INTEGRAL THEOREMS

Green's Theorem

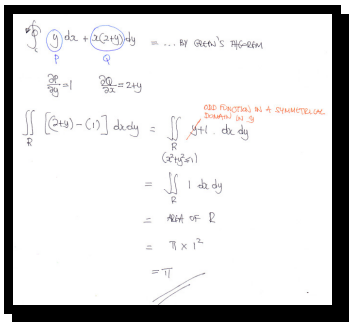
Question 1

Use Green's Theorem on the plane to evaluate the line integral

$$\oint_C \left[y\,dx + x(2+y)\,dy \right] ,$$

where C is a circle of radius 1, centre at the origin O, traced anticlockwise.

$\boxed{\pi}$



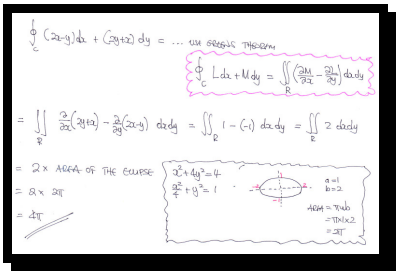
Question 2

Use Green's Theorem on the plane to evaluate the line integral

$$\oint_C (2x-y)\,dx+(2y+x)\,dy ,$$

where C is the path around the ellipse with equation $x^2+4y^2=4$, taken in an anticlockwise direction.

$\boxed{4\pi}$



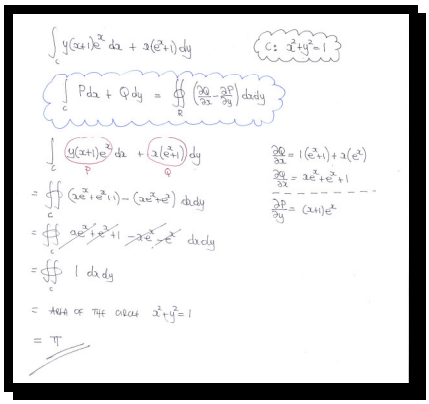
Question 3

Use Green's Theorem on the plane to evaluate the line integral

$$\oint_C y(x+1)\mathrm{e}^x\,dx \;+\; x\left(\mathrm{e}^x+1\right)\,dy\,,$$

where C is a circle of radius 1, centre at the origin O, traced anticlockwise.

π



Question 4

The functions F and G are defined as

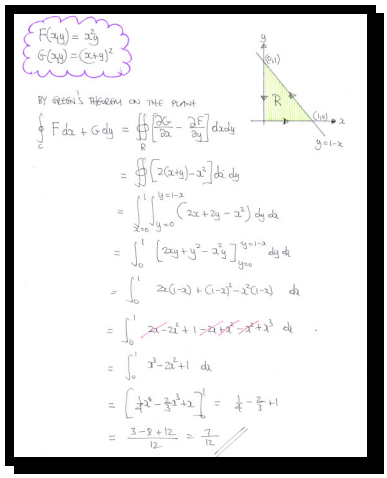
$$F(x,y)=x^2y \quad \text{and} \quad G(x,y)=(x+y)^2$$

The anticlockwise path along the perimeter of the triangle whose vertices are located at $(0,0)$, $(1,0)$ and $(0,1)$, is denoted by C.

Use Green's Theorem on the plane to evaluate the line integral

$$\int_C (F\,dx+G\,dy).$$

$$\frac{7}{12}$$



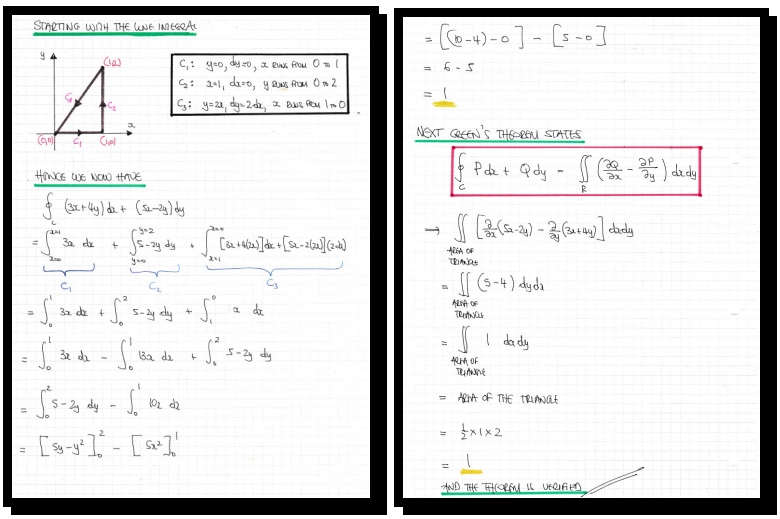
Question 5

The contour C is the boundary of a triangle with vertices at the points with Cartesian coordinates $(0,0)$, $(1,0)$ and $(1,2)$, traced in an anticlockwise direction.

Verify Green's Theorem on the plane for the line integral

$$\oint_C (3x+4y)\,dx+(5x-2y)\,dy\,.$$

[ ], [both sides yield 1]



Question 6

The functions $P(x,y)$ and $Q(x,y)$ have continuous first order partial derivatives.

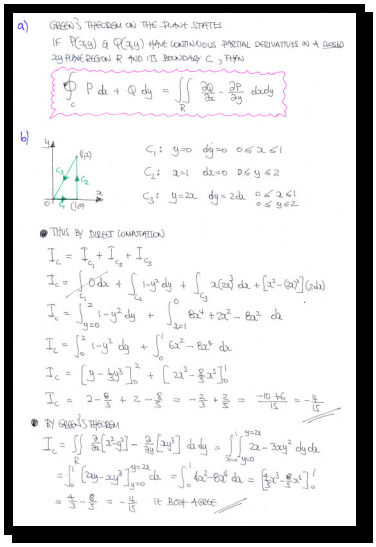
a) State formally Green's theorem in the plane, with reference to P and Q.

The contour C is the boundary of a triangle with vertices at the points with Cartesian coordinates $(0,0)$, $(1,0)$ and $(1,2)$.

b) Verify Green's Theorem on the plane for the line integral

$$\int_C \left(xy^3\right)dx+\left(x^2-y^2\right)dy\,.$$

both sides yield $-\frac{4}{15}$



Question 7

The functions $P(x,y)$ and $Q(x,y)$ have continuous first order partial derivatives.

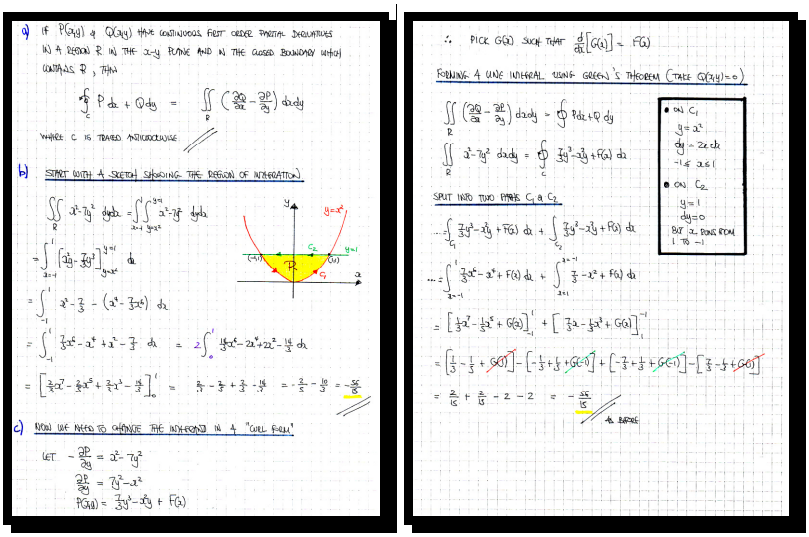
a) State formally Green's theorem in the plane, with reference to the functions, P and Q.

b) Evaluate the integral

$$\int_{-1}^{1}\int_{x^2}^{1}\left(x^2-7y^2\right)dy\,dx\,.$$

c) By considering a line integral over a suitable contour C, use Green's theorem in the plane to independently verify the answer to part **(b)**.

$$\boxed{}\,,\quad \boxed{-\frac{56}{15}}$$



Question 8

The closed curve C bounds the finite region R in the x-y plane defined as

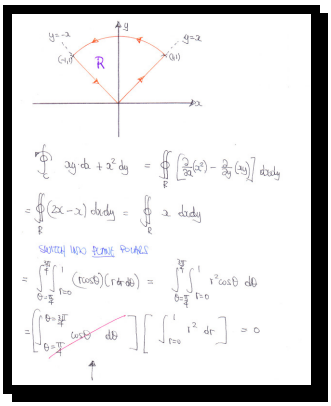
$$R(x,y)=\{x+y\geq 0\ \cap\ x-y\leq 0\ \cap\ x^2+y^2\leq 1\}.$$

Evaluate the line integral

$$\oint_C \left(xy\,dx\ +\ x^2\,dy\right),$$

where C is traced anticlockwise.

$\boxed{0}$



Question 9

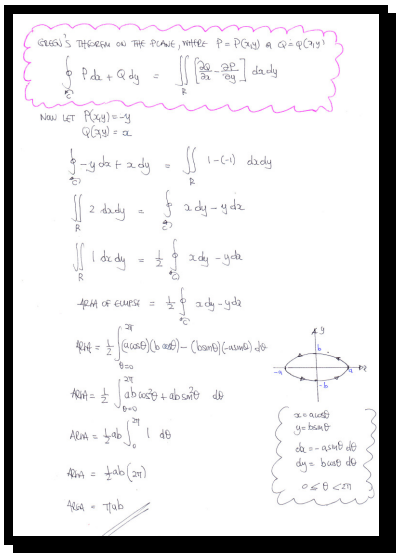
An ellipse has Cartesian equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where a and b positive constants.

