BESSEL EQUATION
and
BESSEL FUNCTIONS

Summary of Bessel Functions

Bessel's Equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-n^2\right)y=0.$$

If n is an integer, the two independent solutions of Bessel's Equation are

- $J_n\left(x\right)$, Bessel function of the first kind,

$$J_n\left(x\right)=\sum_{p=0}^{\infty}\left[\frac{(-1)^p}{(n+p)!\,p!}\left(\frac{x}{2}\right)^{2p+n}\right]$$

Generating function for $J_n\left(x\right)$

$$\mathrm{e}^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}\left[t^nJ_n\left(x\right)\right]$$

- $Y_n\left(x\right)$, Bessel function of the second kind

$$Y_n\left(x\right)=\frac{2}{\pi}\ln\left(\tfrac{1}{2}x\right)J_n\left(x\right)-\frac{1}{\pi}\left(\tfrac{1}{2}x\right)^{-n}\sum_{p=0}^{n-1}\left[\frac{(n-1-p)!}{p!}\left(\tfrac{1}{2}x\right)^{2p}\right]$$

$$+\frac{1}{\pi}\left(\tfrac{1}{2}x\right)^{n}\sum_{p=0}^{n-1}\left[\frac{(-1)^p}{(p+n)!\,p!}\left(\tfrac{1}{2}x\right)^{2p}\left[2\gamma-\sum_{m=1}^{p}\left(\frac{1}{m}\right)-\sum_{m=1}^{p+n}\left(\frac{1}{m}\right)\right]\right]$$

Other relations for $J_n(x)$, $n \in \mathbb{Z}$.

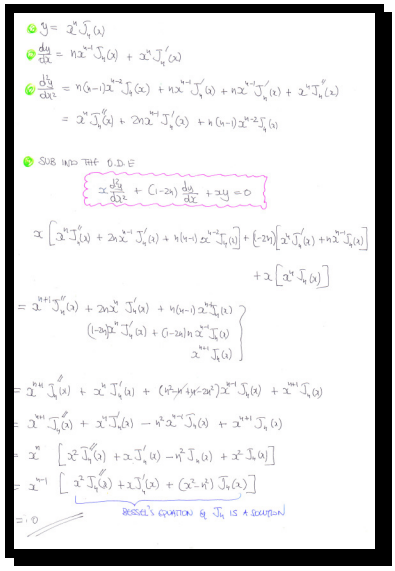
- $J_{-n}(x) = (-1)^n J_n(x)$.
- $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x) \right]$
- $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x) \right]$
- $J_n(x) = \sum_{p=0}^{\infty} \left[\frac{(-1)^p}{(n+p)!\, p!} \left(\frac{x}{2} \right)^{2p+n} \right]$
- $J_0(x) = \sum_{p=0}^{\infty} \left[\frac{(-1)^p}{(p!)^2} \left(\frac{x}{2} \right)^{2p} \right] = 1 - \frac{x^2}{2^2 (1!)^2} + \frac{x^4}{2^4 (2!)^2} - \frac{x^6}{2^6 (3!)^2} + \ldots$
- $J_1(x) = \sum_{p=0}^{\infty} \left[\frac{(-1)^p}{(p+1)!\, p!} \left(\frac{x}{2} \right)^{2p+1} \right] = \frac{x}{2^1 0!1!} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} - \frac{x^7}{2^7 3!4!} + \ldots$

Question 1

$$x\frac{d^2y}{dx^2}+(1-2n)\frac{dy}{dx}+xy=0\,,\quad x\neq 0\,.$$

Show that $x^n J_n(x)$ is a solution of the above differential equation.

proof



Question 2

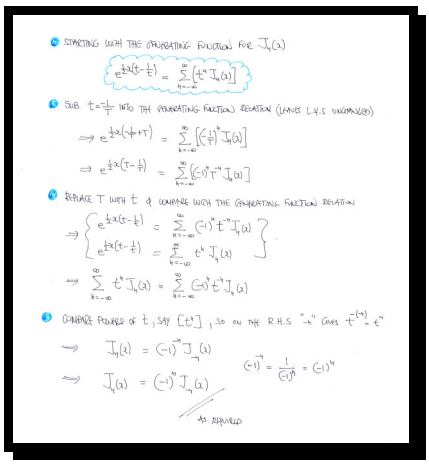
Starting from the generating function of the Bessel function of the first kind

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}\left[t^n J_n\left(x\right)\right],\ n\in\mathbb{Z},$$

show that

$$J_n\left(x\right)=\left(-1\right)^n J_{-n}\left(x\right).$$

proof



Question 3

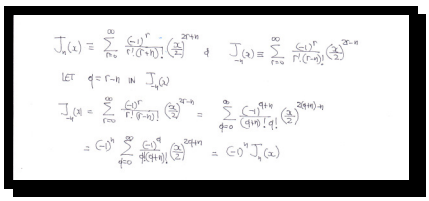
Starting from the series definition of the Bessel function of the first kind

$$J_n(x)=\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(n+r)!r!}\left(\frac{x}{2}\right)^{2r+n}\right],\ n\in\mathbb{Z},$$

show that

$$J_{-n}(x)=(-1)^n J_n(x).$$

proof



Question 4

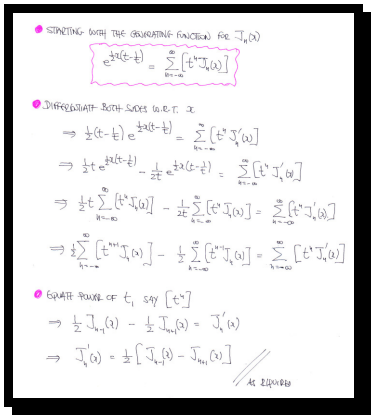
Starting from the generating function of the Bessel function of the first kind

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}\left[t^n J_n\left(x\right)\right],\ n\in\mathbb{Z},$$

show that

$$J_n'\left(x\right)=\frac{1}{2}\left[J_{n-1}\left(x\right)-J_{n+1}\left(x\right)\right].$$

proof



Question 5

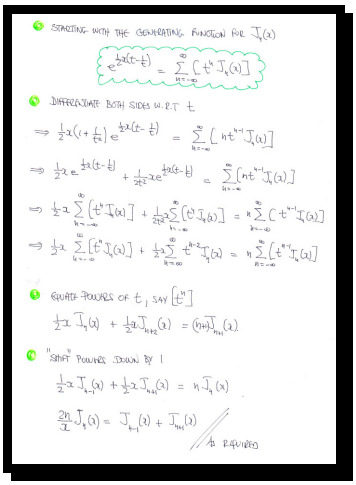
Starting from the generating function of the Bessel function of the first kind

$$\mathrm{e}^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}\left[t^n J_n\left(x\right)\right],\ n\in\mathbb{Z},$$

show that

$$\frac{2n}{x}J_n\left(x\right)=J_{n-1}\left(x\right)+J_{n+1}\left(x\right).$$

proof



Question 6

The generating function of the Bessel function of the first kind is

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right],\ n\in\mathbb{Z}.$$

a) By differentiating the generating function relation with respect to x, show that

$$\frac{1}{2}J_{n-1}(x)-\frac{1}{2}J_{n+1}(x)=J_n'(x).$$

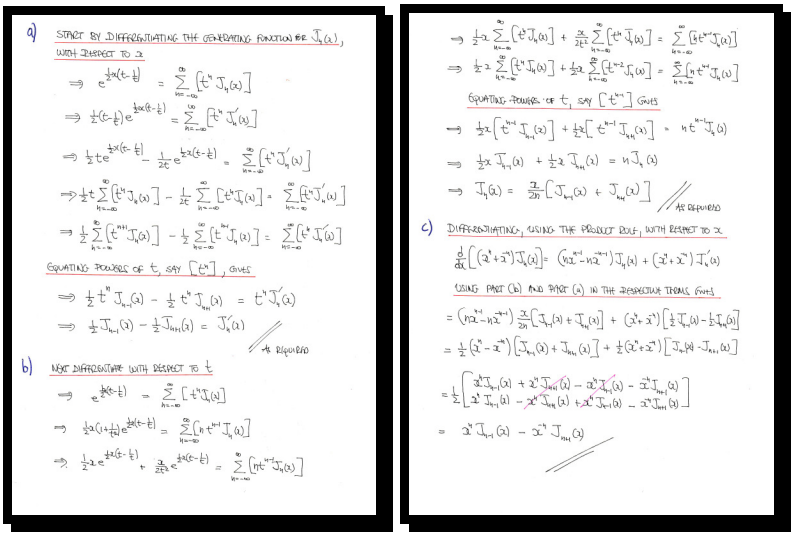
b) By differentiating the generating function relation with respect to t, show that

$$J_n(x)=\frac{x}{2n}\left[J_{n-1}(x)+J_{n+1}(x)\right].$$

c) Hence find a simplified expression for

$$\frac{d}{dx}\left[\left(x^n+x^{-n}\right)J_n(x)\right].$$

$\square$, $\boxed{\frac{d}{dx}\left[\left(x^n+x^{-n}\right)J_n(x)\right]=x^n J_{n-1}(x)-x^{-n}J_{n+1}(x)}$



