DIFFERENCE EQUATIONS

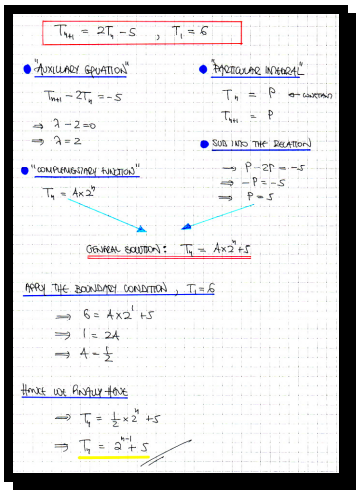
Question 1 (**)

A sequence of numbers T_1, T_2, T_3, T_4, T_5, … , T_n, … is generated by the recurrence relation

$$T_{n+1} = 2T_n - 5, \quad T_1 = 6.$$

Determine an expression for the n^{th} term of this sequence.

$\boxed{}$, $\boxed{T_n = 2^{n-1} + 5}$



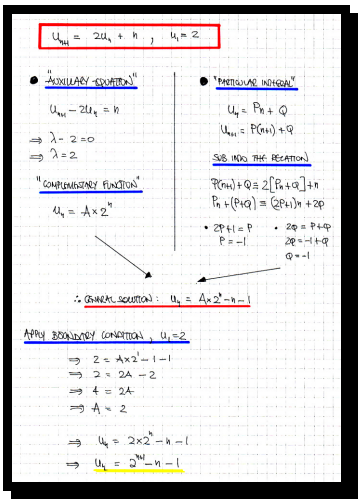
Question 2 (**)

A sequence of numbers u_1, u_2, u_3, u_4, $u_5 \ldots$, u_n, $\ldots$ is generated by the recurrence relation

$$u_{n+1} = 2u_n + n, \quad u_1 = 2.$$

Determine an expression for the n^{th} term of this sequence.

$\square$, $\boxed{u_n = 2^{n+1} - n - 1}$



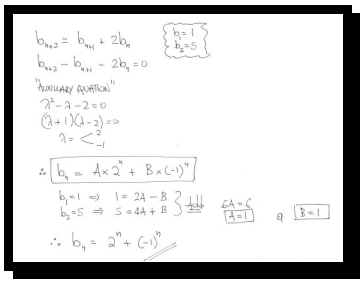
Question 3 (+)**

A sequence of numbers b_1, b_2, b_3, b_4, $b_5 \ldots$, b_n, $\ldots$ is generated by

$$b_{n+2} = 2b_n + n, \quad b_1 = 1, \quad b_2 = 5.$$

Determine an expression for the n^{th} term of this sequence.

$$\boxed{b_n = 2^n + (-1)^n}$$



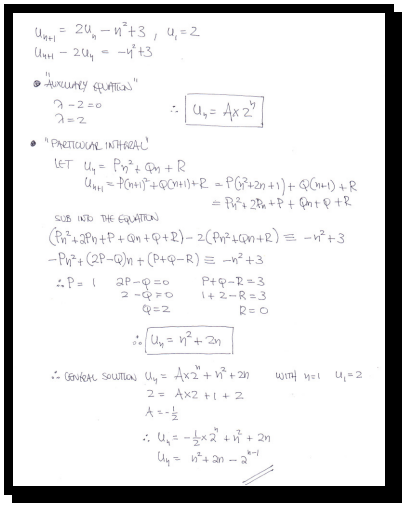
Question 4 (***)

A sequence of numbers u_1, u_2, u_3, u_4, $u_5 \ldots$, u_n, $\ldots$ is generated by the recurrence relation

$$u_{n+1} = 2u_n - n^2 + 3, \quad u_1 = 2.$$

Determine an expression for the n^{th} term of this sequence.

$$\boxed{u_n = n^2 + 2n - 2^{n-1}}$$



Question 5 (***)

A sequence of numbers t_1, t_2, t_3, t_4, $t_5 \ldots$, t_n, $\ldots$ is generated by the recurrence relation

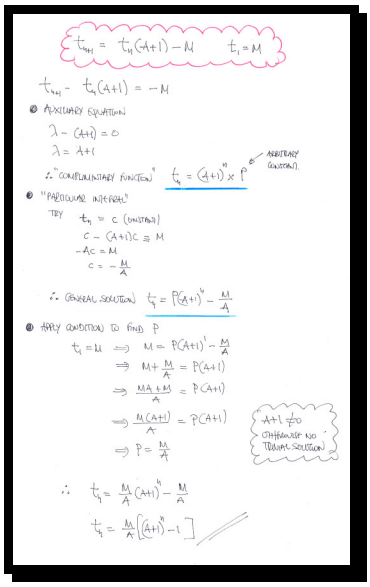
$$t_{n+1} = (A+1)t_n - M \ , \quad t_1 = M \ ,$$

where A and M are positive constants.

