COMPLEX VARIABLES

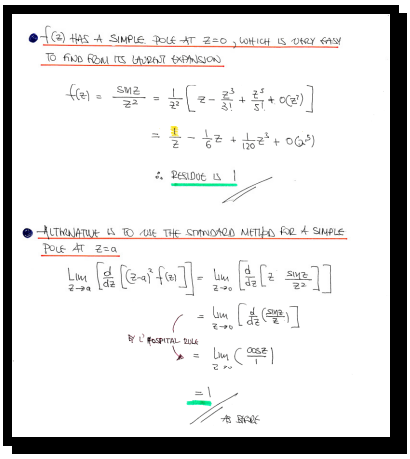
CALCULATIONS OF RESIDUES

Question 1

$$f(z)\equiv\frac{\sin z}{z^2},\ z\in\mathbb{C}.$$

Find the residue of the pole of $f(z)$.

$\boxed{}$, $\boxed{res(z=0)=1}$

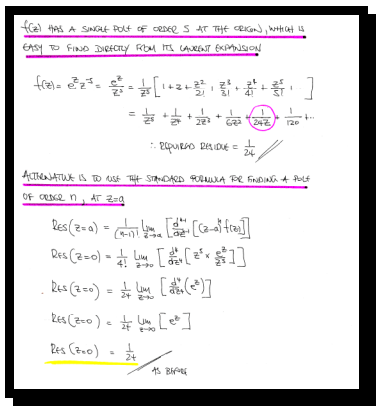


Question 2

$$f(z)\equiv e^z z^{-5},\ z\in\mathbb{C}.$$

Find the residue of the pole of $f(z)$.

$\boxed{}$, $\boxed{res(z=0)=\frac{1}{24}}$

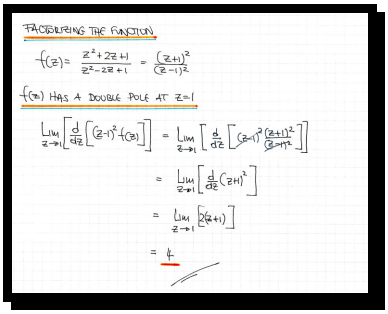


Question 3

$$f(z)\equiv\frac{z^2+2z+1}{z^2-2z+1},\ z\in\mathbb{C}.$$

Find the residue of the pole of $f(z)$.

$\boxed{}$, $\boxed{res(z=1)=4}$

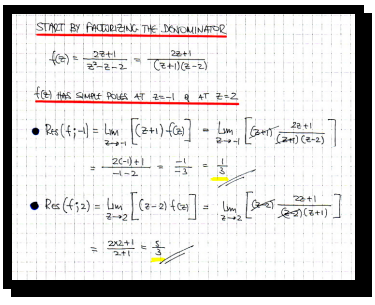


Question 4

$$f(z)\equiv\frac{2z+1}{z^2-z-2},\ z\in\mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$\boxed{}$, $\boxed{res(z=2)=\frac{5}{3}}$, $\boxed{res(z=-1)=\frac{1}{3}}$

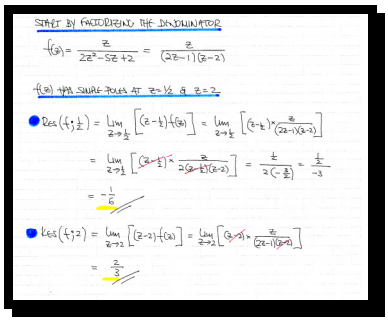


Question 5

$$f(z) \equiv \frac{z}{2z^2 - 5z + 2}, \ z \in \mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$\square$, $\boxed{res\left(z = \frac{1}{2}\right) = -\frac{1}{6}}$, $\boxed{res(z = 2) = \frac{2}{3}}$