Question 7

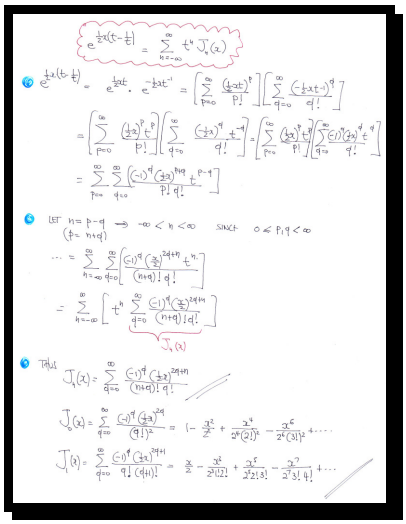
Starting from the generating function of the Bessel function of the first kind

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right],\ n\in\mathbb{Z},$$

determine the series expansion of $J_n(x)$, and hence show that

- $J_0(x)=1-\dfrac{x^2}{2^2(1!)^2}+\dfrac{x^4}{2^4(2!)^2}-\dfrac{x^6}{2^6(3!)^2}+\ldots$
- $J_1(x)=\dfrac{x}{2^1 0!1!}-\dfrac{x^3}{2^3 1!2!}+\dfrac{x^5}{2^5 2!3!}-\dfrac{x^7}{2^7 3!4!}+\ldots$

$$\boxed{J_n(x)=\sum_{p=0}^{\infty}\left[\frac{(-1)^p}{(n+p)!p!}\left(\frac{x}{2}\right)^{2p+n}\right]}$$



Question 8

The generating function of the Bessel function of the first kind is

$$\mathrm{e}^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}\left[t^n J_n\left(x\right)\right],\ n\in\mathbb{Z}.$$

Use the generating function relation, to show that for $n\geq 0$

a) $J_{-n}\left(x\right)=\left(-1\right)^n J_n\left(x\right)$

b) $J_{n-1}\left(x\right)-J_{n+1}\left(x\right)=2J_n'\left(x\right).$

c) $J_{n-1}\left(x\right)+J_{n+1}\left(x\right)=\dfrac{2n}{x}J_n\left(x\right).$

Use parts **(b)** and **(c)** to find simplified expressions for

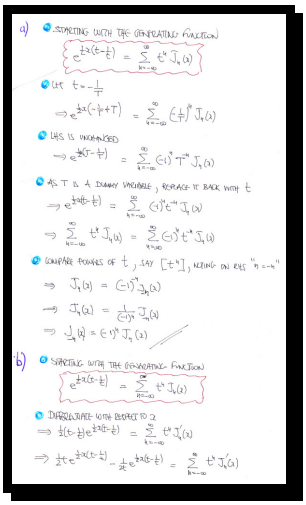
d) $\dfrac{d}{dx}\left[x^n J_n\left(x\right)\right].$

e) $\dfrac{d}{dx}\left[x^{-n} J_n\left(x\right)\right]$

f) Use parts **(d)** and **(e)** to show that the positive zeros of $J_n\left(x\right)$ interlace with those of $J_{n+1}\left(x\right)$.

$$\boxed{\frac{d}{dx}\left[x^n J_n\left(x\right)\right]=x^n J_{n-1}\left(x\right)},\ \boxed{\frac{d}{dx}\left[x^{-n} J_n\left(x\right)\right]=-x^{-n} J_{n+1}\left(x\right)}$$

[solution overleaf]



a) STARTING WITH THE GENERATING FUNCTION

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}t^n J_n(x)$$

LET $t=-\frac{1}{T}$

$$\Rightarrow e^{\frac{1}{2}x\left(-\frac{1}{T}+T\right)}=\sum_{n=-\infty}^{\infty}\left(-\frac{1}{T}\right)^n J_n(x)$$

LHS IS UNCHANGED

$$\Rightarrow e^{\frac{1}{2}x\left(T-\frac{1}{T}\right)}=\sum_{n=-\infty}^{\infty}(-1)^n T^{-n}J_n(x)$$

AS T IS A DUMMY VARIABLE, REPLACE IT BACK WITH t

$$\Rightarrow e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}(-1)^n t^{-n}J_n(x)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty}t^n J_n(x)=\sum_{n=-\infty}^{\infty}(-1)^n t^{-n}J_n(x)$$

COMPARE POWERS OF t, SAY $\left[t^n\right]$, NOTING ON RHS "$n\to -n$"

$$\Rightarrow J_n(x)=(-1)^n J_{-n}(x)$$

$$\Rightarrow J_{-n}(x)=\frac{1}{(-1)^n}J_n(x)$$

$$\Rightarrow J_{-n}(x)=(-1)^n J_n(x)$$

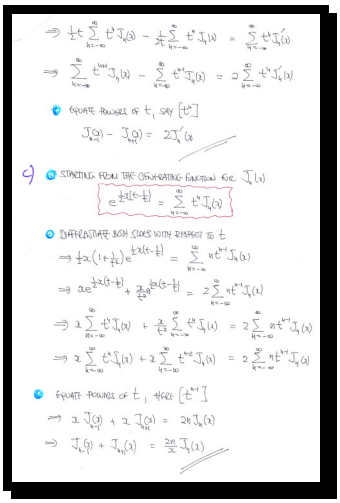
b) STARTING WITH THE GENERATING FUNCTION

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}t^n J_n(x)$$

DIFFERENTIATE WITH RESPECT TO x

$$\Rightarrow \tfrac{1}{2}\left(t-\tfrac{1}{t}\right)e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}t^n J_n'(x)$$

$$\Rightarrow \tfrac{1}{2}t\,e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}-\tfrac{1}{2t}e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}t^n J_n'(x)$$



$$\Rightarrow \tfrac{1}{2}t\sum_{n=-\infty}^{\infty}t^n J_n(x)-\tfrac{1}{2t}\sum_{n=-\infty}^{\infty}t^n J_n(x)=\sum_{n=-\infty}^{\infty}t^n J_n'(x)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty}t^{n+1}J_n(x)-\sum_{n=-\infty}^{\infty}t^{n-1}J_n(x)=2\sum_{n=-\infty}^{\infty}t^n J_n'(x)$$

EQUATE POWERS OF t, SAY $\left[t^n\right]$

$$J_{n-1}(x)-J_{n+1}(x)=2J_n'(x)$$

c) STARTING FROM THE GENERATING FUNCTION FOR $J_n(x)$

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}t^n J_n(x)$$

DIFFERENTIATE BOTH SIDES WITH RESPECT TO t

$$\Rightarrow \tfrac{1}{2}x\left(1+\tfrac{1}{t^2}\right)e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}nt^{n-1}J_n(x)$$

$$\Rightarrow \tfrac{1}{2}x\,e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}+\tfrac{1}{2}x\tfrac{1}{t^2}e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}nt^{n-1}J_n(x)$$

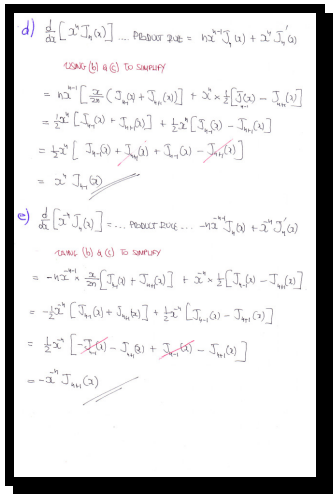
$$\Rightarrow \tfrac{1}{2}x\sum_{n=-\infty}^{\infty}t^n J_n(x)+\tfrac{1}{2}x\sum_{n=-\infty}^{\infty}t^{n-2}J_n(x)=\sum_{n=-\infty}^{\infty}nt^{n-1}J_n(x)$$

$$\Rightarrow \tfrac{1}{2}x\sum_{n=-\infty}^{\infty}t^n J_n(x)+\tfrac{1}{2}x\sum_{n=-\infty}^{\infty}t^{n-2}J_n(x)=2\sum_{n=-\infty}^{\infty}nt^{n-1}J_n(x)$$