Use Green's theorem in the plane, to show that the area of the ellipse is πab.

proof



Question 10

It is given that the vector function $\mathbf{F}$ satisfies

$$\mathbf{F}=\left(\sin x^3-xy\right)\mathbf{i}+\left(x+y^3\sin y\right)\mathbf{j}.$$

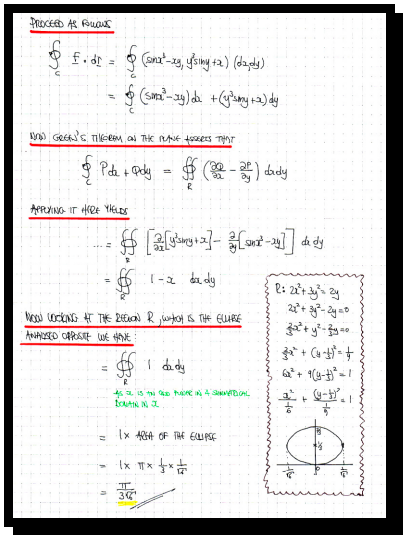
Evaluate the line integral

$$\oint_C \mathbf{F}\cdot\mathbf{dr}\,,$$

where C is the ellipse with cartesian equation

$$2x^2+3y^2=2y\,.$$

$\square$, $\boxed{\dfrac{\pi}{3\sqrt{6}}}$



Question 11

It is given that the vector function $\mathbf{F}$ satisfies

$$\mathbf{F}=\left[x\cos x\right]\mathbf{i}+\left[15xy+\ln\left(1+y^3\right)\right]\mathbf{j}.$$

Evaluate the line integral

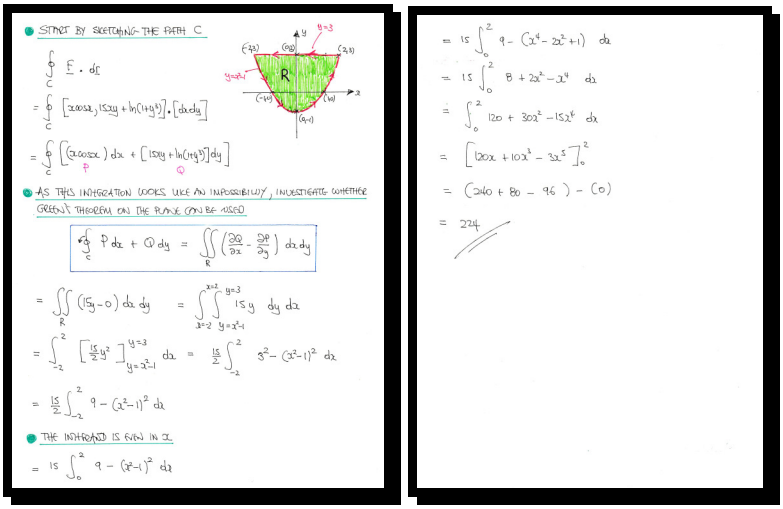
$$\oint_C \mathbf{F}\cdot\mathbf{dr},$$

where C is the curve

$$\left\{(x,y):y=3,\ -2\leq x\leq 2\right\}\cup\left\{(x,y):y=x^2-1,\ -2\leq x\leq 2\right\},$$

traced in an anticlockwise direction.

, 224



Gauss' Theorem

also known as the Divergence Theorem

Created by T. Madas

Question 1

$$\mathbf{A}(x,y,z)\equiv(2x+y-z)\mathbf{i}+\left(xy^2z\right)\mathbf{j}+(xy-2yz)\mathbf{k}\,.$$

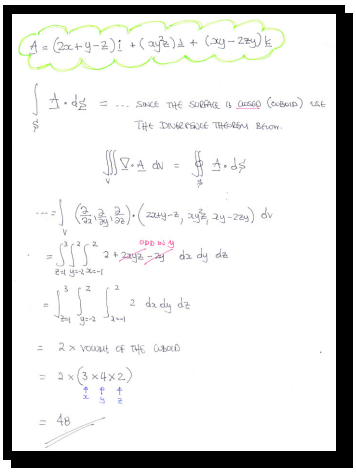
Evaluate the integral

$$\iint_S \mathbf{A}\cdot\mathbf{dS}\,,$$

where S is the **closed** surface enclosing the finite region V, defined by

$$-1\le x\le 2\,,\quad -2\le y\le 2\,,\quad 1\le z\le 3\,.$$

48



Question 2

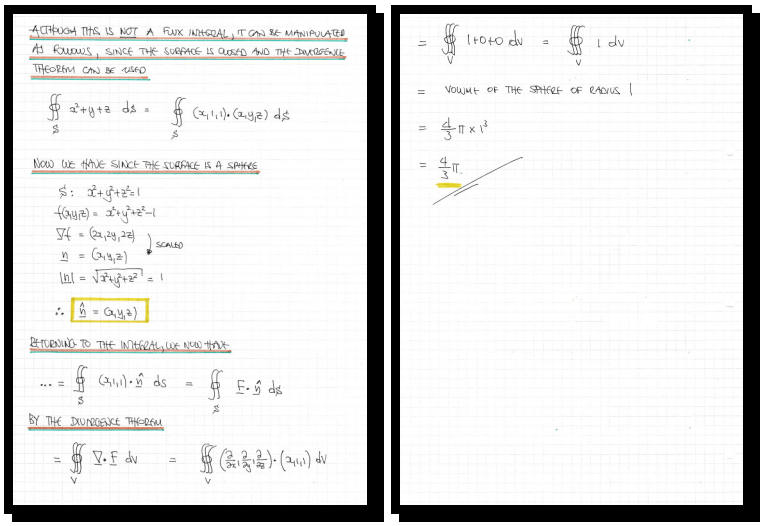
The surface S is the sphere with Cartesian equation

$$x^2 + y^2 + z^2 = 1$$

Use the Divergence Theorem to evaluate

$$\iint_S \left(x^2 + y + z \right) dS \, .$$

$\square$, $\boxed{\frac{4}{3}\pi}$



Question 3

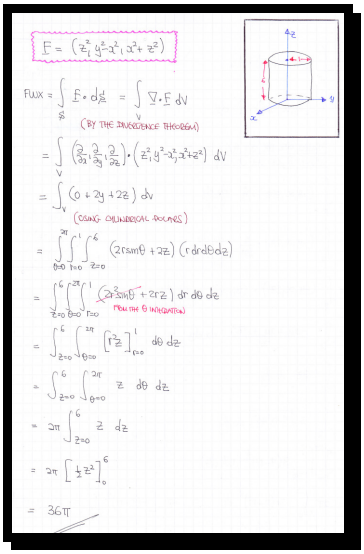
$$\mathbf{F}(x,y,z) \equiv z^2\mathbf{i} + \left(y^2 - x^2\right)\mathbf{j} + \left(x^2 + z^2\right)\mathbf{k}\,.$$

Evaluate the integral

$$\iint_S \mathbf{F}\cdot\mathbf{dS}\,,$$

where S is the surface of a cylinder of radius 1, whose axis is the z axis, between $z=0$ and $z=6$.

36π



Question 4

$$\mathbf{F}(x,y,z) \equiv xy\mathbf{i} + y\mathbf{j} + 4\mathbf{k}\,.$$

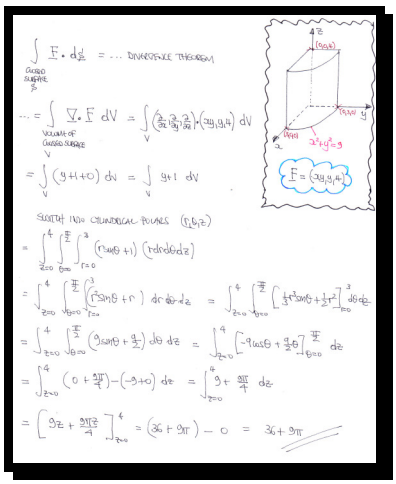
Evaluate the integral

$$\oiint_S \mathbf{F}\cdot\mathbf{dS}\,,$$

where S is the **closed** surface enclosing the finite region V, defined by

$$x^2 + y^2 \le 9\,, \quad x \ge 0\,, \quad y \ge 0\,, \quad 0 \le z \le 4\,.$$

$$\boxed{9\pi + 36}$$



Question 5

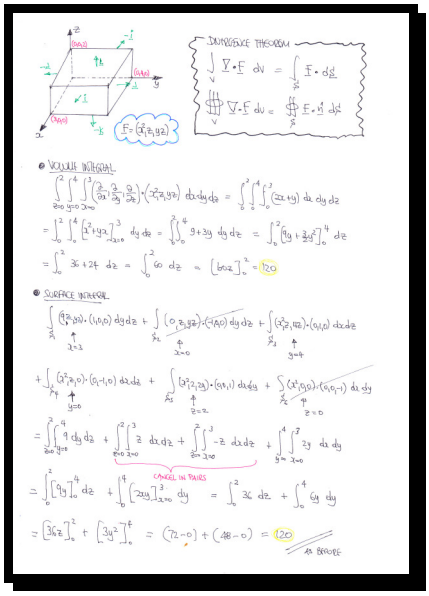
The vector field **F** exists inside and around the finite region V, defined by the inequalities

$$0 \le x \le 3, \quad 0 \le y \le 4 \quad \text{and} \quad 0 \le z \le 2.$$

Use V to verify the Divergence Theorem of Gauss, given further that

$$\mathbf{F}(x, y, z) \equiv x^2\mathbf{i} + z\mathbf{j} + yz\mathbf{k}.$$

both sides yield 120



Question 6

$$\mathbf{F}(x,y,z)\equiv\left(x+y^2\right)\mathbf{i}+\left(2y+xz\right)\mathbf{j}+\left(3z+xyz\right)\mathbf{k}\,.$$

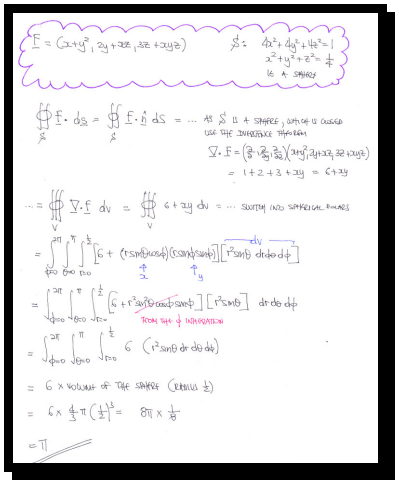
Evaluate the integral

$$\iint_S \mathbf{F}\cdot\mathbf{dS}\,,$$

where S is the surface with Cartesian equation

$$4x^2+4y^2+4z^2=1\,.$$

π



Question 7

A smooth vector field $\mathbf{A}$, exists in and on the boundary of a smooth closed surface S, and $\hat{\mathbf{n}}$ is an outward unit vector to S.

a) Show that

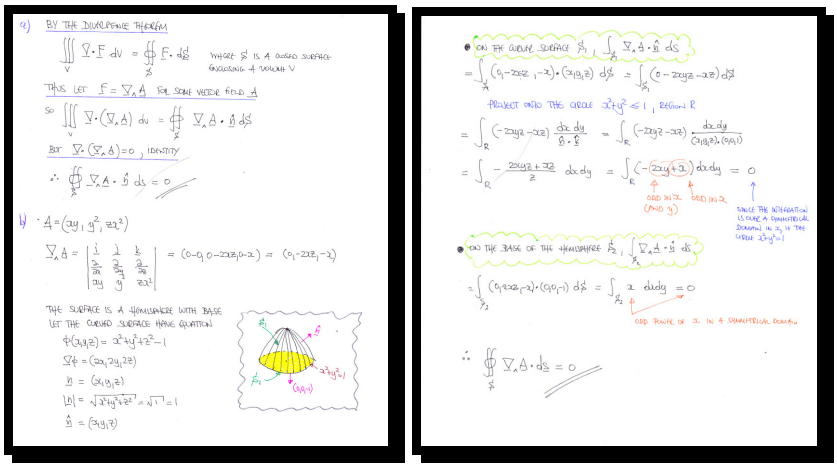
$$\int_S \nabla \wedge \mathbf{A} \cdot \hat{\mathbf{n}} \, dS = 0$$