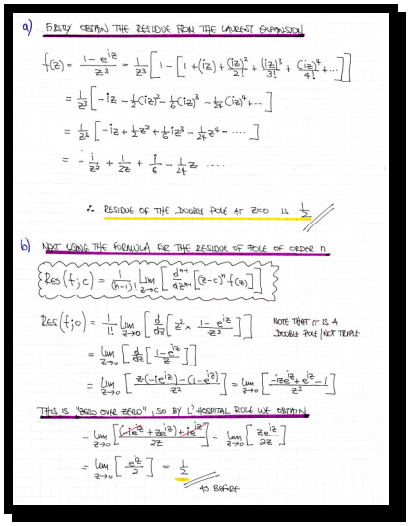
Question 6

$$f(z)\equiv\frac{1-\mathrm{e}^{\mathrm{i}z}}{z^3},\ z\in\mathbb{C}.$$

a) Find the first four terms in the Laurent expansion of $f(z)$ and hence state the residue of the pole of $f(z)$.

b) Determine the residue of the pole of $f(z)$ by an alternative method

$\square$, $\boxed{res(z=0)=\frac{1}{2}}$



Question 7

$$f(z)\equiv\frac{z^2+4}{z^3+2z^2+2z},\ z\in\mathbb{C}.$$

Find the residue of each of the three poles of $f(z)$.

$$\boxed{res(z=0)=2},\ \boxed{res(z=-1+\mathrm{i})=\frac{1}{2}(-1+3\mathrm{i})},\ \boxed{res(z=-1-\mathrm{i})=-\frac{1}{2}(1+3\mathrm{i})}$$

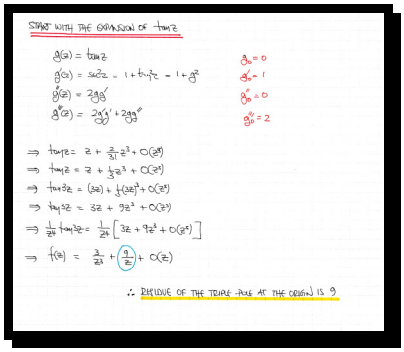


Question 8

$$f(z)\equiv\frac{\tan 3z}{z^4},\ z\in\mathbb{C}.$$

Find the residue of the pole of $f(z)$.

$$\boxed{},\ \boxed{res(z=0)=9}$$



Question 9

$$f(z)\equiv\frac{z^2-2z}{\left(z^2+4\right)(z+1)^2},\ z\in\mathbb{C}.$$

Find the residue of each of the three poles of $f(z)$.

$$\boxed{res(z=2\mathrm{i})=\frac{1}{25}(7+\mathrm{i})},\ \boxed{res(z=-2\mathrm{i})=\frac{1}{25}(7-\mathrm{i})},\ \boxed{res(z=-1)=-\frac{14}{25}}$$

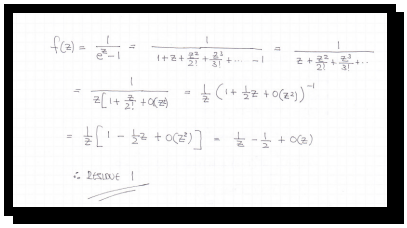


Question 10

$$f(z)\equiv\frac{1}{\mathrm{e}^z-1},\ z\in\mathbb{C}.$$

Find the residue of the pole of $f(z)$, at the origin.

$$\boxed{res(z=0)=1}$$

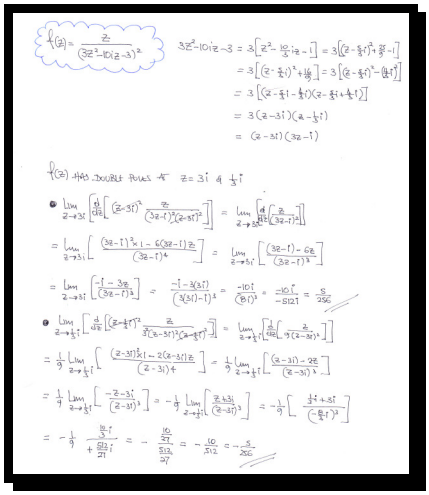


Question 11

$$f(z)\equiv\frac{z}{\left(3z^2-10\mathrm{i}\,z-3\right)^2},\ z\in\mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$$\boxed{res\left(z=3\mathrm{i}\right)=\frac{5}{256}},\ \boxed{res\left(z=\frac{1}{3}\mathrm{i}\right)=-\frac{5}{256}}$$