EQUATE POWERS OF t, HERE $\left[t^{n-1}\right]$

$$\Rightarrow \tfrac{1}{2}x\,J_{n-1}(x)+\tfrac{1}{2}x\,J_{n+1}(x)=2nJ_n(x)$$

$$\Rightarrow J_{n-1}(x)+J_{n+1}(x)=\tfrac{2n}{x}J_n(x)$$



d) $\frac{d}{dx}\left[x^n J_n(x)\right]$ ... PRODUCT RULE $=nx^{n-1}J_n(x)+x^n J_n'(x)$

USING (b) & (c) TO SIMPLIFY

$$=nx^{n-1}\left[\tfrac{x}{2n}\left(J_{n-1}(x)+J_{n+1}(x)\right)\right]+x^n\times\tfrac{1}{2}\left[J_{n-1}(x)-J_{n+1}(x)\right]$$

$$=\tfrac{1}{2}x^n\left[J_{n-1}(x)+J_{n+1}(x)\right]+\tfrac{1}{2}x^n\left[J_{n-1}(x)-J_{n+1}(x)\right]$$

$$=\tfrac{1}{2}x^n\left[J_{n-1}(x)+J_{n+1}(x)+J_{n-1}(x)-J_{n+1}(x)\right]$$

$$=x^n J_{n-1}(x)$$

e) $\frac{d}{dx}\left[x^{-n}J_n(x)\right]=$ ... PRODUCT RULE ... $-nx^{-n-1}J_n(x)+x^{-n}J_n'(x)$

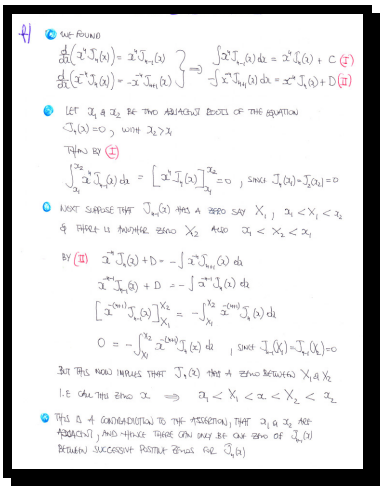
USING (b) & (c) TO SIMPLIFY

$$=-nx^{-n-1}\times\tfrac{x}{2n}\left[J_{n-1}(x)+J_{n+1}(x)\right]+x^{-n}\times\tfrac{1}{2}\left[J_{n-1}(x)-J_{n+1}(x)\right]$$

$$=-\tfrac{1}{2}x^{-n}\left[J_{n-1}(x)+J_{n+1}(x)\right]+\tfrac{1}{2}x^{-n}\left[J_{n-1}(x)-J_{n+1}(x)\right]$$

$$=\tfrac{1}{2}x^{-n}\left[-J_{n+1}(x)-J_{n+1}(x)\right]$$

$$=-x^{-n}J_{n+1}(x)$$



f) WE FOUND

$$\frac{d}{dx}\left(x^n J_n(x)\right)=x^n J_{n-1}(x)\quad\Rightarrow\quad\int x^n J_{n-1}(x)\,dx=x^n J_n(x)+C\ (\mathrm{I})$$

$$\frac{d}{dx}\left(x^{-n}J_n(x)\right)=-x^{-n}J_{n+1}(x)\quad\Rightarrow\quad-\int x^{-n}J_{n+1}(x)\,dx=x^{-n}J_n(x)+D\ (\mathrm{II})$$

LET α_1 & α_2 BE TWO ADJACENT ZEROS OF THE EQUATION $J_n(x)=0$, WITH $\alpha_2>\alpha_1$

TAKING BY (I)

$$\int_{\alpha_1}^{\alpha_2}x^n J_{n-1}(x)\,dx=\left[x^n J_n(x)\right]_{\alpha_1}^{\alpha_2}=0\ ,\ \text{SINCE } J_n(\alpha_1)=J_n(\alpha_2)=0$$

NEXT SUPPOSE THAT $J_{n-1}(x)$ HAS A ZERO SAY X_1, $\alpha_1<X_1<\alpha_2$ & THERE IS ANOTHER ZERO X_2 ALSO $\alpha_1<X_2<\alpha_2$

BY (II) $\quad x^{-n}J_n(x)+D=-\int x^{-n}J_{n+1}(x)\,dx$

$$x^{-n}J_n(x)+D=-\int x^{-n}J_{n+1}(x)\,dx$$

$$\left[x^{-(n+1)}J_n(x)\right]_{X_1}^{X_2}=-\int_{X_1}^{X_2}x^{-(n+1)}J_n(x)\,dx$$

$$0=-\int_{X_1}^{X_2}x^{-(n+1)}J_n(x)\,dx\ ,\ \text{SINCE } J_n(X_1)=J_n(X_2)=0$$

BUT THIS WORK IMPLIES THAT $J_n(x)$ HAS A ZERO BETWEEN X_1 & X_2

I.E. ONE THIS ZERO α $\Rightarrow$ $\alpha_1<X_1<\alpha<X_2<\alpha_2$

THIS IS A CONTRADICTION TO THE ASSERTION, THAT α_1 & α_2 ARE ADJACENT, AND HENCE THERE CAN ONLY BE ONE ZERO OF $J_n(x)$ BETWEEN SUCCESSIVE POSITIVE ZEROS FOR $J_n(x)$

Question 9

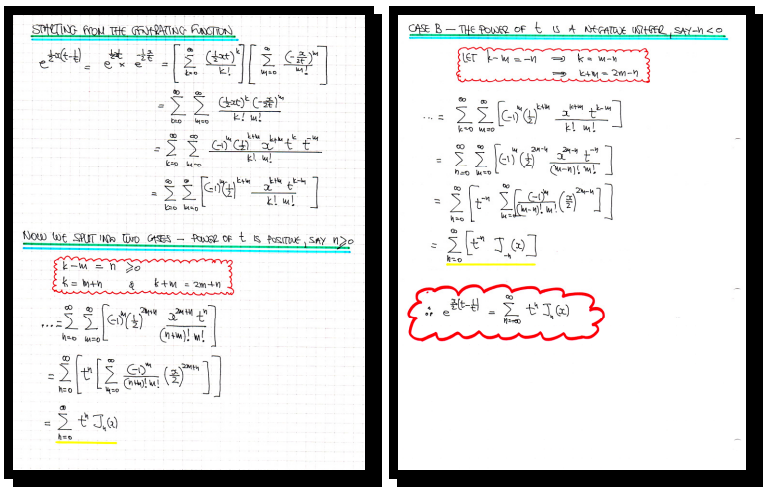
The Bessel function of the first kind is defined by the series

$$J_n(x)=\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(n+r)!r!}\left(\frac{x}{2}\right)^{2r+n}\right],\ n\in\mathbb{Z}.$$

Use the above definition to show

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right],\quad n\in\mathbb{Z}.$$

, proof



Question 10

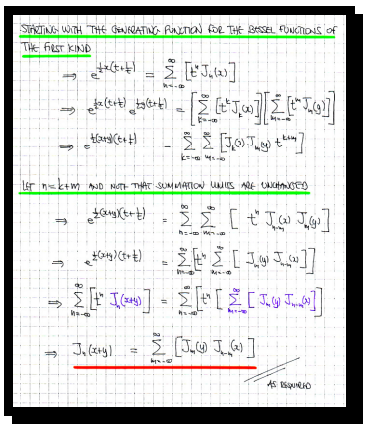
The generating function of the Bessel function of the first kind is

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right],\ n\in\mathbb{Z}.$$

Use the generating function relation, to show that

$$J_n(x+y)=\sum_{m=-\infty}^{\infty}\left[J_m(y)\,J_{n-m}(x)\right].$$

[ ], [proof]



Question 11

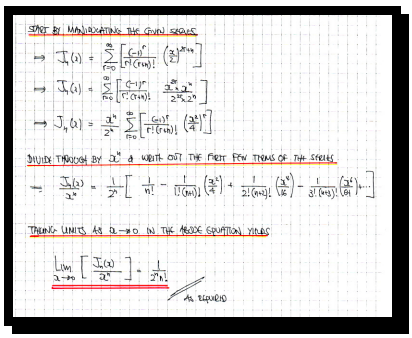
The Bessel function of the first kind is defined by the series

$$J_n(x)=\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(n+r)!r!}\left(\frac{x}{2}\right)^{2r+n}\right],\ n\in\mathbb{Z}.$$