Determine an expression for the n^{th} term of this sequence is given by

$$t_n = \frac{M}{A}\left[(A+1)^n - 1\right].$$

proof



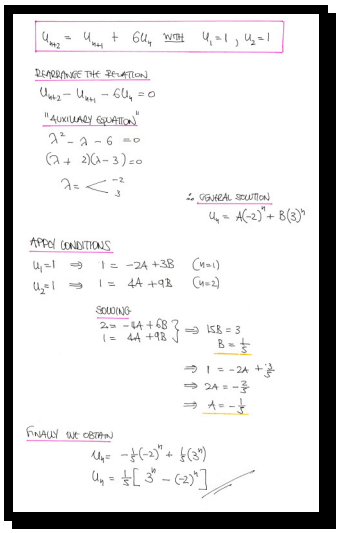
Question 6 (***)

A sequence of numbers $u_1, u_2, u_3, u_4, u_5 \ldots, u_n, \ldots$ is generated by the following recurrence relation

$$u_{n+2} = u_{n+1} + 6u_n, \quad u_1 = 1, \quad u_2 = 1.$$

Determine an expression for the n^{th} term of this sequence.

$\square$, $\boxed{u_n = \frac{1}{5}\left[3^n - (-2)^n\right]}$



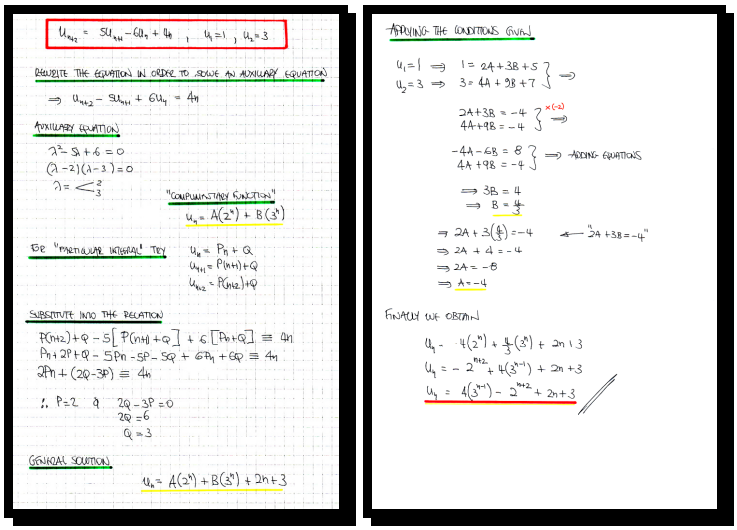
Question 7 (***)

A sequence of numbers u_1, u_2, u_3, u_4, $u_5 \ldots$, u_n, $\ldots$ is generated by the recurrence relation

$$u_{n+2} = 5u_{n+1} - 6u_n + 4n, \quad u_1 = 1, \; u_2 = 3.$$

Determine an expression for the n^{th} term of this sequence.

$\boxed{}$, $\boxed{u_n = 4\times 3^{n-1} - 2^{n+2} + 2n + 3}$



Question 8 **(***)**

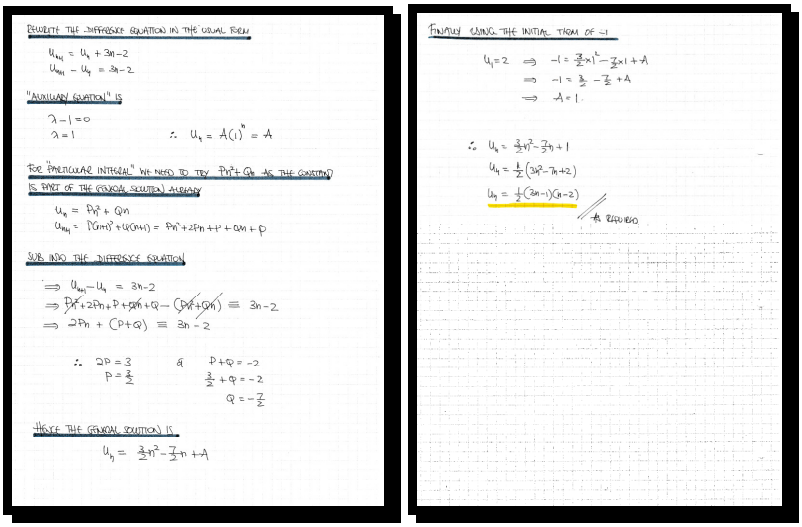
A sequence of numbers u_1, u_2, u_3, u_4, $u_5 \ldots$, u_n, $\ldots$ is generated by the recurrence relation