You may find the Divergence Theorem useful in this part.

b) Prove the validity of the result of part **(a)** if

- $\mathbf{A} = xy\,\mathbf{i} + y^2\,\mathbf{j} + zx^2\,\mathbf{k}$
- $S : x^2 + y^2 + z^2 = 1,\ z \geq 0.$

proof



Question 8

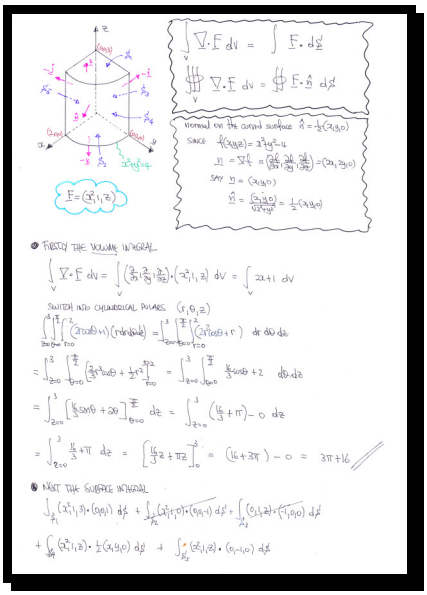
A vector field, $\mathbf{F}$, exists inside and around the finite region V, defined by

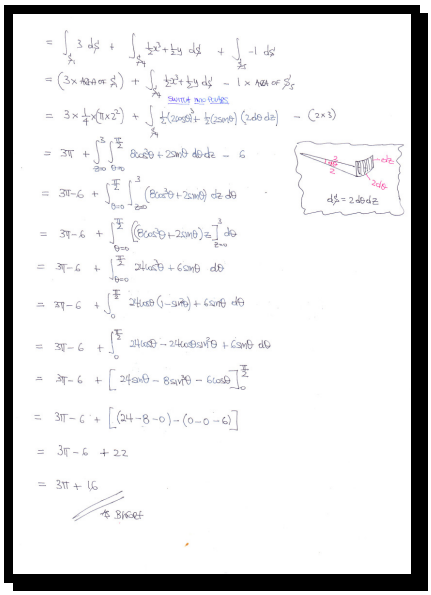
$$x^2+y^2=4, \quad x\geq 0, \quad y\geq 0, \quad 0\leq z\leq 3.$$

Use V to verify the Divergence Theorem of Gauss, given further that

$$\mathbf{F}(x,y,z)\equiv x^2\mathbf{i}+\mathbf{j}+z\mathbf{k}.$$

both sides yield $3\pi+16$





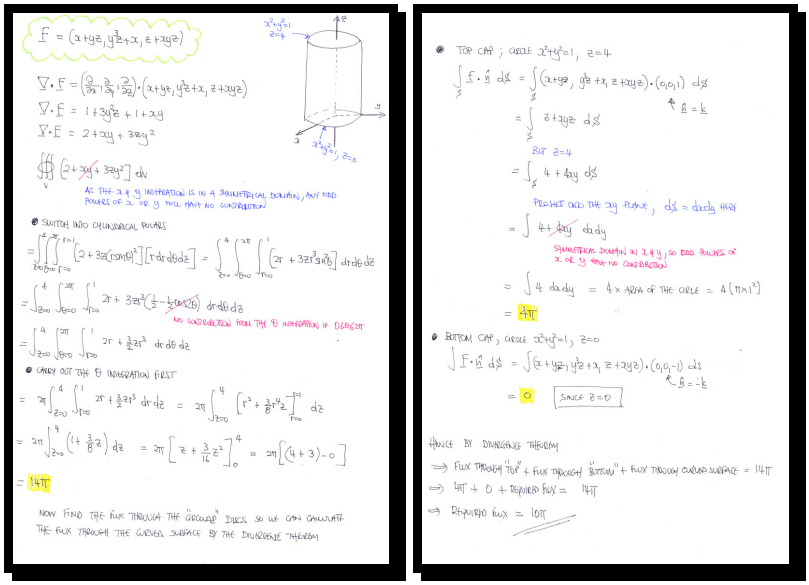
Question 9

$$\mathbf{F}(x,y,z)\equiv(x+yz)\mathbf{i}+\left(y^3z+x\right)\mathbf{j}+(z+xyz)\mathbf{k}$$

Use the Divergence Theorem of Gauss to find the flux through the **open** surface with Cartesian equation

$$x^2+y^2=1,\quad 0\le z\le 4.$$

$$\boxed{10\pi}$$



Question 10

A vector field, $\mathbf{F}$, exists inside and around the sphere S, with Cartesian equation

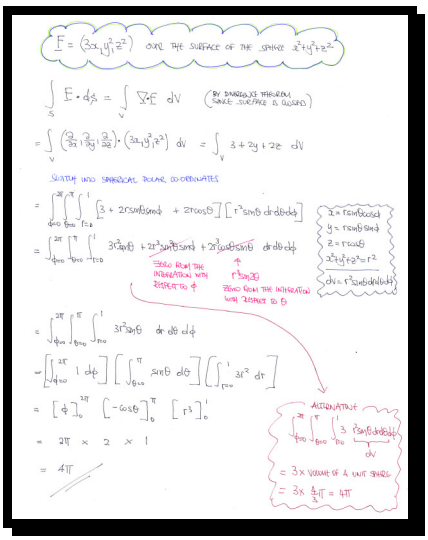
$$x^2+y^2+z^2=1.$$

Evaluate the surface integral

$$\int_S \mathbf{F}\cdot\mathbf{dS},$$

where $\mathbf{F}(x,y,z)=3x\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}$.

$\boxed{4\pi}$



Question 11

a) State Gauss' Divergence Theorem for closed surfaces, fully defining all the quantities involved.

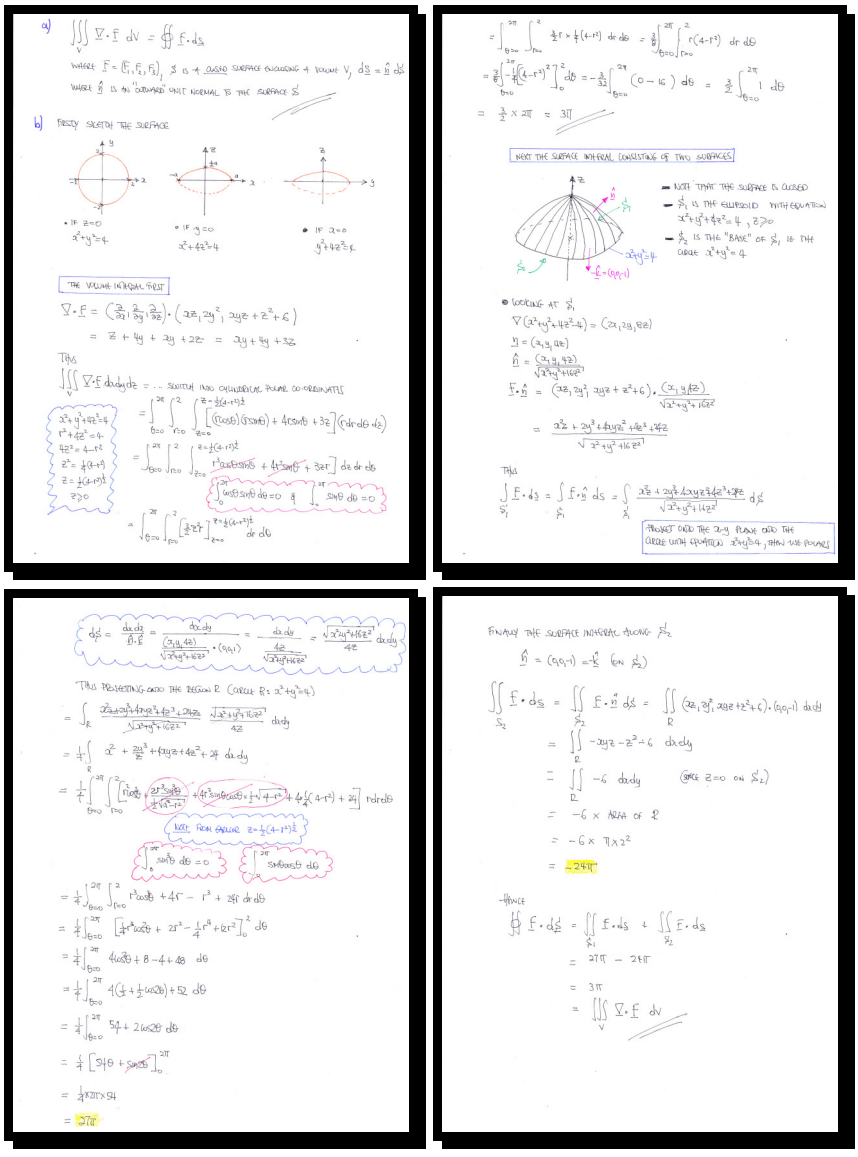
b) Verify Gauss' Divergence Theorem for closed surfaces for the vector field

$$\mathbf{F} = xz\mathbf{i} + 2y^2\mathbf{j} + \left(xyz + z^2 + 6\right)\mathbf{k}$$

for the **finite** region defined as

$$x^2 + y^2 + 4z^2 = 4\,,\ z \geq 0\,.$$

both sides yield 3π



Question 12

The region V is defined as

$$x^2+y^2+(z+4)^2 \le 25,\ z \ge 0.$$

a) Use cylindrical polar coordinates (r,θ,z) to find the volume of this region.

b) Use Gauss' Divergence Theorem for closed surfaces, with an appropriate vector field, to verify the answer obtained in part **(a)**

$$\boxed{\frac{14}{3}\pi}$$



Question 13

a) State Gauss' Divergence Theorem for closed surfaces, fully defining all the quantities involved.

b) Hence show that for a smooth scalar field $\varphi = \varphi(x, y, z)$,

$$\iiint_V \nabla\varphi \, dV = \oiint_S \varphi \hat{\mathbf{n}} \, dS ,$$

where S is a closed surface enclosing a volume V, and $\hat{\mathbf{n}}$ is an outward unit normal field to S.