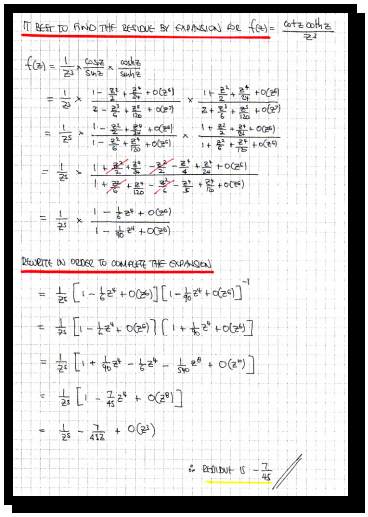


Question 12

$$f(z)\equiv\frac{\cot z\,\coth z}{z^3},\ z\in\mathbb{C}.$$

Find the residue of the pole of $f(z)$ at $z=0$.

$\boxed{}$, $\boxed{res(z=0)=-\frac{7}{45}}$

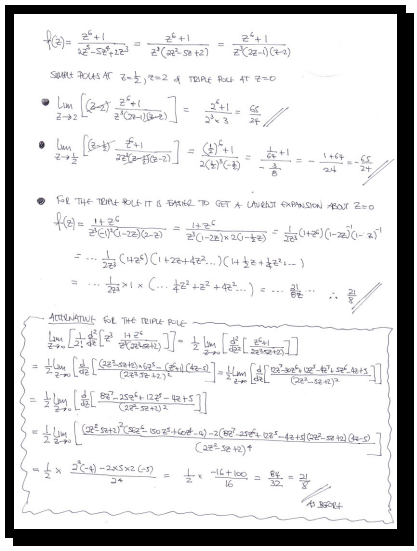


Question 13

$$f(z)\equiv\frac{z^6+1}{2z^5-5z^4+2z^3},\ z\in\mathbb{C}.$$

Find the residue of each of the three poles of $f(z)$.

$$\boxed{res\left(z=\frac{1}{2}\right)=-\frac{65}{24}},\ \boxed{res\left(z=2\right)=\frac{65}{24}},\ \boxed{res\left(z=0\right)=\frac{21}{8}}$$

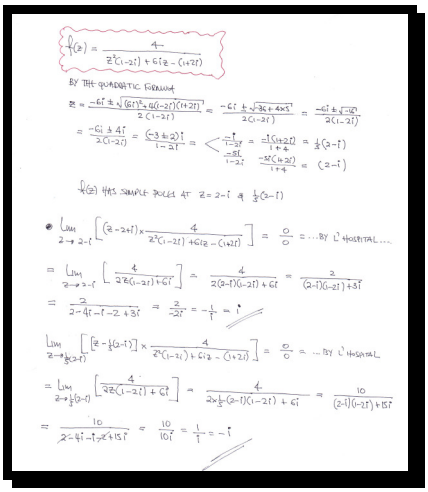


Question 14

$$f(z)\equiv\frac{4}{z^2(1-2\mathrm{i})+6z\,\mathrm{i}-(1+2\mathrm{i})},\ z\in\mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$$\boxed{res(z=2-\mathrm{i})=\mathrm{i}},\ \boxed{res\left(z=\frac{1}{5}(2-\mathrm{i})\right)=-\mathrm{i}}$$