Use the above definition to show

$$\lim_{x\to 0}\left[\frac{J_n(x)}{x^n}\right]=\frac{1}{2^n\,n!},\ n\in\mathbb{Z}.$$

, proof



Question 12

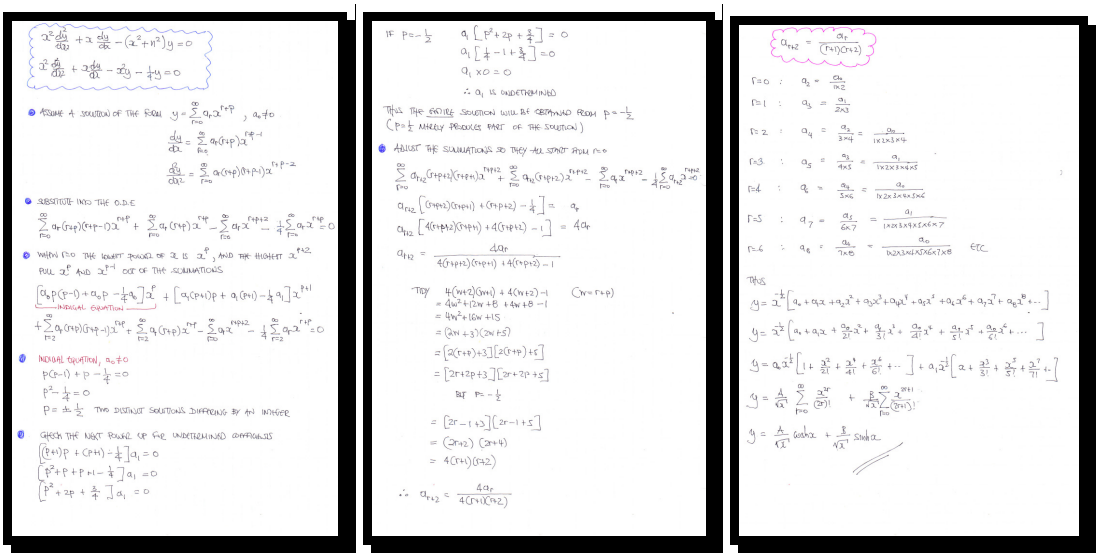
$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-\left(x^2+n^2\right)y=0.$$

The above differential equation is known as modified Bessel's Equation.

Use the Frobenius method to show that the general solution of this differential equation, for $n=\frac{1}{2}$, is

$$y=x^{-\frac{1}{2}}\left[A\cosh x+B\sinh x\right].$$

proof



Question 13

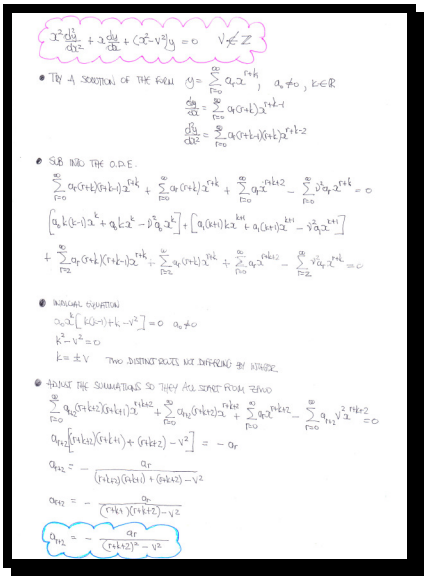
Find the two independent solutions of Bessel's equation

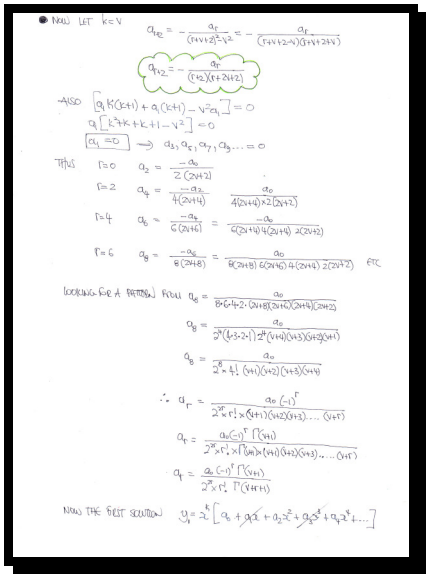
$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-\nu^2\right)y=0\,,\ \nu\notin\mathbb{Z}\,.$$

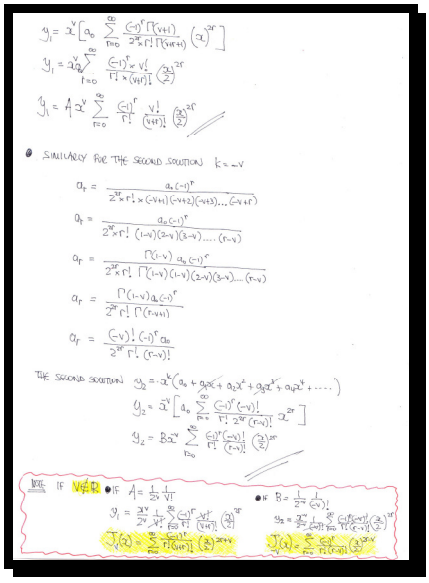
Give the answer as exact simplified summations.

$$y=Ax^{\nu}\sum_{r=0}^{\infty}\left[\frac{(-1)^r\,\nu!}{r!(\nu+r)!}\left(\frac{1}{2}x\right)^{2r}\right]\ \text{or}\ J_{\nu}(x)=\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!(\nu+r)!}\left(\frac{1}{2}x\right)^{2r+\nu}\right]$$

$$y=Bx^{-\nu}\sum_{r=0}^{\infty}\left[\frac{(-1)^r\,(-\nu)!}{r!(r-\nu)!}\left(\frac{1}{2}x\right)^{2r}\right]\ \text{or}\ J_{-\nu}(x)=\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!(r-\nu)!}\left(\frac{1}{2}x\right)^{2r-\nu}\right]$$







Question 14

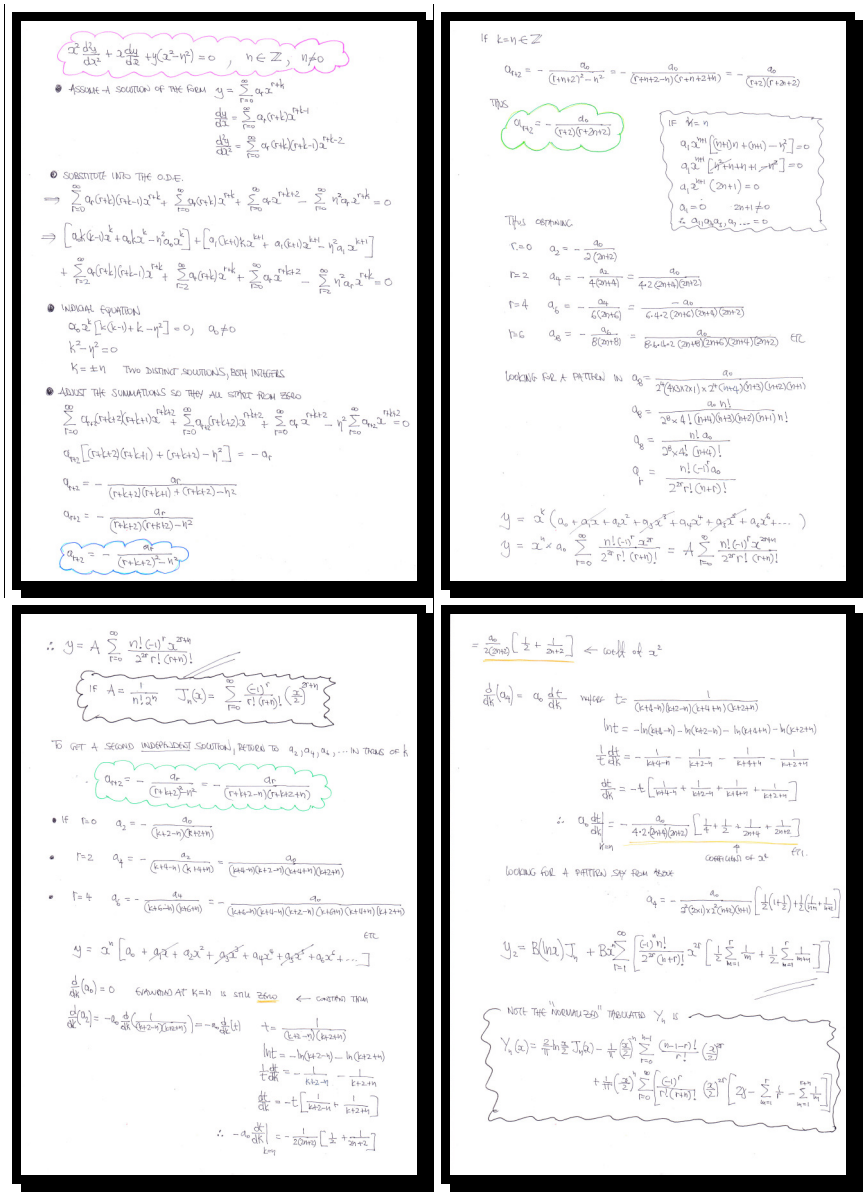
Find the two independent solutions of Bessel's equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-n^2\right)y=0,\ n\in\mathbb{Z}.$$