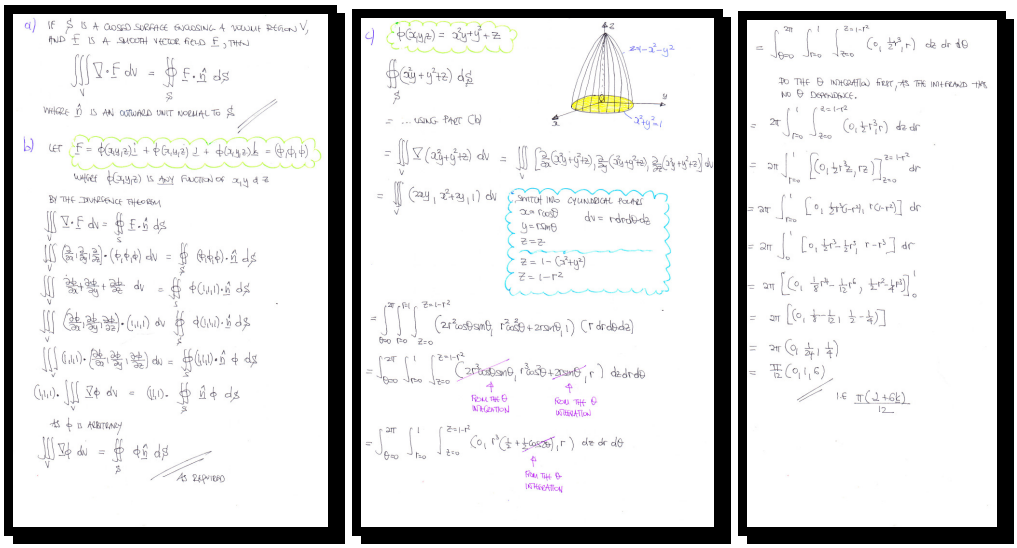
c) Evaluate

$$\oiint_S \left(x^2 y + y^2 + z\right)\hat{\mathbf{n}} \, dS ,$$

where S is the paraboloid with equation

$$z = 1 - x^2 - y^2 , \ z \geq 0 .$$

$$\boxed{\frac{\pi}{12}(\mathbf{j} + 6\mathbf{k})}$$



Question 14

a) State Gauss' Divergence Theorem for closed surfaces, fully defining all the quantities involved.

The vector field $\mathbf{E}$ s given as

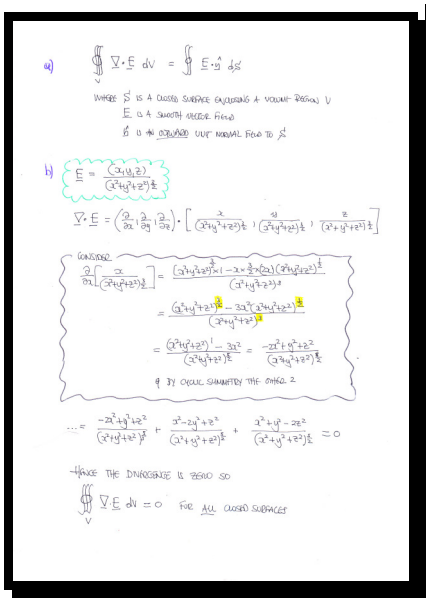
$$\mathbf{E}=\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\left(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\right).$$

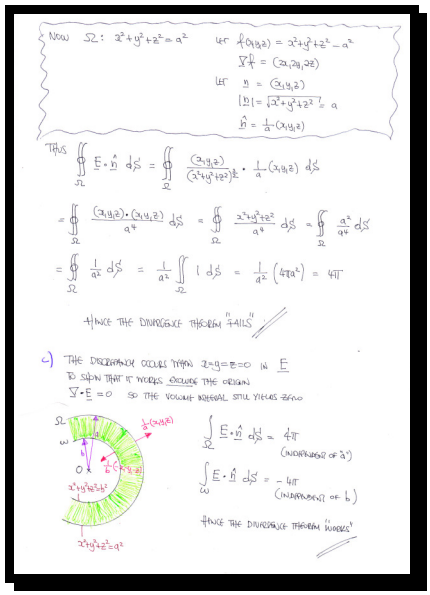
b) Show that Gauss' Divergence Theorem for closed surfaces "fails" for $\mathbf{E}$ and the surface with Cartesian equation

$$x^2+y^2+z^2=a^2\,,\ a>0\,.$$

c) Explain carefully why the theorem "fails".

proof





Question 15

The surface S is the sphere with Cartesian equation

$$x^2+y^2+z^2=4$$

a) By using Spherical Polar coordinates, (r,θ,φ), evaluate by direct integration the following surface integral

$$I=\iint_S \left(x^4+xy^2+z\right)dS\,.$$

b) Verify the answer of part **(a)** by using the Divergence Theorem.

$$\frac{256\pi}{5}$$



Question 16

The surface Ω is the sphere with Cartesian equation

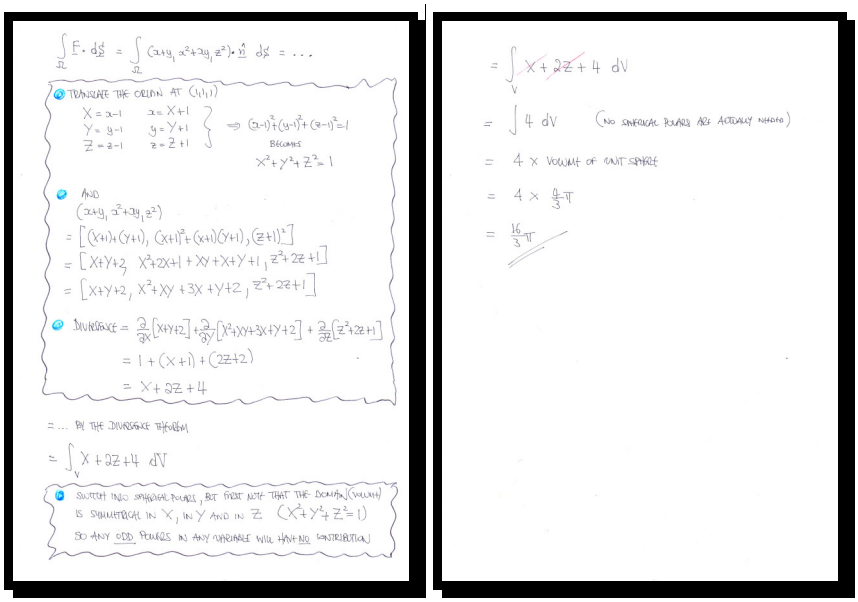
$$(x-1)^2+(y-1)^2+(z-1)^2=1$$

Use the Divergence Theorem to evaluate

$$\iint_{\Omega}\left[(x+y)\mathbf{i}+\left(x^2+xy\right)\mathbf{j}+z^2\mathbf{k}\right]\cdot\mathbf{dS},$$

where $\mathbf{dS}$ is a unit surface element on Ω.

$$\frac{16}{3}\pi$$



Question 17

The vector field $\mathbf{u}$ is given in spherical polar coordinates (r,θ,φ) by

$$\mathbf{u}(r,\theta,\varphi)=\left(r^2\cos^2\varphi\right)\hat{\mathbf{r}}+\left(r\cos^2\varphi\right)\hat{\boldsymbol{\varphi}}.$$

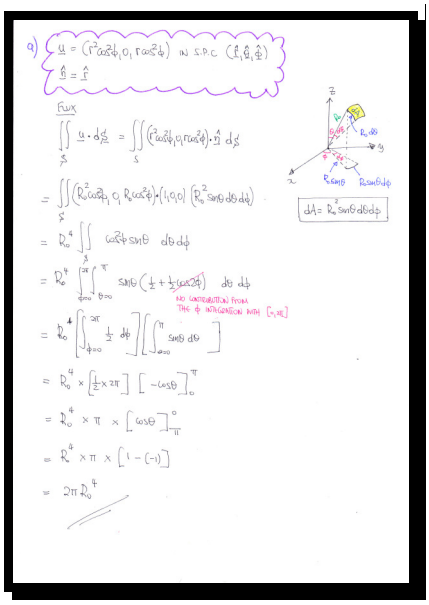
a) Find the flux of $\mathbf{u}$ through a spherical surface of radius R_0.

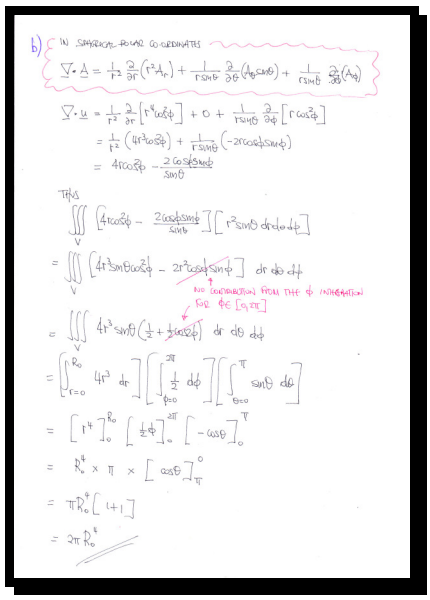
b) Verify the answer to part **(a)** by calculating an appropriate volume integral.

You may assume that in spherical polar coordinates

$$\nabla\cdot\left(A_r,A_\theta,A_\varphi\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(A_\varphi\right)$$

$$\boxed{2\pi R_0^4}$$





Question 18

a) State Gauss' Divergence Theorem for closed surfaces, fully defining all the quantities involved.

b) Hence show that for a smooth vector field $\mathbf{A}=\mathbf{A}(x,y,z)$, with $\nabla\cdot\mathbf{A}=0$,

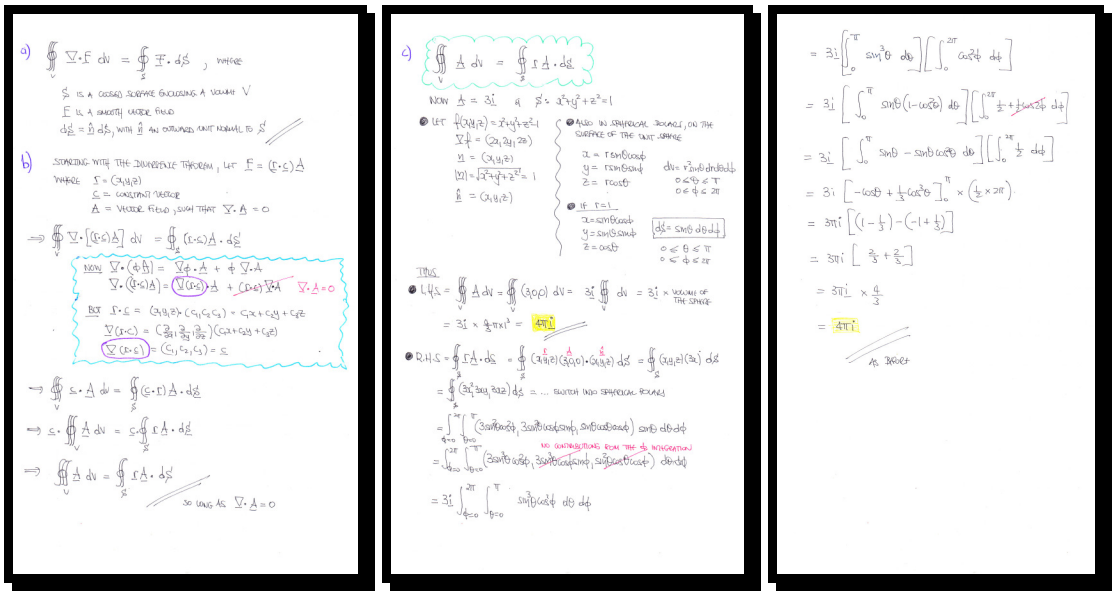
$$\iiint_V \mathbf{A}\,dV = \oiint_S \mathbf{r}\,\mathbf{A}\cdot\hat{\mathbf{n}}\,dS,$$

where S is a closed surface enclosing a volume V, $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, and $\hat{\mathbf{n}}$ is an outward unit normal field to S.