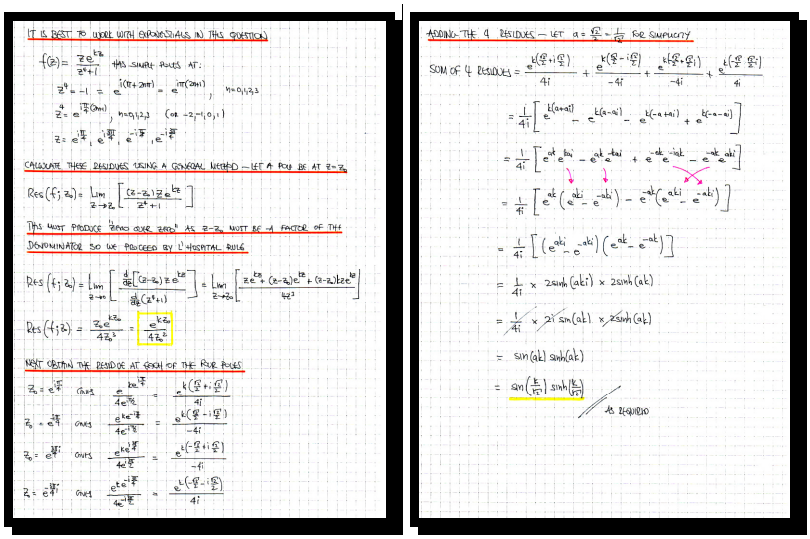
Question 15

$$f(z)\equiv\frac{z\,\mathrm{e}^{kz}}{z^4+1},\ z\in\mathbb{C},\ k\in\mathbb{R},\ k>0.$$

Show that the sum of the residues of the four poles of $f(z)$, is

$$\sin\left(\frac{k}{\sqrt{2}}\right)\sinh\left(\frac{k}{\sqrt{2}}\right).$$

[ ], [proof]



LAURENT SERIES

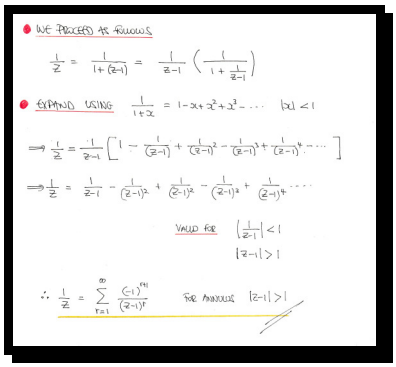
Question 1

Determine a Laurent series for

$$f(z)=\frac{1}{z},$$

which is valid in the infinite annulus $|z-1|>1$.

$$\boxed{\phantom{\frac{1}{z}=\sum_{r=1}^{\infty}\left[\frac{(-1)^{r+1}}{(z-1)^{r}}\right]}}, \quad \boxed{\frac{1}{z}=\sum_{r=1}^{\infty}\left[\frac{(-1)^{r+1}}{(z-1)^{r}}\right]}$$



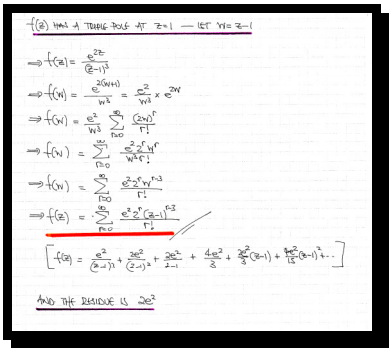
Question 2

Determine a Laurent series for

$$f(z)=\frac{e^{2z}}{(z-1)^3},$$

about its singularity, and hence state the residue of $f(z)$ about its singularity.

