \text{constant}$.

proof



Question 2

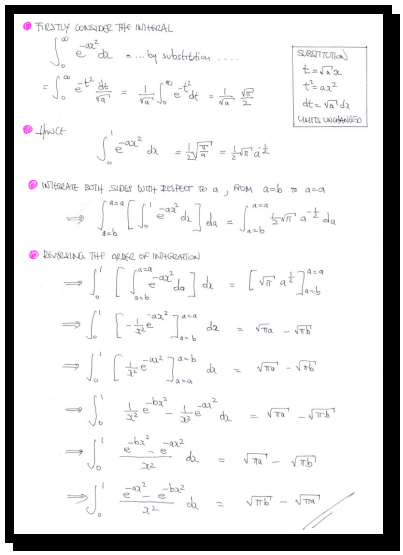
By integrating both sides of an appropriate integral relationship, with suitable limits, show that

$$\int_0^{\infty} \frac{\mathrm{e}^{-ax^2}-\mathrm{e}^{-bx^2}}{x^2}\,dx = \sqrt{\pi b}-\sqrt{\pi a}\,,$$

where $b > a > 0$.

You may assume that $\displaystyle\int_0^{\infty} \mathrm{e}^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}$.

proof



Question 3

The integral I is defined as

$$I = \int_0^{\infty} \mathrm{e}^{kx} \sin x \ dx \, .$$

where k is a constant.

a) Use a suitable method to show that

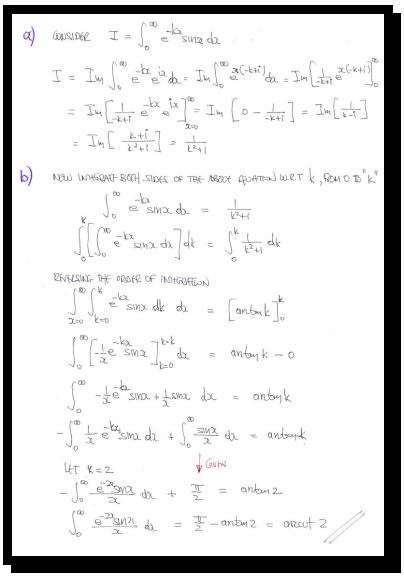
$$I = \frac{1}{k^2+1} \, .$$

b) By integrating both sides of an appropriate integral relationship with respect to k, with suitable limits, show further that

$$\int_0^{\infty} \frac{\mathrm{e}^{-2x} \sin x}{x} \ dx \ = \ \operatorname{arccot} 2 \, .$$

You may assume that $\displaystyle \int_0^{\infty} \frac{\sin x}{x} \ dx \ = \frac{\pi}{2} \, .$

proof



Question 4

By integrating both sides of an appropriate integral relationship with respect to b, with suitable limits, show that

$$\int_0^{\infty} \frac{\mathrm{e}^{-x}\sinh bx}{x}\,dx = \frac{1}{2}\ln\left[\frac{1+b}{1-b}\right].$$

proof