Give the answer as exact simplified summations.

$$y=Ax^n\sum_{r=0}^{\infty}\left[\frac{(-1)^r\,n!}{r!(n+r)!}\left(\frac{1}{2}x\right)^{2r}\right]\ \text{or}\ J_n(x)=\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!(n+r)!}\left(\frac{1}{2}x\right)^{2r+n}\right]$$

$$y=B(\ln x)x^n\sum_{r=0}^{\infty}\left[\frac{(-1)^r\,n!}{r!(n+r)!}\left(\frac{1}{2}x\right)^{2r}\right]+Bx^n\sum_{r=1}^{\infty}\left[\frac{(-1)^r\,n!}{r!(n+r)!}\left(\frac{1}{2}x\right)^{2r}\left[\frac{1}{2}\sum_{m=1}^{r}\frac{1}{m}+\frac{1}{2}\sum_{m=1}^{r}\frac{1}{m+n}\right]\right]$$



Question 15

Find the two independent solutions of Bessel's equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-n^2\right)y=0,\ n=0.$$

Give the answer as exact simplified summations.

$$y=A\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(r!)^2}\left(\frac{1}{2}x\right)^{2r}\right] \text{ or } J_0(x)=\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(r!)^2}\left(\frac{1}{2}x\right)^{2r}\right]$$

$$y=B(\ln x)\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(r!)^2}\left(\frac{1}{2}x\right)^{2r}\right]+B\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(r!)^2}\left(\frac{1}{2}x\right)^{2r}\sum_{m=1}^{r}\frac{1}{m}\right]$$



Question 16

The generating function of the Bessel function of the first kind is

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right],\ n\in\mathbb{Z}.$$

a) Use the generating function, to show that for $n\geq 0$

i. $J_{-n}(x)=(-1)^n J_n(x)$

ii. $J_{n-1}(x)-J_{n+1}(x)=2J_n'(x)$.

iii. $J_{n-1}(x)+J_{n+1}(x)=\dfrac{2n}{x}J_n(x)$.

b) Use part **(a)** deduce that

i. $\dfrac{d}{dx}\left[x^n J_n(x)\right]=x^n J_{n-1}(x)$.

ii. $\dfrac{d}{dx}\left[x^{1-n} J_{n-1}(x)\right]=-x^{1-n}J_n(x)$

c) Use part **(b)** to show further that

$$x^2J_n''(x)+xJ_n'(x)+\left(x^2-n^2\right)J_n(x)=0.$$

proof

[solution overleaf]



Question 17

$$t\frac{d^2y}{dt^2}+\frac{dy}{dt}+ty\,,\ t>0\,.$$

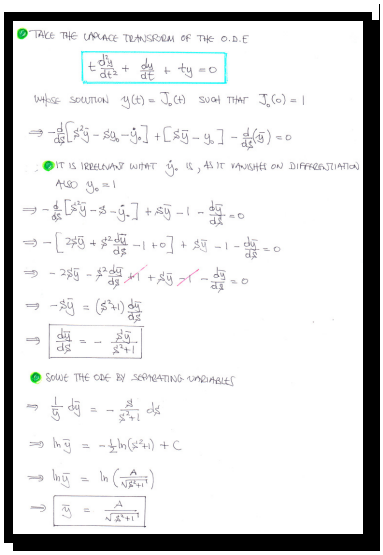
The Bessel function of order zero, $J_0(t)$, is a solution of the above differential equation.

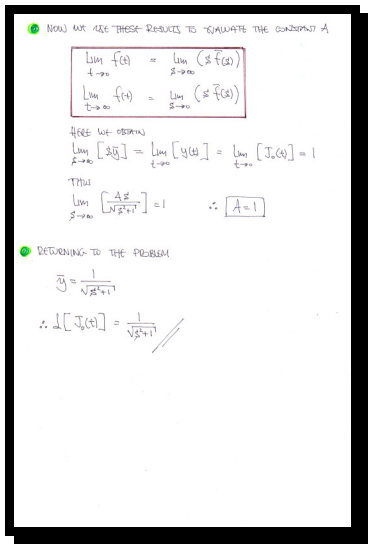
It is further given that $\lim_{t\to 0}\left[J_0(t)\right]=1$.

By taking the Laplace transform of the above differential equation, show that

$$\mathcal{L}\left[J_0(t)\right]=\frac{1}{\sqrt{s^2+1}}\,.$$

proof





Question 18

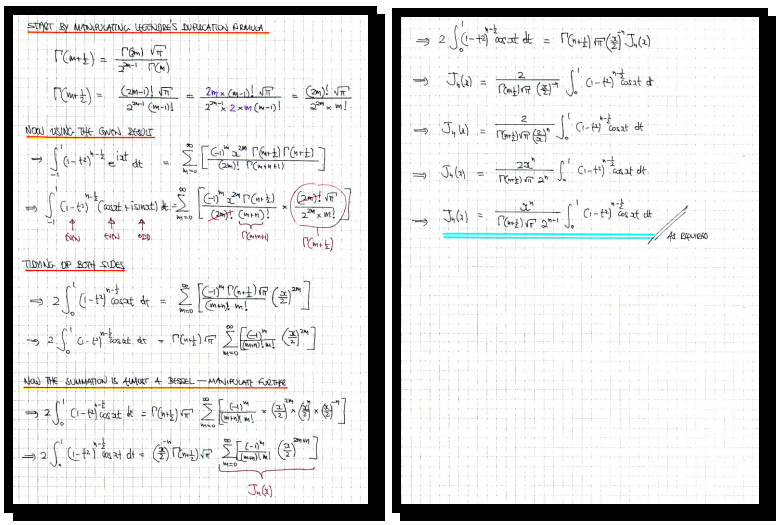
It can be shown that for $n \in \mathbb{N}$

$$\int_{-1}^{1} \left(1-t^2\right)^{n-\frac{1}{2}} \mathrm{e}^{\mathrm{i}xt} \ dt = \sum_{m=0}^{\infty} \left[\frac{(-1)^m \ \Gamma\left(m+\frac{1}{2}\right) \Gamma\left(n+\frac{1}{2}\right)}{(2m)! \ \Gamma(m+n+1)} \right].$$

Use Legendre's duplication formula for the Gamma Function to show

$$J_n(x) = \frac{x^n}{2^{n-1} \sqrt{\pi} \ \Gamma\left(n+\frac{1}{2}\right)} \int_0^1 \left(1-t^2\right)^{n-\frac{1}{2}} \cos(xt) \ dt \ .$$

[ ], [proof]



Question 19

Legendre's duplication formula for the Gamma Function states

$$\Gamma\left(n+\frac{1}{2}\right)\equiv\frac{\Gamma(2n)\sqrt{\pi}}{2^{2n-1}\Gamma(n)},\ n\in\mathbb{N}.$$

a) Prove the validity of the above formula.