c) Verify the validity of the result of part **(b)** if $\mathbf{A}=3\mathbf{i}$ and S is the sphere with equation

$$x^2+y^2+z^2=1.$$

both sides yield $4\pi\mathbf{i}$



Stokes' Theorem

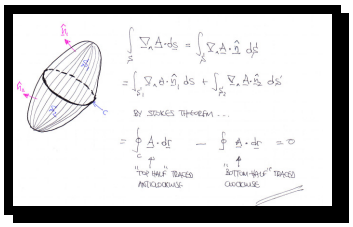
Question 1

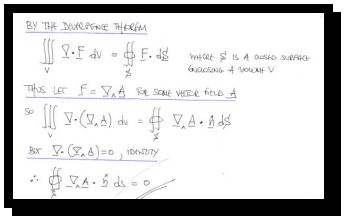
If $\mathbf{F}$ is a smooth vector field, S is a smooth closed surface, and $\hat{\mathbf{n}}$ is an outward unit normal vector to S, show that

$$\int_S \nabla \wedge \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = 0$$

You may find Stokes' Theorem or the Divergence Theorem useful in this question.

proof





Question 2

a) State Stokes' Integral Theorem for open surfaces, fully defining all the quantities involved.

b) Show that for a smooth scalar field φ and a constant vector $\mathbf{A}$

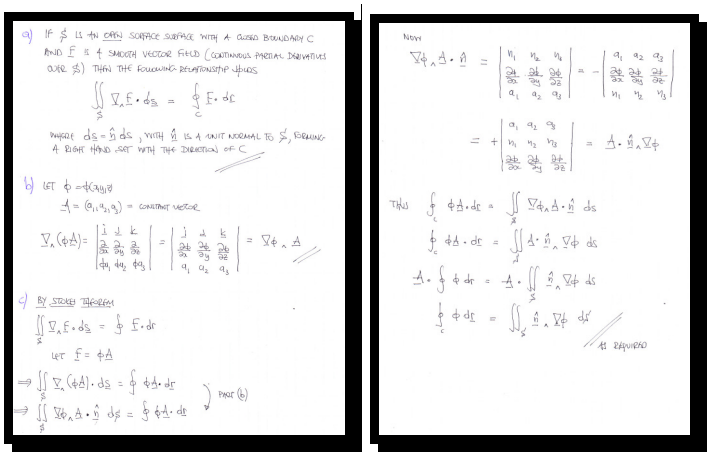
$$\nabla \wedge (\varphi \mathbf{A}) = \nabla \varphi \wedge \mathbf{A} .$$

The open smooth surface S has boundary c and unit normal field $\hat{\mathbf{n}}$.

c) Use part (a) and (b) to prove

$$\oint_c \varphi \, d\mathbf{r} = \int_S \hat{\mathbf{n}} \wedge \nabla \varphi \, dS .$$

proof



Question 3

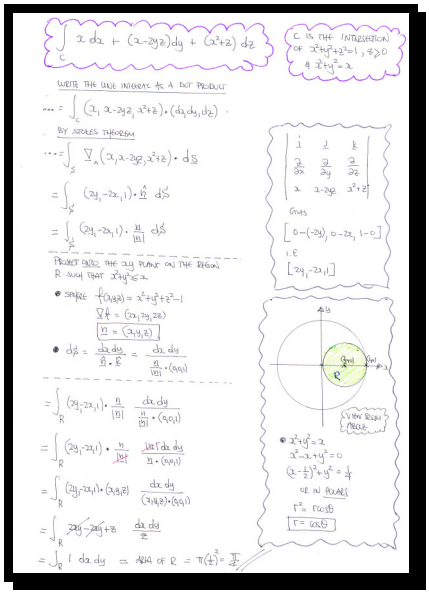
Evaluate the line integral

$$\oint_C \left[x\,dx + (x - 2yz)\,dy + \left(x^2 + z\right)dz \right],$$

where C is the intersection of the surfaces with respective Cartesian equations

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0 \qquad \text{and} \qquad x^2 + y^2 = x, \quad z \geq 0.$$

$$\frac{\pi}{4}$$



Question 4

It is given that the vector field $\mathbf{F}$ satisfies

$$\mathbf{F} = y^2\,\mathbf{i} + z^2\,\mathbf{j} + x^2\,\mathbf{k}\,.$$

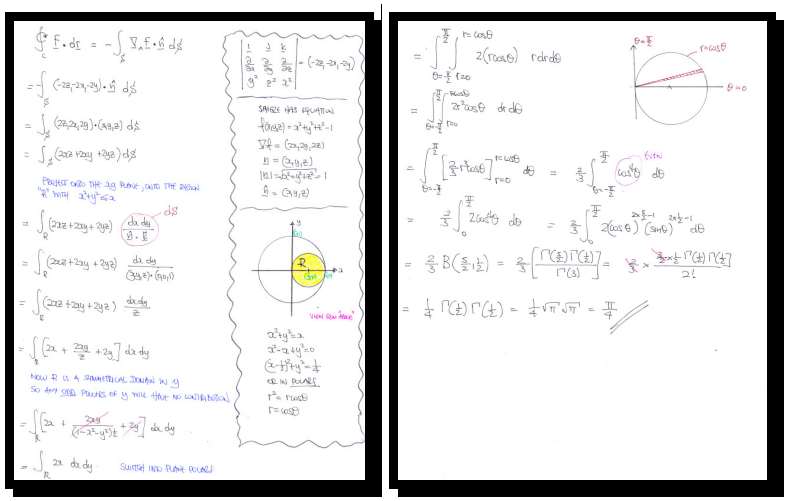
Evaluate the line integral

$$\oint_C \mathbf{F}\cdot\mathbf{dr}\,,$$

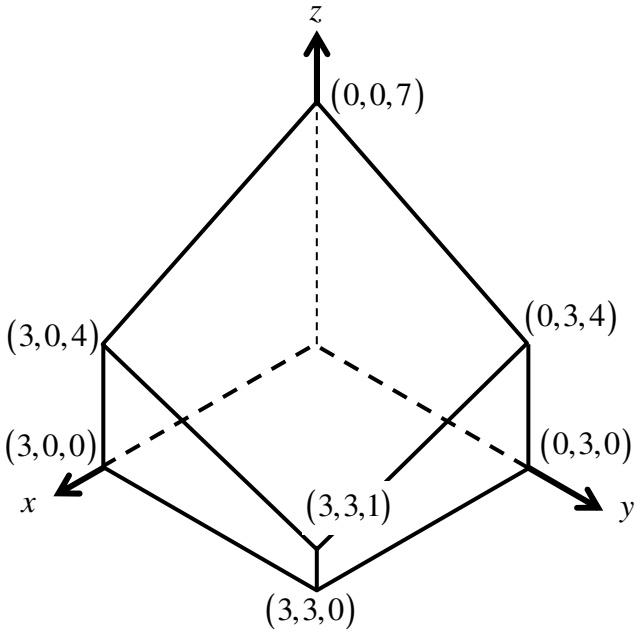
where C is the intersection of the surfaces with respective Cartesian equations

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0 \qquad \text{and} \qquad x^2 + y^2 = x, \quad z \geq 0\,.$$

$$\frac{\pi}{4}$$



Question 5



The figure above shows the finite region V defined by the intersection of the planes

$$x+y+z=7,\ x=3,\ y=3,\ x=0,\ y=0 \text{ and } z=0.$$

The open surface S encloses V except the plane face with equation $z=0$.

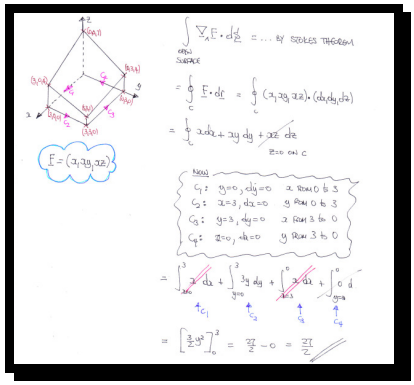
The vector field, $\mathbf{F}(x,y,z)\equiv x\mathbf{i}+xy\mathbf{j}+xz\mathbf{k}$, exists on and around S.

Evaluate the surface integral

$$\int_S \nabla\wedge\mathbf{F}\cdot\mathbf{dS},$$

where $\mathbf{dS}=\hat{\mathbf{n}}\,dS$, where $\hat{\mathbf{n}}$ is an outward unit normal vector to S.

$$\boxed{\int_S \nabla\wedge\mathbf{F}\cdot\mathbf{dS}=\frac{27}{2}}$$



Question 6

a) State Stokes' Integral Theorem for open two sided surfaces, fully defining all the quantities involved.

The vector field

$$\mathbf{v} = yz\,\mathbf{k}$$

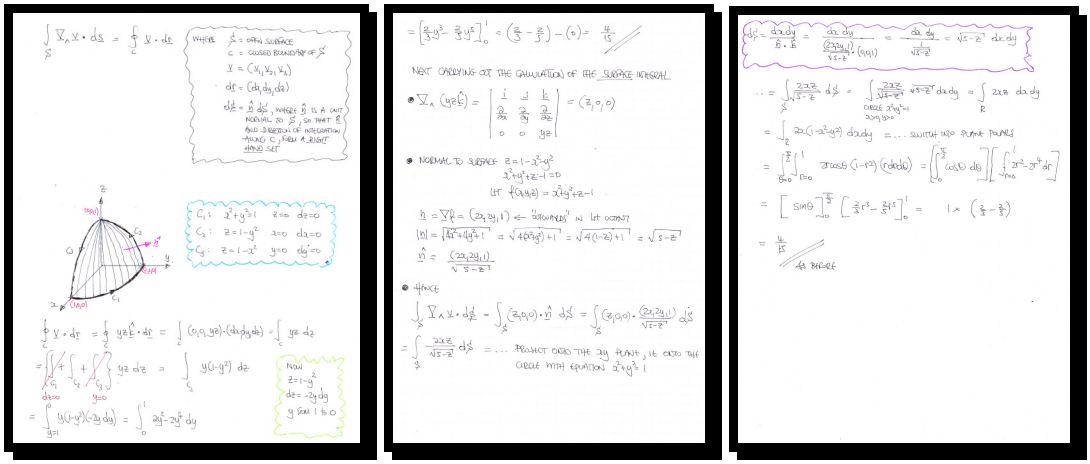
exists around the open surface S, with closed boundary C.

The equation of S is

$$z = 1 - x^2 - y^2,\ x \geq 0,\ y \geq 0,\ z \geq 0.$$

b) Use $\mathbf{v}$ and S to verify the validity of Stokes' Theorem.

both sides yield $\frac{4}{15}$



Question 7

The vector field

$$\mathbf{F} = z\,\mathbf{i} + xy\,\mathbf{j} + xz\,\mathbf{k}$$

exists around the open two sided surface S, with closed boundary C.

