# IYGB

## Special Paper O

## Time: 3 hours 30 minutes

**Candidates may NOT use any calculator.**

**Information for Candidates**

This practice paper follows the Advanced Level Mathematics Core Syllabus.
Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

**Advice to Candidates**

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

**Scoring**

Total Score $=T$, Number of non attempted questions $=N$, Percentage score $=P$.

$P=\frac{1}{2}T+N$ (rounded up to the nearest integer)

Distinction $P\geq 70$, Merit $55\leq P\leq 69$, Pass $40\leq P\leq 54$

**Question 1**

Solve the following equation for $x$.

$$\frac{x}{x-z}+\frac{y}{y-z}=2, \quad x\neq z,\ y\neq z. \qquad (4)$$

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**Question 2**

Find the rational solution of the following equation

$$\frac{2+9\sqrt{x}}{2\sqrt{3}-\sqrt{3x}}=\sqrt{3}+2\sqrt{2},\ x\in\mathbb{Q}. \qquad (8)$$

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**Question 3**

The points $A(3,3,2)$, $B(6,4,3)$ and $C(5,1,4)$ are referred with respect to a fixed origin $O$. The point $M$ is the midpoint of $AC$.

**a)** Show that $\overrightarrow{BM}$ is perpendicular to $\overrightarrow{AC}$. $(3)$

The point $D$ is such so that $ABCD$ is a kite with an area of $6\sqrt{6}$.

The straight line $BD$ is a line of symmetry for the kite $ABCD$.

**b)** Find the coordinates of $D$. $(7)$

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**Question 4**

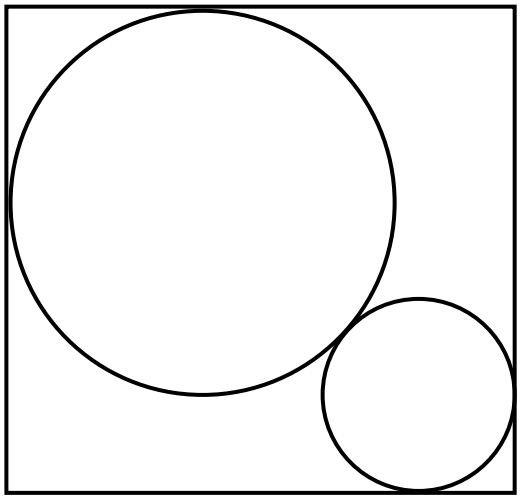
Given that

$$y=\frac{\sqrt{x}}{1+\sqrt{x}},$$

show that $\frac{dy}{dx}=f(y)$, where $f(y)$ is a function to be determined. $(8)$

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**Question 5**



Two circles of different radii are touching each other externally.

The two circles are enclosed by a square so that all $4$ sides of the squares are tangents to the circles, as shown in the figure above.

Given that the radius of the smaller circle is $r$ and the radius of the larger circle is $2r$, determine the exact area of the square in terms of $r$. $(\mathbf{8})$

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**Question 6**

Sum the following series of infinite terms.

$$\frac{1}{3}+\frac{3}{9}+\frac{7}{27}+\frac{15}{81}+\frac{31}{243}+\frac{63}{729}+... \qquad (\mathbf{7})$$

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**Question 7**

It is given that

$$2\cos\theta+\sin\theta=1.$$

Determine the possible values of

$$7\cos\theta+6\sin\theta. \qquad (\mathbf{12})$$

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**Question 8**

The curve $C$ has parametric equations

$$x = \frac{3t-1}{t^2-1}, \quad y = \frac{t}{t^2-1}, \quad t \in \mathbb{R}.$$

Show by eliminating the parameter $t$, that a Cartesian equation of $C$ is

$$(x-2y)(x-4y) = x-3y \qquad \textbf{(11)}$$

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**Question 9**

A curve $C$ is defined in the largest possible real number domain and has equation

$$y = \frac{x^3}{1-x^4}.$$

Sketch the graph of $C$. **(11)**

The sketch must include

- the coordinates of any points where the graph of $C$ meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

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**Question 10**

The finite region $R$ is by the coordinate axes and the curve with equation

$$y = \arccos x, \ -1 \le x \le 1.$$

The region $R$ is rotated by $2\pi$ radians in the $x$ axis forming a solid of revolution.