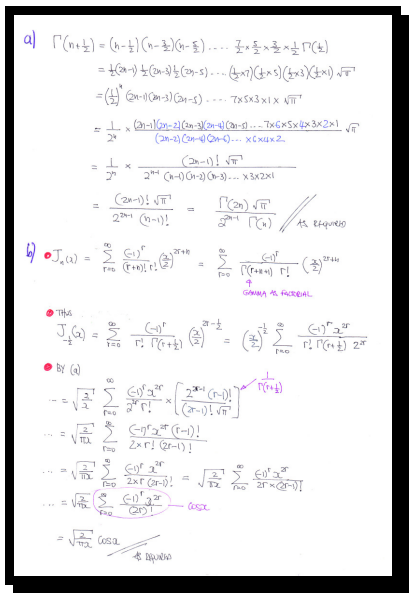
b) Hence show that

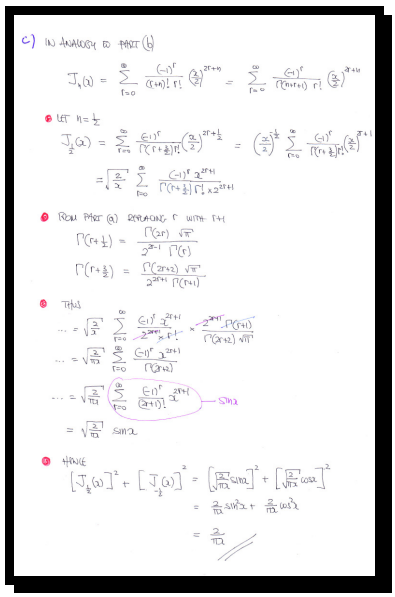
$$J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cos x\,.$$

c) Determine an exact simplified expression for

$$\left[J_{-\frac{1}{2}}(x)\right]^2+\left[J_{\frac{1}{2}}(x)\right]^2.$$

$$\boxed{\frac{2}{\pi x}}$$





Question 20

a) By using techniques involving the Beta function and the Gamma function, show that

$$\int_0^{\frac{\pi}{2}} (\cos\theta)^{2k+1}\, d\theta = \frac{(k!)^2\, 2^{2k}}{(2k+1)!}.$$

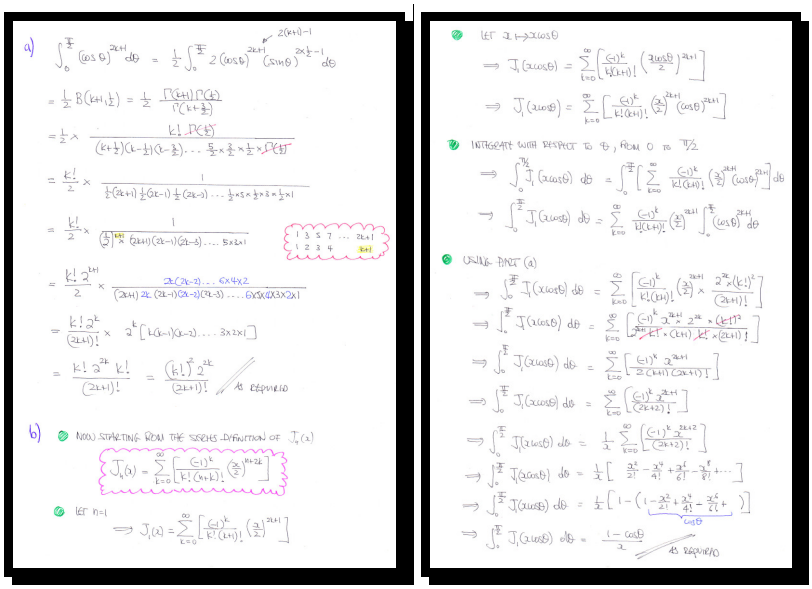
The series definition of the Bessel function of the first kind

$$J_n(x) = \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(n+r)!r!}\left(\frac{x}{2}\right)^{2r+n}\right],\ n\in\mathbb{Z}.$$

b) Use the above definition and the result of part **(a)**, to show that

$$\int_0^{\frac{\pi}{2}} J_1(x\cos\theta)\, d\theta = \frac{1-\cos\theta}{x},$$

proof



Question 21

The Bessel function $J_n(\alpha x)$ satisfies the differential equation

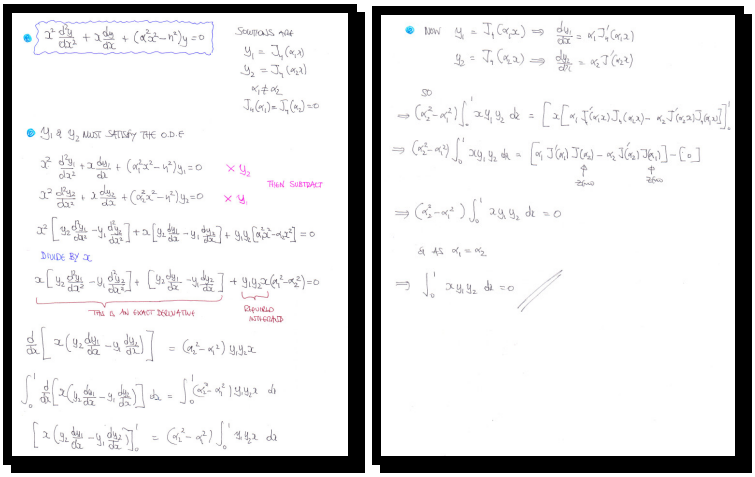
$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(\alpha^2x^2-n^2\right)y=0\,,\ n\in\mathbb{Z}\,,$$

where α is a non zero constant.

If $J_n(\alpha_1 x)$ and $J_n(\alpha_2 x)$ satisfy $J_n(\alpha_1)=J_n(\alpha_2)=0$, with $\alpha_1\neq\alpha_2$, show that

$$\int_0^1 x\,J_n(\alpha_1 x)J_n(\alpha_2 x)\,dx=0\,.$$

proof



Question 22

The series definition of the Bessel function of the first kind

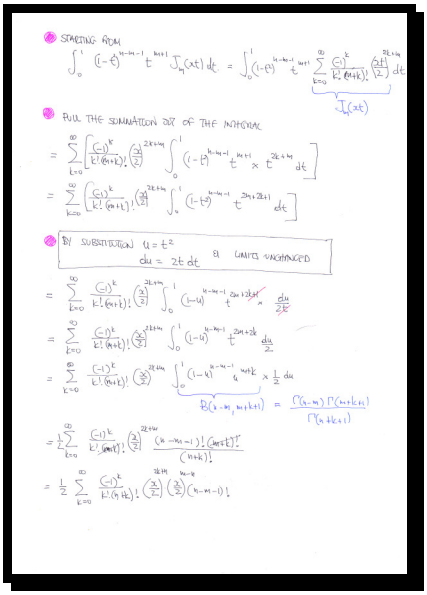
$$J_n(x)=\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(n+r)!r!}\left(\frac{x}{2}\right)^{2r+n}\right],\ n\in\mathbb{Z}.$$

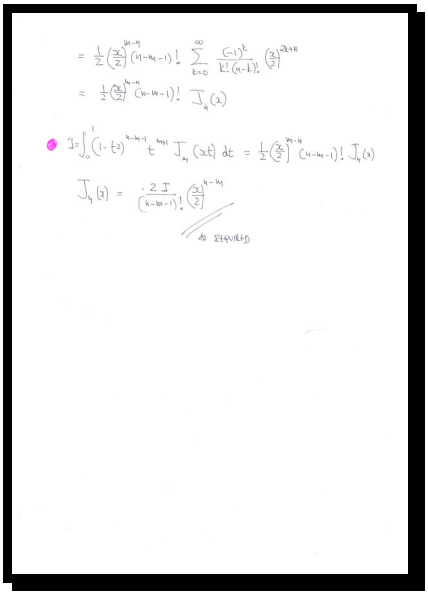
Use the above definition to show that

$$J_n(x)=\frac{2\,I}{(n-m-1)!}\left(\frac{x}{2}\right)^{n-m},$$

where $I=\int_0^1(1-t)^{n-m-1}\,t^{m+1}\,J_m(xt)\,dt$, $n>m>-1$.

proof





Question 23

$$I=\int_{-1}^{1}\left(1-t^2\right)^{n-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\,xt}\ \ dt$$

a) By using the series definition of the exponential function and converting the integrand into a Beta function, show that

$$I=\sum_{m=0}^{\infty}\left[\frac{(-1)^m\,x^{2m}}{(2m)!}\ \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(n+m+1)}\right].$$