$$u_{n+1} = u_n + 3n - 2, \quad u_1 = -1.$$

Determine an expression for the n^{th} term of this sequence.

Give the answer in the form $u_n = \frac{1}{2}(an+b)(n+c)$, where a, b and c are integers.

$\square$, $\boxed{u_n = \frac{1}{2}(3n-1)(n-2)}$



Question 9 **(***)**

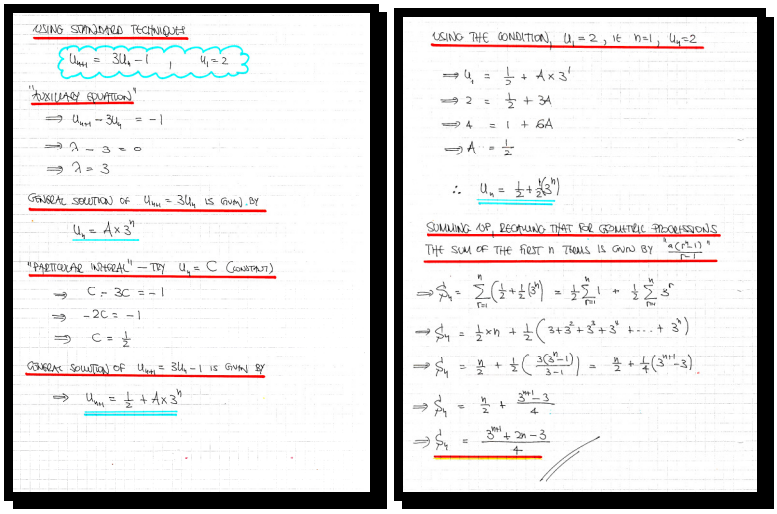
A sequence of numbers, u_1, u_2, u_3, u_4, ... , is defined by

$$u_{n+1}=3u_n-1, \quad u_1=2.$$

Determine, in terms of n, a simplified expression for

$$\sum_{r=1}^{n} u_r .$$

$\square$, $\boxed{S_n=\frac{1}{4}\left[3^{n+1}+2n-3\right]}$



Question 10 (***)

A sequence of numbers u_1, u_2, u_3, u_4, $u_5 \ldots$, u_n, $\ldots$ is generated by the recurrence relation

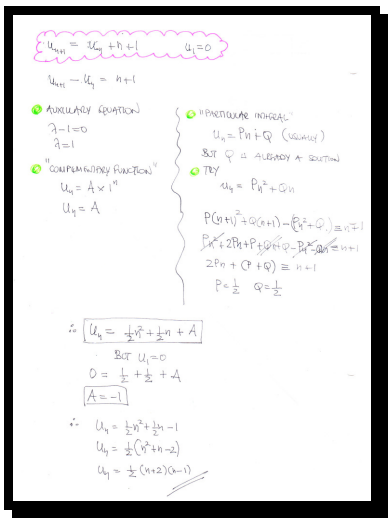
$$u_{n+1} = u_n + n + 1, \quad u_1 = 0.$$

Determine an expression for the n^{th} term of this sequence is given by

$$u_n = \frac{1}{2}(n + A)(n + B),$$

where A and B are integers to be found.

$$\boxed{u_n = \frac{1}{2}(n+2)(n-1)}$$



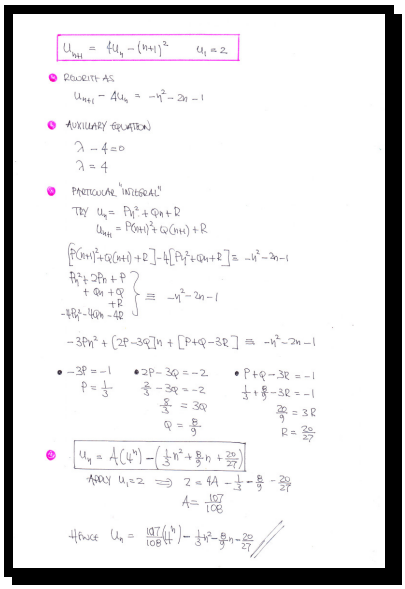
Question 11 (***)