S is defined as

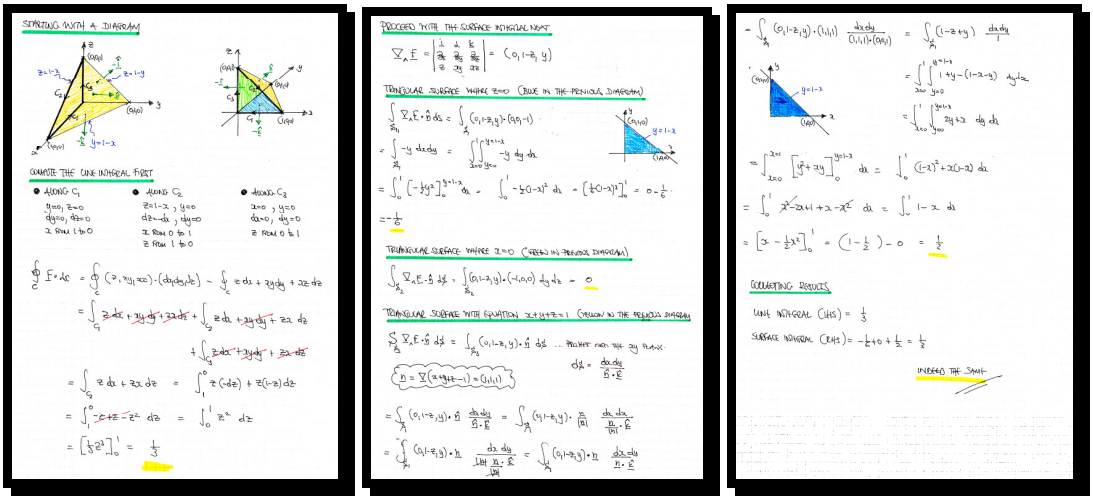
- $x + y + z = 1,\ x \geq 0,\ y \geq 0,\ z \geq 0.$
- $x = 0,\ z \leq 1 - y,\ y \geq 0,\ z \geq 0.$
- $z = 0,\ y \leq 1 - x,\ x \geq 0,\ y \geq 0.$

Show that

$$\oint_C \mathbf{F}\cdot\mathbf{dr} = \int_S \nabla \wedge \mathbf{F}\cdot\hat{\mathbf{n}}\, dS$$

where $\hat{\mathbf{n}}$ is an outward unit normal to S.

, both sides yield $\frac{1}{3}$



Question 8

It is given that the vector field $\mathbf{F}$ satisfies

$$\mathbf{F} = 8z\,\mathbf{i} + 4x\,\mathbf{j} + y\,\mathbf{k}\,.$$

Evaluate the line integral

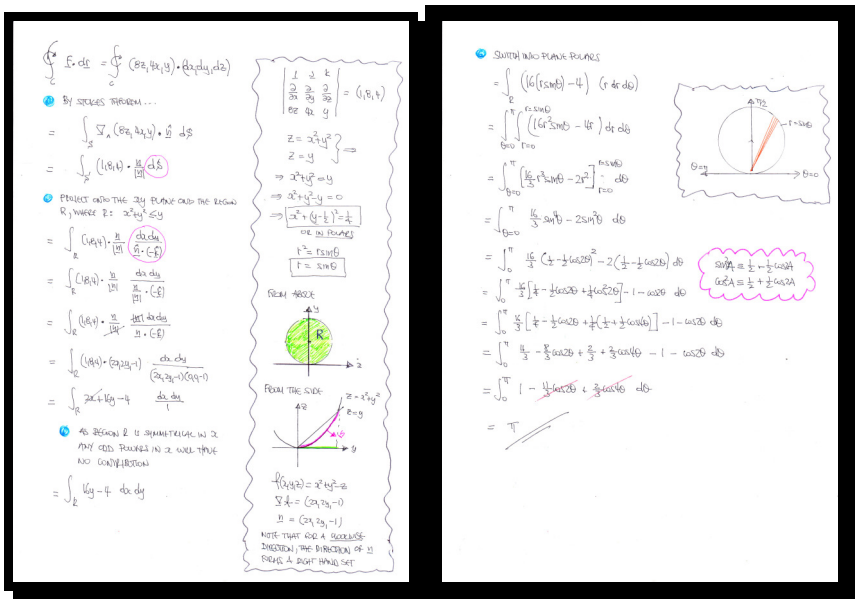
$$\oint_C \mathbf{F}\cdot\mathbf{dr}\,,$$

where C is the intersection of the surfaces with respective Cartesian equations

$$z = y^2 + x^2 \qquad \text{and} \qquad x^2 + y^2 = y\,, \quad z \geq 0\,.$$

You may find Stokes' Theorem useful in this question.

π



Question 9

The surface S has Cartesian equation

$$(z-1)^2 = x^2 + y^2, \quad 1 \le z \le 3.$$

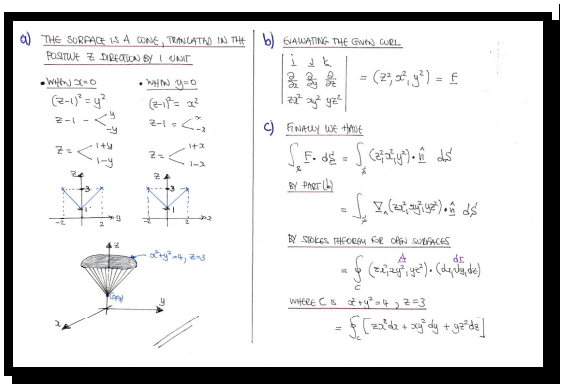
a) Sketch the graph of S.

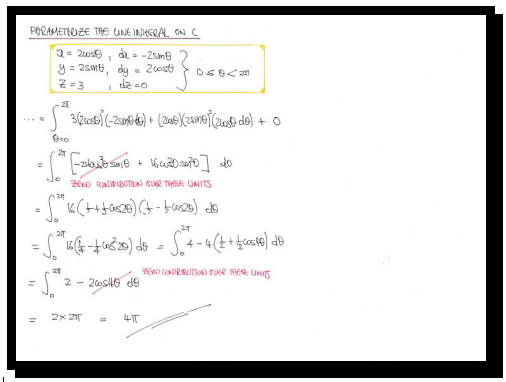
b) Evaluate $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ zx^2 & xy^2 & yz^2 \end{vmatrix}$.

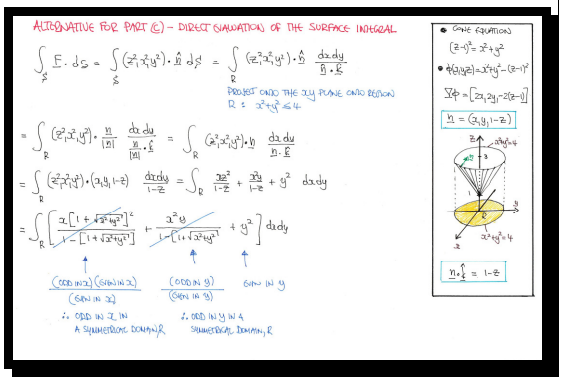
c) Given that $\mathbf{F} = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$, evaluate the integral

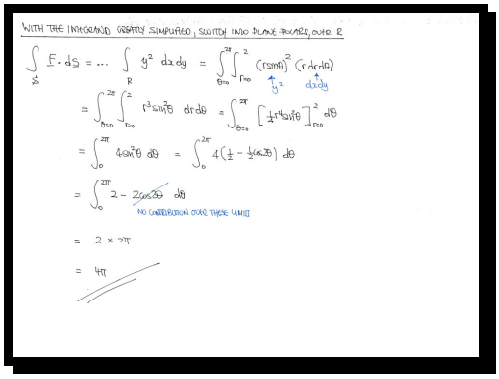
$$\int_S \mathbf{F} \cdot \mathbf{dS}.$$

[ ], 4π









Question 10

The vector field $\mathbf{F}$ exists around the open surface S, with closed boundary C.

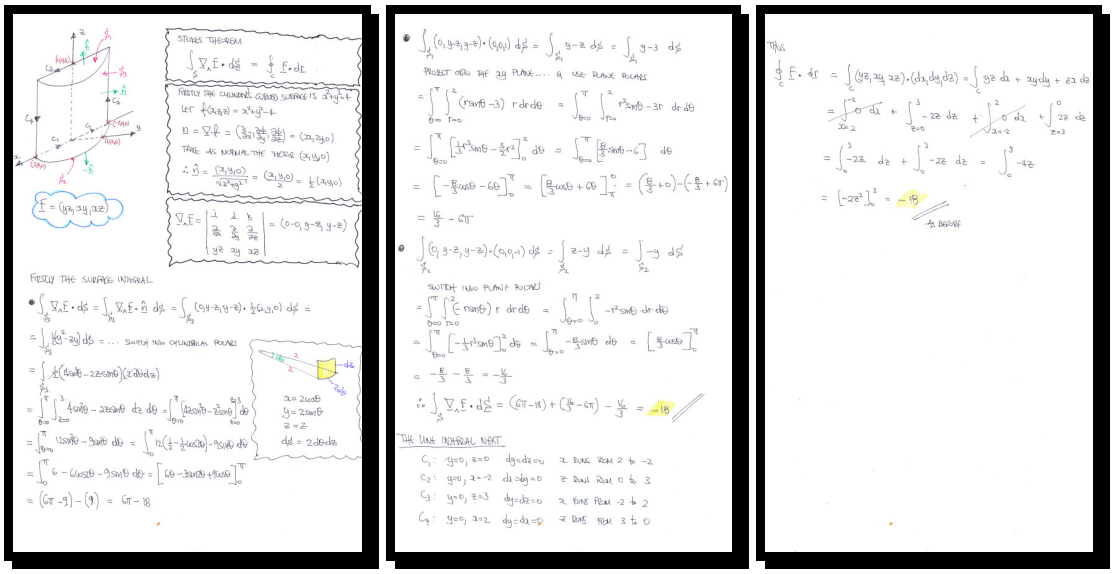
The open surface consists of the following three faces.

- The cylindrical surface $x^2+y^2=4$, $y\geq 0$ and $0\leq z\leq 3$.
- The plane face $x^2+y^2=4$, $y\geq 0$ and $z=0$.
- The plane face $x^2+y^2=4$, $y\geq 0$ and $z=3$.