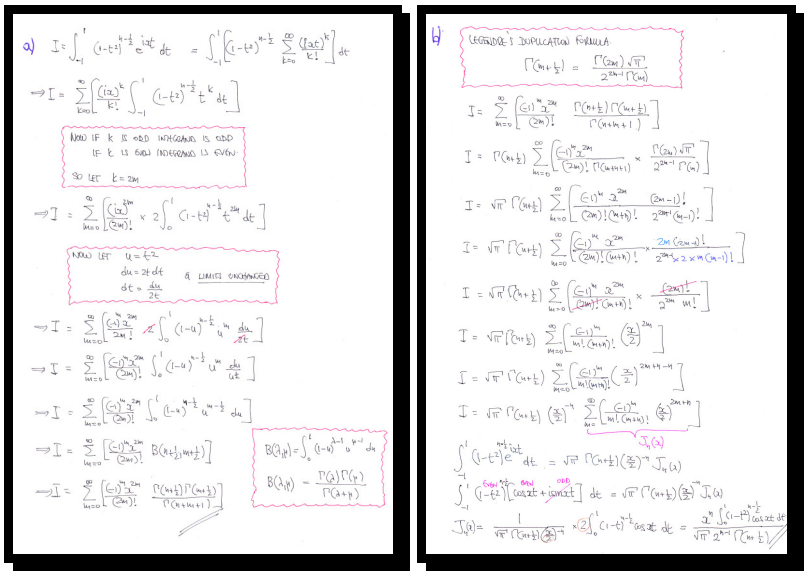
Legendre's duplication formula for the Gamma Function states

$$\Gamma\left(m+\frac{1}{2}\right)\equiv\frac{\Gamma(2m)\sqrt{\pi}}{2^{2m-1}\Gamma(m)},\ m\in\mathbb{N}.$$

b) Use the above formula and the result of part **(a)** to show further

$$J_n(x)=\frac{x^n}{2^{n-1}\sqrt{\pi}\ \Gamma\left(n+\frac{1}{2}\right)}\int_0^1\left(1-t^2\right)^{n-\frac{1}{2}}\cos(xt)\ \ dt\,.$$

proof



Question 24

The Bessel function of the first kind $J_n(x)$, satisfies

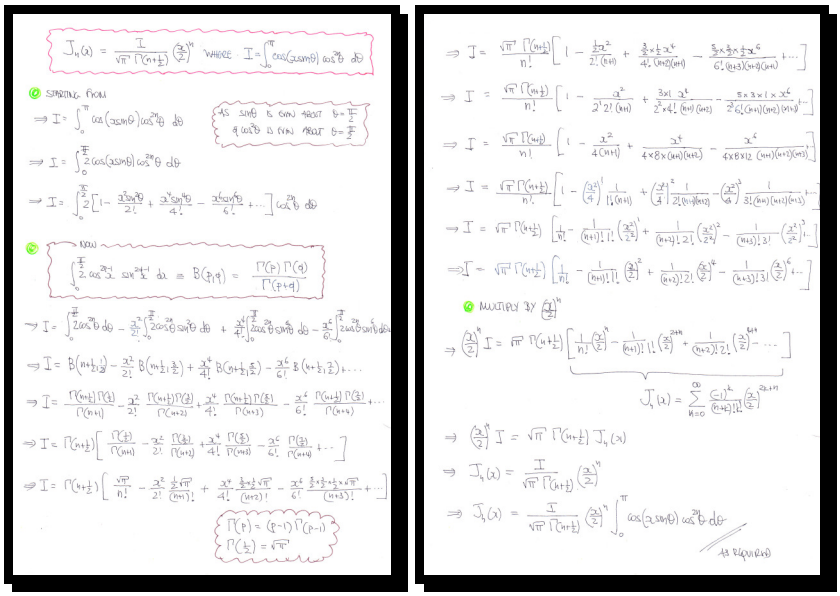
$$J_n(x)=\sum_{p=0}^{\infty}\left[\frac{(-1)^p}{(n+p)!\,p!}\left(\frac{x}{2}\right)^{2p+n}\right].$$

Show that

$$J_n(x)=\frac{I}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^n,$$

where $I=\displaystyle\int_0^{\pi}\cos(x\sin\theta)\cos^{2n}\theta\ \ d\theta$.

proof



Question 25

The generating function of the Bessel function of the first kind is

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right],\ n\in\mathbb{Z}.$$

a) Use the generating function, to show that for $n\geq 0$

i. $J_{-n}(x)=(-1)^n J_n(x)$

ii. $J_{n-1}(x)-J_{n+1}(x)=2J_n'(x)$.

iii. $J_{n-1}(x)+J_{n+1}(x)=\dfrac{2n}{x}J_n(x)$.

b) Given that $y=J_n(\lambda x)$ satisfies the differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(\lambda^2x^2-n^2\right)y=0,\quad n=0,1,2,3,...$$

verify that

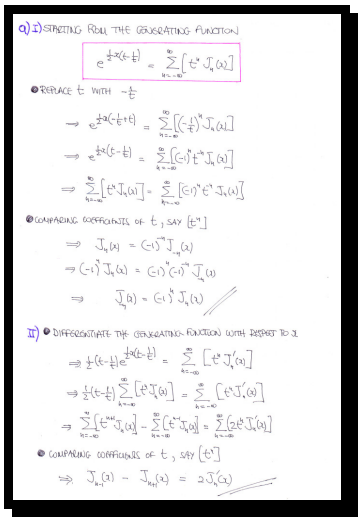
$$\frac{d}{dx}\left[x^2\left(\frac{dy}{dx}\right)^2\right]+\left(\lambda^2x^2-n^2\right)\frac{d}{dx}\left(y^2\right)=0,$$

and hence show that if λ_i is a non zero root of $J_n(\lambda)=0$

$$2\int_0^1 x\left[J_n(\lambda_i x)\right]^2\ dx=\left[J_{n-1}(\lambda_i)\right]^2=\left[J_{n+1}(\lambda_i)\right]^2.$$

proof

[solution overleaf]



a) I) STARTING FROM THE GENERATING FUNCTION

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{k=-\infty}^{\infty}\left[t^k J_k(x)\right]$$

● REPLACE t WITH $-\frac{1}{t}$

$$\Rightarrow e^{\frac{1}{2}x\left(-\frac{1}{t}+t\right)}=\sum_{k=-\infty}^{\infty}\left[(-1)^k t^{-k} J_k(x)\right]$$

$$\Rightarrow e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{k=-\infty}^{\infty}\left[(-1)^k t^{-k} J_k(x)\right]$$

$$\Rightarrow \sum_{k=-\infty}^{\infty}\left[t^k J_k(x)\right]=\sum_{k=-\infty}^{\infty}\left[(-1)^k t^{-k} J_k(x)\right]$$

● COMPARING COEFFICIENTS OF t^k, SAY $\left[t^k\right]$

$$\Rightarrow J_k(x)=(-1)^k J_{-k}(x)$$

$$\Rightarrow (-1)^k J_k(x)=(-1)^{-k} J_{-k}(x)$$

$$\Rightarrow J_{-k}(x)=(-1)^k J_k(x)$$

II) ● DIFFERENTIATE THE GENERATING FUNCTION WITH RESPECT TO x

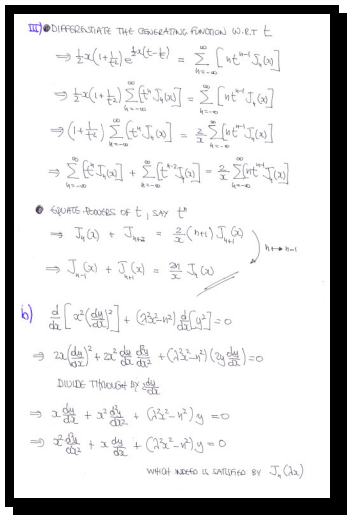
$$\Rightarrow \frac{1}{2}\left(t-\frac{1}{t}\right)e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{k=-\infty}^{\infty}\left[t^k J_k'(x)\right]$$

$$\Rightarrow \frac{1}{2}\left(t-\frac{1}{t}\right)\sum_{k=-\infty}^{\infty}\left[t^k J_k(x)\right]=\sum_{k=-\infty}^{\infty}\left[t^k J_k'(x)\right]$$

$$\Rightarrow \sum_{k=-\infty}^{\infty}\left[t^{k+1} J_k(x)\right]-\sum_{k=-\infty}^{\infty}\left[t^{k-1} J_k(x)\right]=\sum_{k=-\infty}^{\infty}\left[2t^k J_k'(x)\right]$$