$\square$, $\boxed{f(z)=\sum_{r=0}^{\infty}\left[\frac{e^2\,2^r\,(z-1)^{r-3}}{r!}\right]}$, $\boxed{res(z=1)=2e^2}$



Question 3

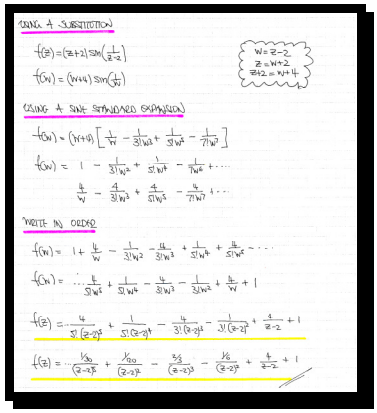
Determine a Laurent series for

$$f(z)=(z+2)\sin\left(\frac{1}{z-2}\right),$$

about $z=2$.

$\boxed{}$,

$$\boxed{f(z)=\ldots+\frac{4}{5!(z-2)^5}+\frac{1}{5!(z-2)^4}-\frac{4}{3!(z-2)^3}-\frac{1}{3!(z-2)^2}+\frac{4}{z-2}+1}$$



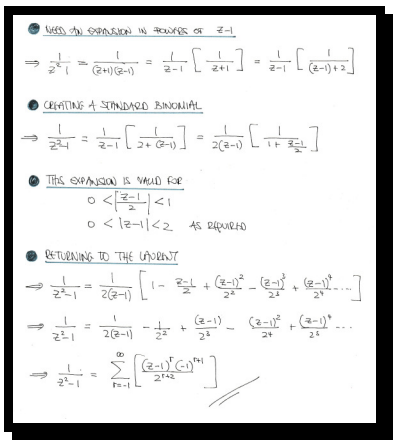
Question 4

Determine a Laurent series for

$$f(z)=\frac{1}{z^2-1},$$

which is valid in the punctured disc $0<|z-1|<2$.

$$\boxed{},\ \boxed{\frac{1}{z^2-1}=\sum_{r=-1}^{\infty}\left[\frac{(z-1)^r(-1)^{r+1}}{2^{r+2}}\right]}$$



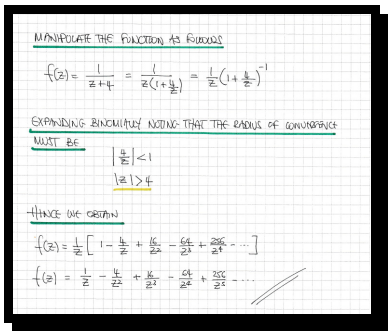
Question 5

Determine a Laurent series for

$$f(z)=\frac{1}{z+4},$$

which is valid for $|z|>4$.

$$\square, \quad \boxed{\frac{1}{z+4}=\sum_{r=0}^{\infty}\left[\frac{(-4)^r}{z^{r+1}}\right]}$$



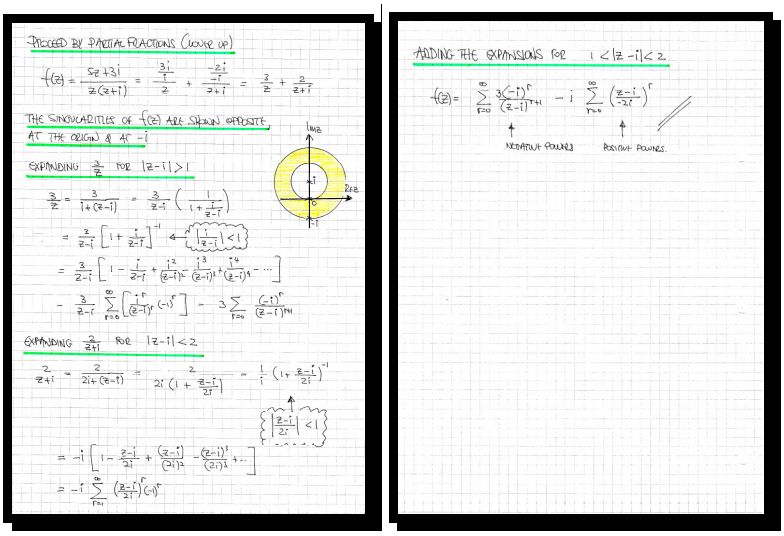
Question 6

Determine a Laurent series for

$$f(z)=\frac{5z+3\mathrm{i}}{z(z+\mathrm{i})},$$

which is valid in the annulus $1<|z-\mathrm{i}|<2$.