Use S and C to verify Stokes' Theorem, given further that

$$\mathbf{F}(x,y,z)\equiv yz\mathbf{i}+xy\mathbf{j}+xz\mathbf{k}\,.$$

both sides yield -18



Question 11

It is given that the vector field $\mathbf{F}$ satisfies

$$\mathbf{F}=8z\,\mathbf{i}+4x\,\mathbf{j}+y\,\mathbf{k}\,.$$

Evaluate the line integral

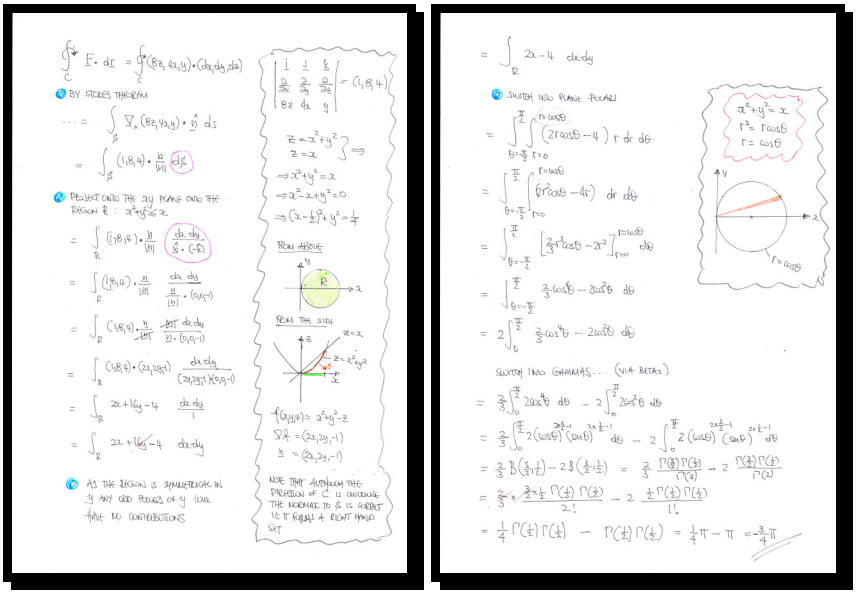
$$\oint_C \mathbf{F}\cdot\mathbf{dr}\,,$$

where C is the intersection of the surfaces with respective Cartesian equations

$$z=x^2+y^2 \qquad \text{and} \qquad x^2+y^2=x\,,\quad z\geq 0\,.$$

You may find Stokes' Theorem useful in this question.

$$\boxed{-\frac{3\pi}{4}}$$



Question 12

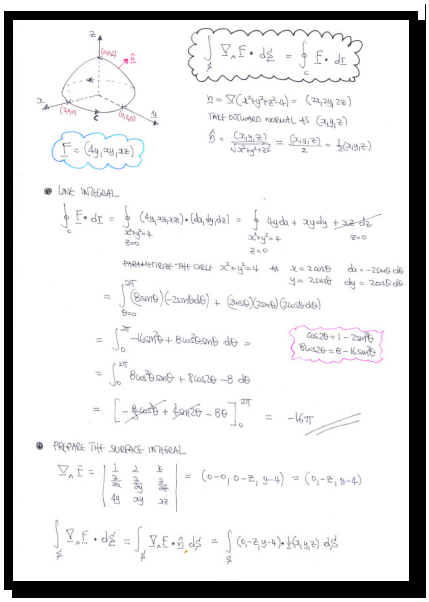
The vector field $\mathbf{F}$ exists around the open surface S, with closed boundary C, whose equation satisfies

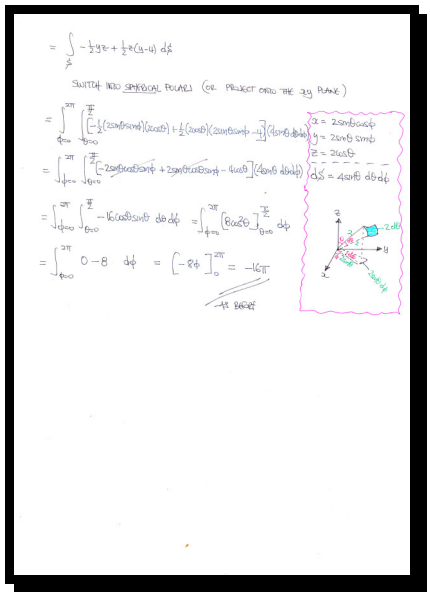
$$x^2+y^2+z^2=4,\ z\geq 0.$$

Use S and C to verify Stokes' Theorem, given further that

$$\mathbf{F}(x,y,z)\equiv 4y\mathbf{i}+xy\mathbf{j}+xz\mathbf{k}.$$

both sides yield -16π





Question 13

The vector field $\mathbf{A}$ exists around the open surface S, with closed boundary C.

$$\mathbf{A}=\left(x^2y\right)\mathbf{i}+\left(xy+xyz\right)\mathbf{j}+\left(xy+xz^2\right)\mathbf{k}$$

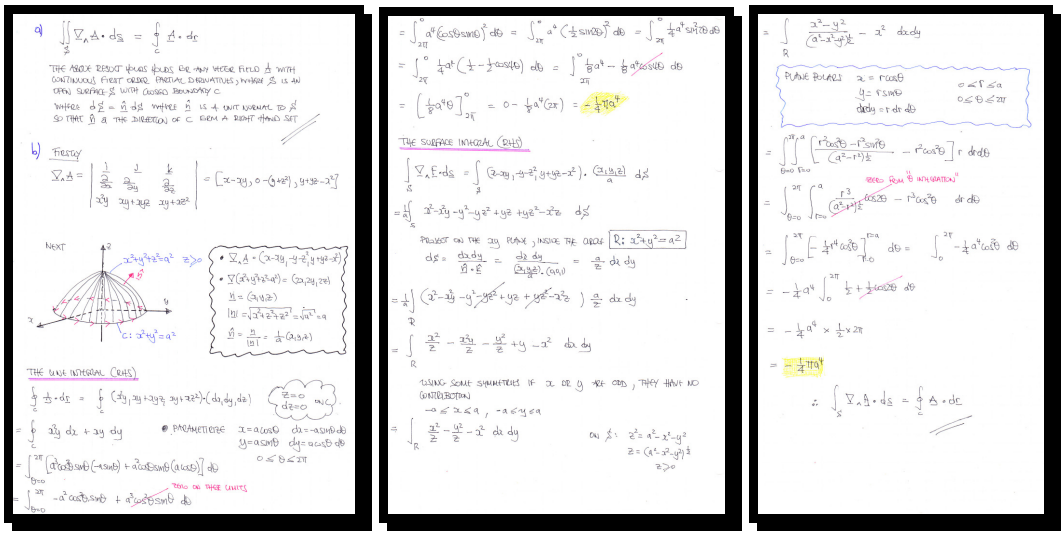
a) State Stokes' Integral Theorem for open surfaces, fully defining all the quantities involved.

The Cartesian equation of S is

$$x^2+y^2+z^2=a^2,\quad a>0,\quad z\geq 0.$$

b) Use $\mathbf{A}$ and S to verify the validity of Stokes' Theorem.

both sides yield $-\frac{1}{4}\pi a^4$



Question 14

The smooth vector field $\mathbf{F}$ exists around the open, two sided, surface S, with closed boundary C.

a) State Stokes' Integral Theorem for open surfaces, fully defining all the quantities involved.

b) Hence show, that if φ is a smooth scalar field defined everywhere, and C is any path between two fixed points, then

$$\int_C \nabla\varphi\cdot\mathbf{dr},$$

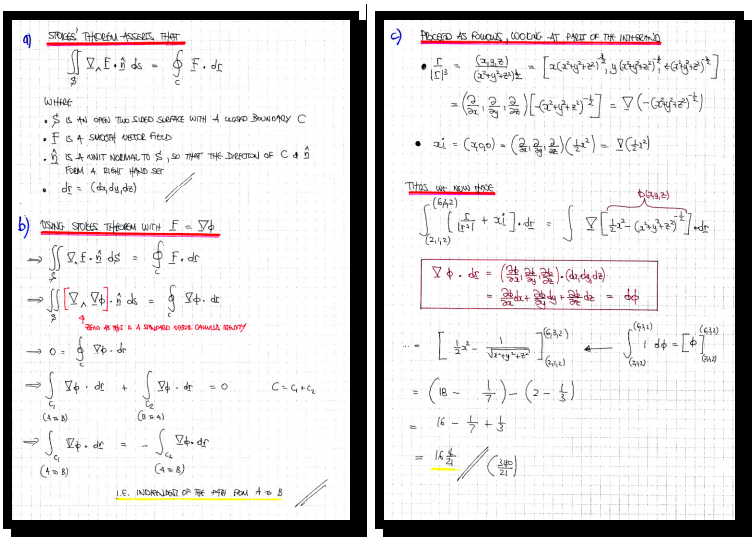
is independent of the path of C.

c) Given further that $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ evaluate

$$\int_C\left[\frac{\mathbf{r}}{|\mathbf{r}|^3}+x\mathbf{i}\right]\cdot\mathbf{dr},$$

where C is the straight line segment from $(2,1,2)$ to $(6,3,2)$.

$$\boxed{},\ \boxed{\frac{340}{21}}$$



Question 15

The smooth vector field $\mathbf{F}$ exists around the open, two sided, surface S, with closed boundary C.

a) State Stokes' Integral Theorem for open surfaces, fully defining all the quantities involved.

b) Hence show that

$$\int_S \hat{\mathbf{n}} \wedge \nabla \varphi \, dS = \oint_C \varphi \, d\mathbf{r},$$

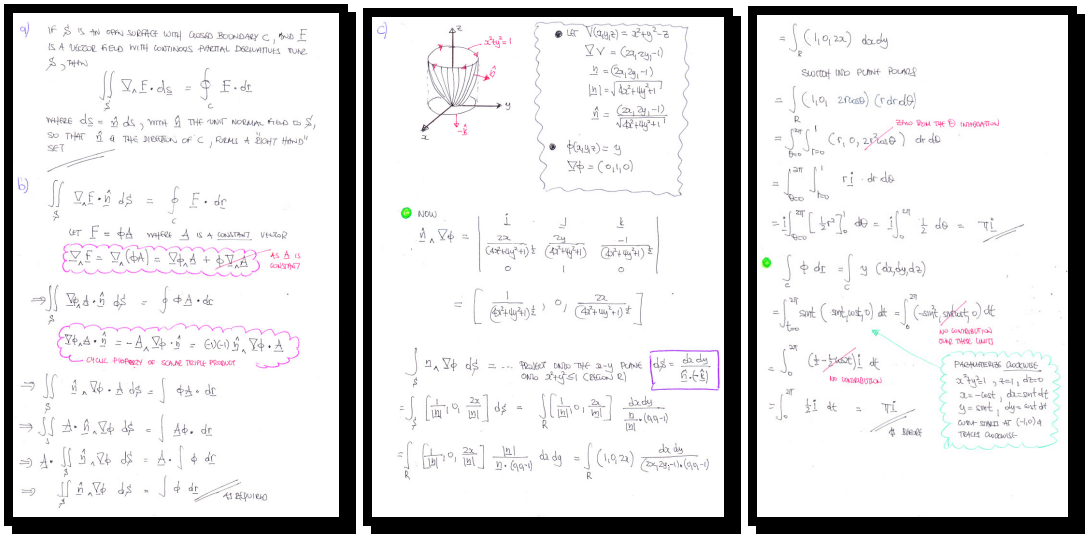
where φ is a smooth scalar function and $\hat{\mathbf{n}}$ is a unit normal vector to S.