● COMPARING COEFFICIENTS OF t^k, SAY $\left[t^k\right]$

$$\Rightarrow J_{k-1}(x)-J_{k+1}(x)=2J_k'(x)$$



III) ● DIFFERENTIATE THE GENERATING FUNCTION W.R.T t

$$\Rightarrow \frac{1}{2}x\left(1+\frac{1}{t^2}\right)e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}=\sum_{k=-\infty}^{\infty}\left[kt^{k-1} J_k(x)\right]$$

$$\Rightarrow \frac{1}{2}x\left(1+\frac{1}{t^2}\right)\sum_{k=-\infty}^{\infty}\left[t^k J_k(x)\right]=\sum_{k=-\infty}^{\infty}\left[kt^{k-1} J_k(x)\right]$$

$$\Rightarrow \left(1+\frac{1}{t^2}\right)\sum_{k=-\infty}^{\infty}\left[t^k J_k(x)\right]=\frac{2}{x}\sum_{k=-\infty}^{\infty}\left[kt^{k-1} J_k(x)\right]$$

$$\Rightarrow \sum_{k=-\infty}^{\infty}\left[t^k J_k(x)\right]+\sum_{k=-\infty}^{\infty}\left[t^{k-2} J_k(x)\right]=\frac{2}{x}\sum_{k=-\infty}^{\infty}\left[kt^{k-1} J_k(x)\right]$$

● EQUATE POWERS OF t, SAY t^k

$$\Rightarrow J_k(x)+J_{k+2}(x)=\frac{2}{x}(k+1)J_{k+1}(x) \quad \text{(}k\mapsto k-1\text{)}$$

$$\Rightarrow J_{k-1}(x)+J_{k+1}(x)=\frac{2k}{x}J_k(x)$$

b) $\dfrac{d}{dx}\left[x^2\left(\dfrac{dy}{dx}\right)^2\right]+\left(\lambda^2x^2-k^2\right)\dfrac{d}{dx}\left[y^2\right]=0$

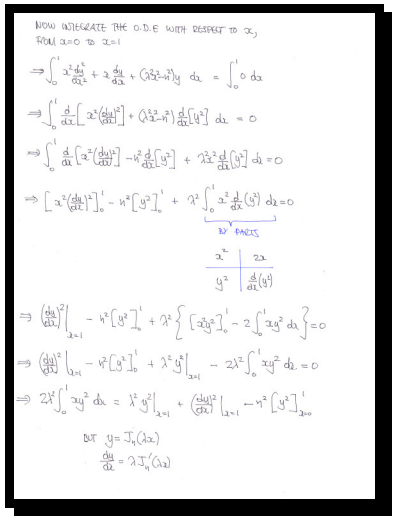
$$\Rightarrow 2x\left(\frac{dy}{dx}\right)^2+2x^2\frac{dy}{dx}\frac{d^2y}{dx^2}+\left(\lambda^2x^2-k^2\right)\left(2y\frac{dy}{dx}\right)=0$$

DIVIDE THROUGH BY $2\dfrac{dy}{dx}$

$$\Rightarrow x\frac{dy}{dx}+x^2\frac{d^2y}{dx^2}+\left(\lambda^2x^2-k^2\right)y=0$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(\lambda^2x^2-k^2\right)y=0$$

WHICH INDEED IS SATISFIED BY $J_k(\lambda x)$



NOW INTEGRATE THE O.D.E WITH RESPECT TO x, FROM $x=0$ TO $x=1$

$$\Rightarrow \int_0^1 x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(\lambda^2x^2-k^2\right)y\,dx=\int_0^1 0\,dx$$

$$\Rightarrow \int_0^1 \frac{d}{dx}\left[x^2\left(\frac{dy}{dx}\right)^2\right]+\left(\lambda^2x^2-k^2\right)\frac{d}{dx}\left[y^2\right]dx=0$$

$$\Rightarrow \int_0^1 \frac{d}{dx}\left[x^2\left(\frac{dy}{dx}\right)^2\right]-k^2\frac{d}{dx}\left[y^2\right]+\lambda^2x^2\frac{d}{dx}\left[y^2\right]dx=0$$

$$\Rightarrow \left[x^2\left(\frac{dy}{dx}\right)^2\right]_0^1-k^2\left[y^2\right]_0^1+\lambda^2\int_0^1 x^2\frac{d}{dx}\left(y^2\right)dx=0$$

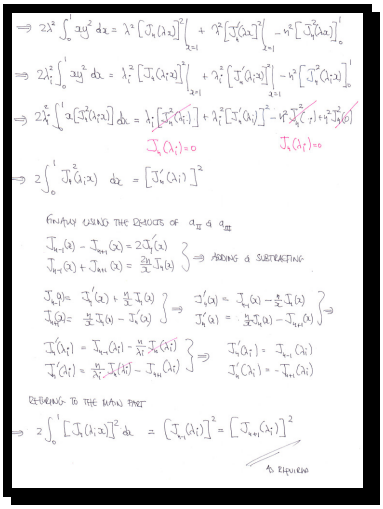
BY PARTS

x^2	$2x$
y^2	$\frac{d}{dx}(y^2)$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2\Big|_{x=1}-k^2\left[y^2\right]_0^1+\lambda^2\left\{\left[x^2y^2\right]_0^1-2\int_0^1 xy^2\,dx\right\}=0$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2\Big|_{x=1}-k^2\left[y^2\right]_0^1+\lambda^2y^2\Big|_{x=1}-2\lambda^2\int_0^1 xy^2\,dx=0$$

$$\Rightarrow 2\lambda^2\int_0^1 xy^2\,dx=\lambda^2y^2\Big|_{x=1}+\left(\frac{dy}{dx}\right)^2\Big|_{x=1}-k^2\left[y^2\right]_0^1$$



$$\Rightarrow 2\lambda^2\int_0^1 xy^2\,dx=\lambda^2\left[J_k(\lambda x)\right]^2\Big|_{x=1}+\lambda^2\left[J_k'(\lambda x)\right]^2\Big|_{x=1}-k^2\left[J_k^2(\lambda x)\right]_0^1$$

$$\Rightarrow 2\lambda_1^2\int_0^1 xy_1^2\,dx=\lambda_1^2\left[J_k(\lambda_1 x)\right]^2\Big|_{x=1}+\lambda_1^2\left[J_k'(\lambda_1 x)\right]^2\Big|_{x=1}-k^2\left[J_k^2(\lambda_1 x)\right]_0^1$$

$$\Rightarrow 2\lambda_1^2\int_0^1 x\left[J_k^2(\lambda_1 x)\right]dx=\lambda_1^2\left[J_k^2(\lambda_1)\right]+\lambda_1^2\left[J_k'(\lambda_1)\right]^2-k^2\cancel{J_k^2(\lambda_1)}+k^2\cancel{J_k^2(0)}$$

$J_k(\lambda_1)=0$ $\qquad$ $J_k(0)=0$

$$\Rightarrow 2\int_0^1 J_k^2(\lambda_1 x)\,dx=\left[J_k'(\lambda_1)\right]^2$$

FINALLY USING THE RESULTS OF a$_{II}$ & a$_{III}$

$$\left.\begin{aligned}J_{k-1}(x)-J_{k+1}(x)&=2J_k'(x)\\ J_{k-1}(x)+J_{k+1}(x)&=\frac{2k}{x}J_k(x)\end{aligned}\right\}\Rightarrow \text{ADDING \& SUBTRACTING}$$

$$\left.\begin{aligned}J_{k-1}(x)&=J_k'(x)+\frac{k}{x}J_k(x)\\ J_{k+1}(x)&=\frac{k}{x}J_k(x)-J_k'(x)\end{aligned}\right\}\Rightarrow \left.\begin{aligned}J_k'(x)&=J_{k-1}(x)-\frac{k}{x}J_k(x)\\ J_k'(x)&=\frac{k}{x}J_k(x)-J_{k+1}(x)\end{aligned}\right\}\Rightarrow$$

$$\left.\begin{aligned}J_k'(\lambda_1)&=J_{k-1}(\lambda_1)-\frac{k}{\lambda_1}J_k(\lambda_1)\\ J_k'(\lambda_1)&=\frac{k}{\lambda_1}J_k(\lambda_1)-J_{k+1}(\lambda_1)\end{aligned}\right\}\Rightarrow \begin{aligned}J_{k-1}(\lambda_1)&=J_{k+1}(\lambda_1)\\ J_k'(\lambda_1)&=-J_{k+1}(\lambda_1)\end{aligned}$$

REFERRING TO THE MAIN PART

$$\Rightarrow 2\int_0^1\left[J_k(\lambda_1 x)\right]^2dx=\left[J_{k-1}(\lambda_1)\right]^2=\left[J_{k+1}(\lambda_1)\right]^2$$

AS REQUIRED