A sequence of numbers u_1, u_2, u_3, u_4, $u_5 \ldots$, u_n, $\ldots$ is generated by the recurrence relation

$$u_{n+1} = 4u_n - (n+1)^2, \quad u_1 = 2.$$

Determine an expression for the n^{th} term of this sequence.

$$\boxed{u_n = \frac{107}{108}\left(4^n\right) - \frac{1}{3}n^2 - \frac{8}{9}n - \frac{20}{27} = \frac{1}{108}\left[107\left(4^n\right) - \left(36n^2 + 96n + 80\right)\right]}$$



Question 12 (*+)**

A recurrence relation is defined as

$$u_{n+1}=(\sin x)u_n+\cos 2x\,,\ u_1=k\,,\ 0<x<\frac{\pi}{2}\,.$$

a) Explain why this recurrence relation will converge to a limit L, for all real values of the constant k.

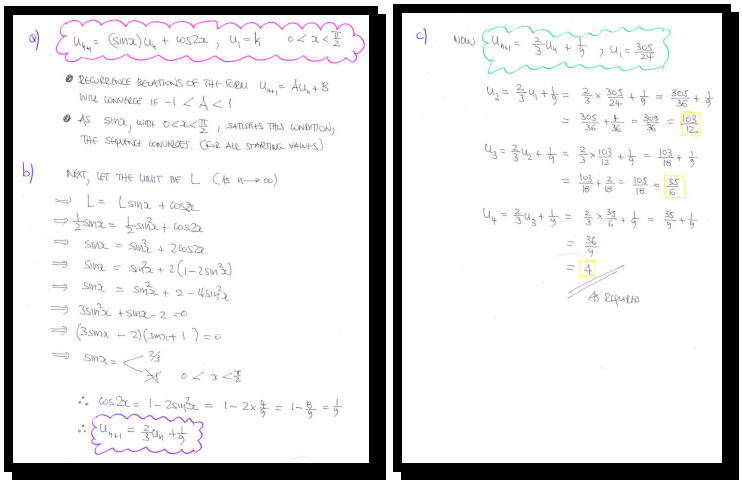
b) Given that $L=\frac{1}{2}\sin x$, write the recurrence relation in the form

$$u_{n+1}=Au_n+B\,,$$

where A and B are rational constants to be found.

c) Given further that $k=\frac{305}{24}$, show that $u_4=4$.

$$\boxed{u_{n+1}=\frac{1}{3}u_n+\frac{1}{9}}$$



Question 13 (****)

The Fibonacci sequence of numbers is generated by the recurrence relation

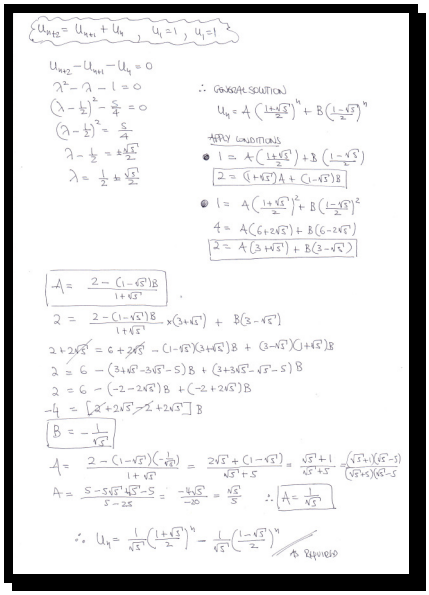
$$u_{n+2} = u_{n+1} + u_n, \quad n \in \mathbb{N},$$

with $u_1 = 1$, $u_2 = 1$.

Solve the above difference equation to show that the n^{th} term of Fibonacci sequence is given by