$$\boxed{\frac{5z+3\mathrm{i}}{z(z+\mathrm{i})}=\sum_{r=0}^{\infty}\left[\frac{3(-\mathrm{i})^r}{(z-\mathrm{i})^{r+1}}\right]-\mathrm{i}\sum_{r=0}^{\infty}\left[\frac{(z-\mathrm{i})^r}{(-2\mathrm{i})^r}\right]}$$



Question 7

Determine a Laurent series for

$$f(z)=\frac{1}{(z-1)(z+2)},$$

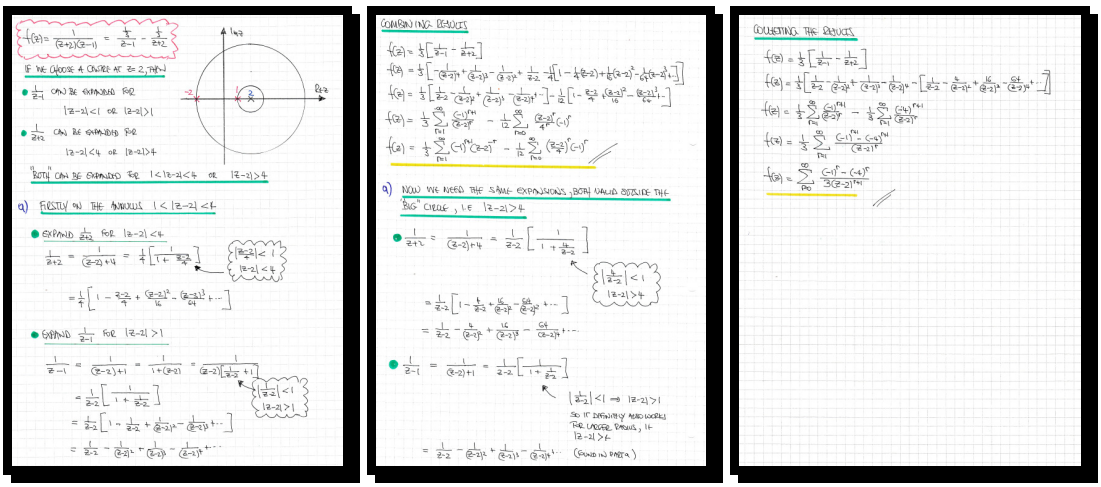
which is valid in ...

a) ... the annulus $1<|z-2|<4$.

b) ... in the region for which $|z-2|>4$.

$\square$, $f(z) = \dfrac{1}{3}\displaystyle\sum_{r=1}^{\infty}\left[(-1)^{r+1}(z-2)^{-r}\right] - \dfrac{1}{12}\sum_{r=0}^{\infty}\left[(-1)^{r}\left(\dfrac{z-2}{4}\right)^{r}\right]$,

$$f(z) = \sum_{r=0}^{\infty}\frac{(-1)^{r}-(-4)^{r}}{3(z-2)^{r+1}}$$



VARIOUS PROBLEMS

Question 1

The complex number $z = c + a\cos\theta + \mathrm{i}b\sin\theta$, $0 \le \theta < 2\pi$, traces a closed contour C, where a, b and c are positive real numbers with $a > c$.

By considering

$$\oint_C \frac{1}{z}\, dz,$$

show that

$$\int_0^{2\pi} \frac{a + c\cos\theta}{(c + a\cos\theta)^2 + (b\sin\theta)^2}\, d\theta = \frac{2\pi}{b}.$$

proof