Determine the exact volume of this solid. **(9)**

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**Question 11**

The acute angles $\theta$ and $\varphi$ satisfy the following equations

$$2\cos\theta = \cos\varphi$$
$$2\sin\theta = 3\sin\varphi .$$

Show clearly that

$$\theta + \varphi = \pi - \arctan\sqrt{15} . \qquad \textbf{(11)}$$

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**Question 12**

The function $f$ is defined as

$$f(x) = -4 + \sqrt{mx+12},\ x \in \mathbb{R},\ x \geq -\frac{m}{12},$$

where $m$ is a positive constant.

It is given that the graph of $f(x)$ and the graph of $f^{-1}(x)$ touch each other.

Solve the equation

$$f(x) = f^{-1}(x) . \qquad \textbf{(12)}$$

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**Question 13**

A circle passes through the points $P(18,0)$ and $Q(32,0)$.

A tangent to this circle passes through the point $S(0,199)$ and touches the circle at the point $T$.

Given that the $y$ axis is a tangent to this circle, determine the coordinates of $T$. **(16)**

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**Question 14**

A curve $C$ has equation

$$y = 8^{2x} - 4^{x-1} + 2^x, \ x \in \mathbb{R}$$

Show that $C$ has no stationary points. **(10)**

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**Question 15**

A function $F$ is defined by the integral

$$F(x) \equiv \int_1^x \frac{e^t}{t} \, dt, \ x \geq 1.$$

Find a simplified expression, in terms of $F$, for

$$\int_1^x \frac{e^t}{t+a} \, dt,$$

where $a$ is a positive constant. **(7)**

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**Question 16**

It is given that

$$(a+2)\sin x + (2a-1)\cos x = 2a+1,$$

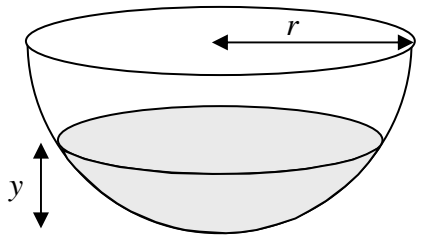
where $a$ is a non zero constant.

Find the exact value of $\tan\left(\frac{1}{2}x\right)$, giving the answer in terms of $a$, where appropriate.

**(10)**

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**Question 17**




A water tank has the shape of a hollow inverted hemisphere of radius $r$ cm.

The tank has a hole at the bottom which allows the water to drain out.

Let $V$, in $\text{cm}^3$, and $y$, in cm, be the volume and the height of the water in the tank, respectively, at time $t$ seconds.

At time $t = 0$ the empty tank is placed under a running water tap. The rate at which the volume of the water in the tank is changing is proportional to the difference between the tank's constant diameter and the height of the water at that instant. When the tank is full the running tap is instantly turned off but the water in the tank continues to leak out from the hole at the bottom.

Show it takes three times as long to empty the tank than it took to fill it up. **(14)**

*You may use without proof that $V$ and $y$ are related by* $V = \frac{1}{3}\pi\left(3ry^2 - y^3\right)$.

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**Question 18**

By suitably rewriting the numerator of the integrand, find a simplified expression for the following integral.

$$\int \frac{3\cos x + 2\sin x}{2\cos x + 3\sin x}\, dx\,. \qquad \textbf{(10)}$$

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**Question 19**

The point $A(1,-1)$ lies on the curve with equation

$$y = \left| x^2 - |2x| \right| - 2x, \ x \in \mathbb{R}.$$

The tangent to the curve at $A$ meets the curve at three more points $B$, $C$ and $D$.

Sketch the curve and its tangent at $A$ in a single set of axes.

Give the coordinates of $B$, $C$ and $D$ in exact form where appropriate. **(12)**

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**Question 20**

The function $f$ is defined as

$$f(n,x) \equiv \sum_{r=0}^{n} \binom{n}{r} r\, x^r (1-x)^{n-r},$$

where $n \in \mathbb{N}$, $x \in \mathbb{R}$, $0 < x < 1$.

Show that $f(n,x) \equiv nx$. **(10)**

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