$$u_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

proof



Question 14 **(****)**

A sequence of numbers is generated by the recurrence relation

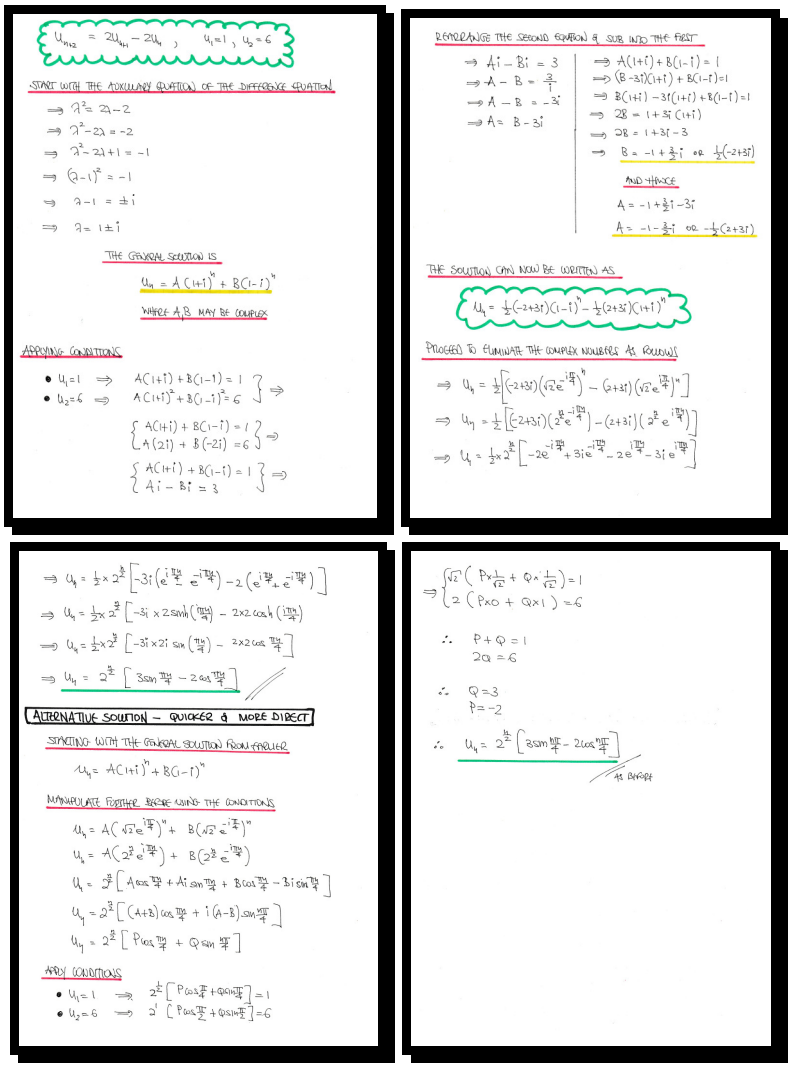
$$u_{n+2}=2u_{n+1}-2u_n\,,\quad n\in\mathbb{N}\,,$$

with $u_1=1$, $u_2=6$.

Determine a simplified expression for the n^{th} term of this sequence.

The final answer may not contain complex numbers

$$\boxed{u_n=\tfrac{1}{2}(-2+3\mathrm{i})(1-\mathrm{i})^n-\tfrac{1}{2}(2+3\mathrm{i})(1+\mathrm{i})^n\;=\;2^{\frac{1}{2}n}\left[3\sin\left(\tfrac{1}{4}n\pi\right)-2\cos\left(\tfrac{1}{4}n\pi\right)\right]},$$



Question 15 **(****)**

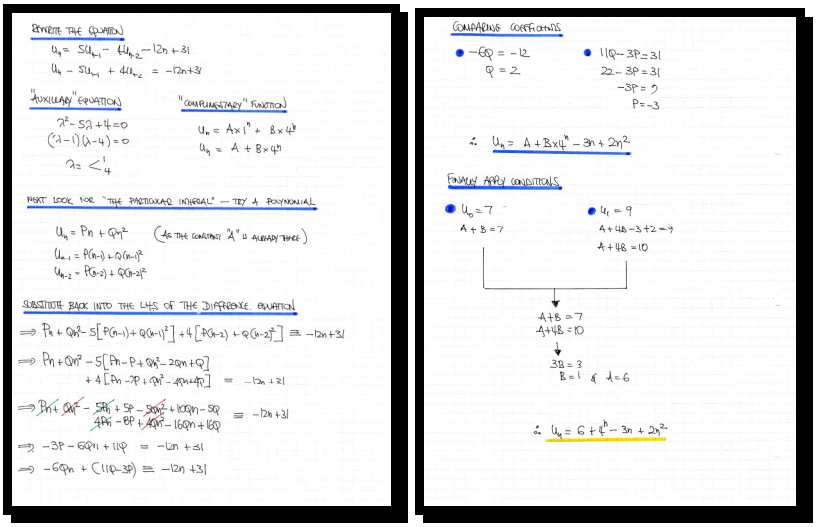
A sequence of numbers is generated by the recurrence relation

$$u_n = 5u_{n-1} - 4u_{n-2} - 12n + 31, \quad n \in \mathbb{N}, \quad n \geq 2$$

with $u_0 = 7$, $u_1 = 9$.