The Cartesian equation of S is

$$z = x^2 + y^2, \quad z \leq 1.$$

c) Use $\varphi(x, y, z) = y$ and S to verify the result of part **(b)**.

both sides yield $\pi\mathbf{i}$



Question 16

The vector field $\mathbf{F}$ exists around the open surface S, with closed boundary C.

a) State Stokes' Integral Theorem for open surfaces, fully defining all the quantities involved.

b) Hence show that

$$\int_S \hat{\mathbf{n}} \wedge \nabla \varphi \, dS = \oint_C \varphi \, d\mathbf{r},$$

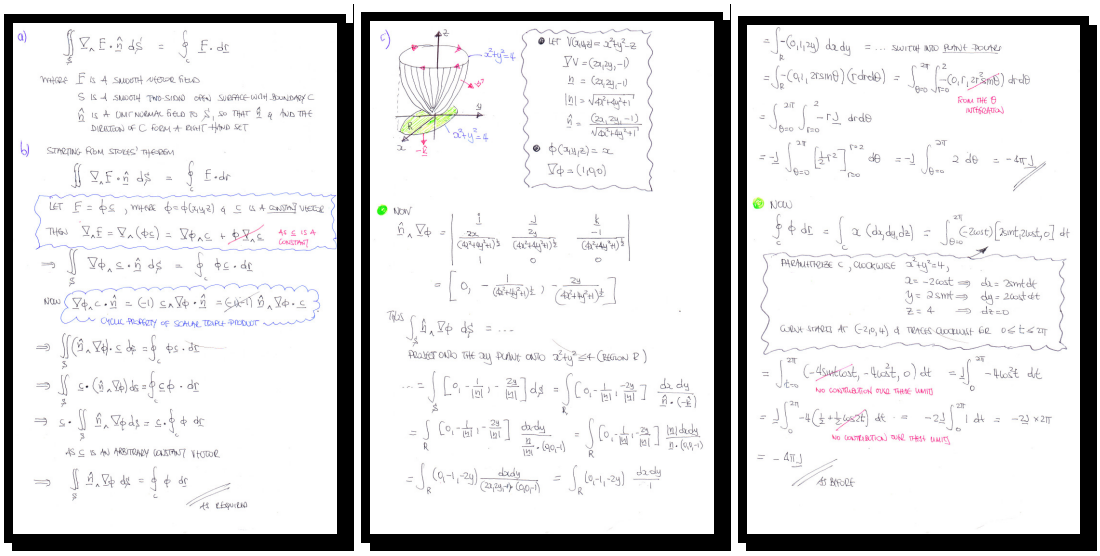
where φ is a smooth scalar function and $\hat{\mathbf{n}}$ is unit normal vector to S.

The Cartesian equation of S is

$$z = x^2 + y^2, \quad z \le 4.$$

c) Use $\varphi(x, y, z) = x$ and S to verify the result of part **(b)**.

both sides yield $-4\pi\mathbf{j}$



Question 17

$\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are vector fields.

a) Prove the validity of the vector identity

$$\mathbf{A}\wedge(\mathbf{B}\wedge\mathbf{C})\equiv\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}.$$

b) Given further that $\mathbf{c}$ is a constant vector and $\mathbf{A}$ a smooth vector field, find a simplified expression for

$$\nabla\wedge(\mathbf{c}\wedge\mathbf{A}).$$

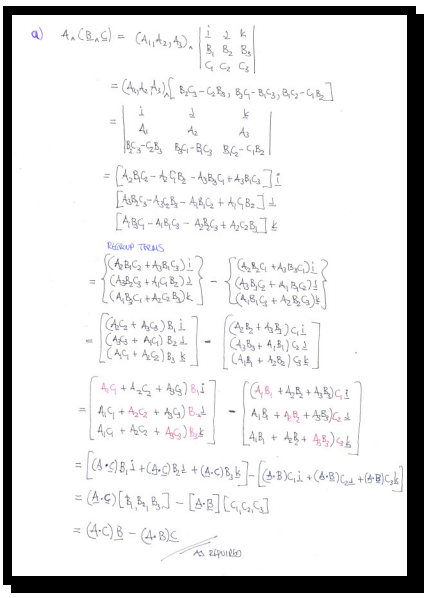
An open two sided surface S has boundary C.

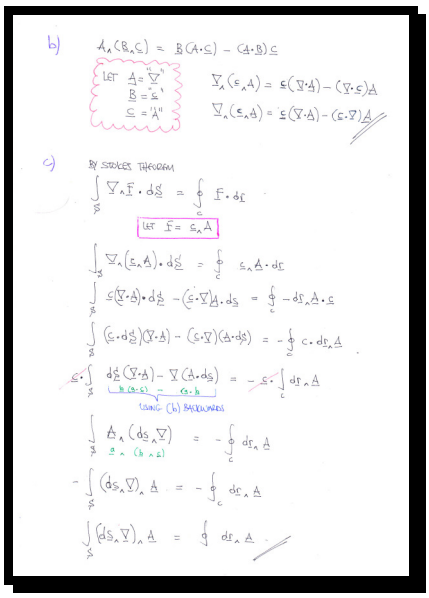
c) Use Stokes' Integral Theorem and the result obtained in part **(b)** to show that

$$\int_S(\mathbf{dS}\wedge\nabla)\wedge\mathbf{A}\ =\ \oint_C d\mathbf{r}\wedge\mathbf{A},$$

where $\mathbf{dS}=\hat{\mathbf{n}}\,dS$ with $\hat{\mathbf{n}}$ a unit normal vector to S, and $\mathbf{dr}=\mathbf{i}\,dx+\mathbf{j}\,dy+\mathbf{k}\,dz$.

$$\boxed{\nabla\wedge(\mathbf{c}\wedge\mathbf{A})=\mathbf{c}(\nabla\cdot\mathbf{A})-(\mathbf{c}\cdot\nabla)\mathbf{A}}$$





Question 18

An open two sided surface S has boundary C.

It is further given that $\mathbf{a}$ is a constant vector and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

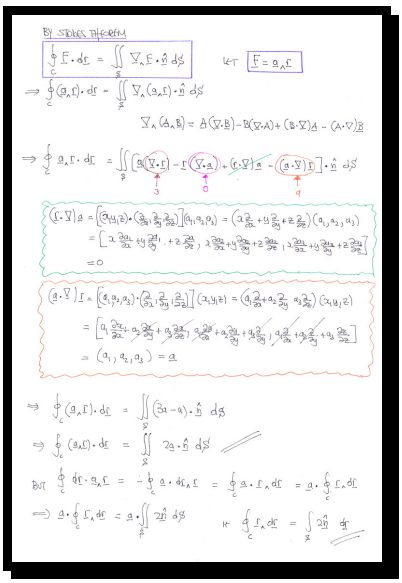
Show that

a) $$\int_S 2\mathbf{a}\cdot\hat{\mathbf{n}}\ dS = \oint_C \mathbf{a}\wedge\mathbf{r}\cdot d\mathbf{r}.$$

b) $$\int_S 2\hat{\mathbf{n}}\ dS = \oint_C \mathbf{r}\wedge d\mathbf{r}.$$

where $\hat{\mathbf{n}}$ a unit normal vector to S, and $\mathbf{dr} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz$.

proof



Question 19

$$\mathbf{A}=2\mathbf{i}-\mathbf{j}+(4y-3)\mathbf{k}\,.$$

The vector field $\mathbf{A}$ exist around the surface S with Cartesian equation

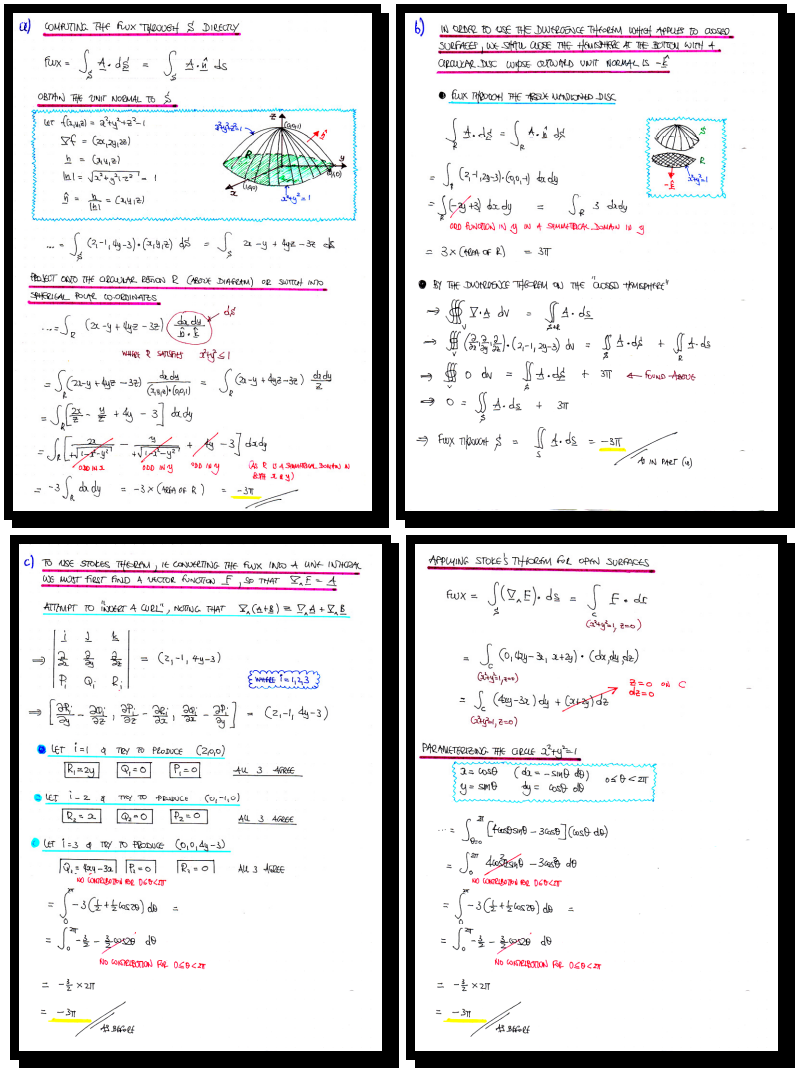
$$x^2+y^2+z^2=1,\ z\geq 0\,.$$

a) Determine the flux of $\mathbf{A}$ through S, where the normal unit field to S is denoted by $\hat{\mathbf{n}}$, such that $\hat{\mathbf{n}}\cdot\mathbf{k}\geq 0$.

b) Obtain the answer of part **(a)** by using the Divergence Theorem.

c) Use Stokes' Theorem to get an expression for the flux of $\mathbf{A}$ through S, as a line integral, and hence verify the answer of part **(a)**.

$\boxed{\phantom{\text{flux}=-3\pi}}$, $\boxed{\text{flux}=-3\pi}$



Created by T. Madas