Determine a simplified expression for the n^{th} term of this sequence.

$\square$, $\boxed{u_n = 4^n + 2n^2 - 3n + 6}$



Question 16 (****)

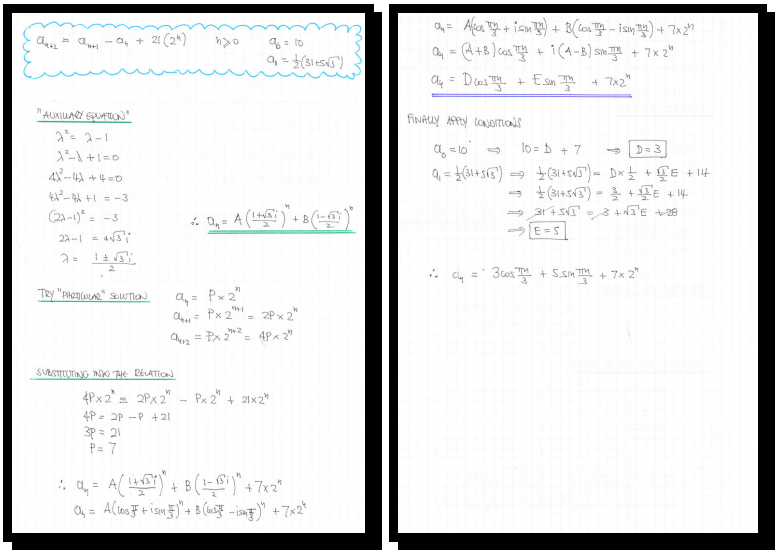
A sequence of numbers is generated by the recurrence relation

$$a_{n+2}=a_{n+1}-a_n+21\left(2^n\right), \quad n\in\mathbb{N},$$

with $a_0=10$, $a_1=\frac{1}{2}\left[31+5\sqrt{3}\right]$.

Determine a simplified expression for the n^{th} term of this sequence.

$$\boxed{a_n=7\times 2^n+3\cos\left(\frac{1}{3}\pi n\right)+5\sin\left(\frac{1}{3}\pi n\right)}$$



Question 17 (****)

A sequence is defined by the recurrence relation

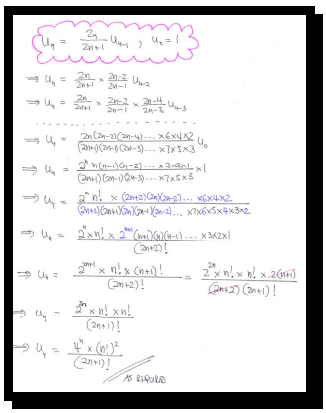
$$u_n = \frac{2n}{2n+1}u_{n-1}\,,\ n\in\mathbb{N}\quad u_0 = 1\,.$$

Show, by direct manipulation, that

$$u_n = \frac{4^n\times(n!)^2}{(2n+1)!}\,.$$

[*you may not use proof by induction*]

proof



Question 18 (****)

A sequence is defined by the recurrence relation

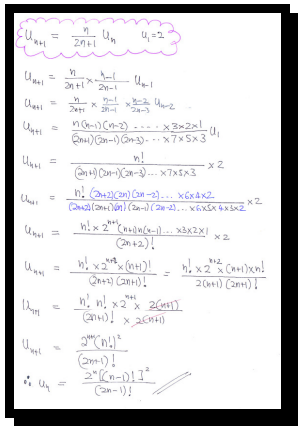
$$u_{n+1}=\frac{n}{2n+1}u_n\,,\ n\in\mathbb{N}\quad u_1=2\,.$$

Show, by direct manipulation, that

$$u_n=\frac{2^n\times\left[(n-1)!\right]^2}{(2n-1)!}\,.$$

[*you may not use proof by induction*]

proof



Question 19 (**+)**

The n^{th} term of a series is given recursively by

$$A_n=\frac{a(2n+1)}{2n+4}A_{n-1},\ n\in\mathbb{N},\ n\geq 1,$$

where a is a positive constant.

Given further that $A_0=1$, show that

$$A_n=\left(\frac{a}{4}\right)^n\binom{2n+2}{n}\frac{1}{n+1}.$$

proof

