IYGB - MATHEMATICAL METHODS 4 - PAPER D - QUESTION 1

REARRANGE THE TRANSFORMATION EQUATIONS, & OBTAIN PARTIAL DERIVATIVES

$$\left.\begin{aligned} x &= u^2+v^2 \\ y &= u^2-v^2 \end{aligned}\right\} \text{ ADDING \& SUBTRACTING } \left.\begin{aligned} x+y &= 2u^2 \\ x-y &= 2v^2 \end{aligned}\right\} \Longrightarrow$$

$$\left.\begin{aligned} u^2 &= \tfrac{1}{2}x+\tfrac{1}{2}y \\ v^2 &= \tfrac{1}{2}x-\tfrac{1}{2}y \end{aligned}\right\} \Longrightarrow$$

- $2u\dfrac{\partial u}{\partial x}=\dfrac{1}{2}$ $\quad$ $2v\dfrac{\partial v}{\partial x}=\dfrac{1}{2}$
- $2u\dfrac{\partial u}{\partial y}=\dfrac{1}{2}$ $\quad$ $2v\dfrac{\partial v}{\partial y}=-\dfrac{1}{2}$

- $\dfrac{\partial u}{\partial x}=\dfrac{1}{4u}$ $\quad$ $\dfrac{2v}{\partial x}=\dfrac{1}{4v}$
- $\dfrac{\partial u}{\partial y}=\dfrac{1}{4u}$ $\quad$ $\dfrac{\partial v}{\partial y}=-\dfrac{1}{4v}$

BY THE CHAIN RULE WE HAVE

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}\frac{1}{4u}+\frac{\partial z}{\partial v}\frac{1}{4v}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}\frac{1}{4u}+\frac{\partial z}{\partial v}\left(-\frac{1}{4v}\right)$$

SUBSTITUTE INTO THE P.D.E.

$$\Rightarrow \frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1$$

$$\Rightarrow \left[\frac{1}{4u}\frac{\partial z}{\partial u}+\frac{1}{4v}\frac{\partial z}{\partial v}\right]+\left[\frac{1}{4u}\frac{\partial z}{\partial u}-\frac{1}{4v}\frac{\partial z}{\partial v}\right]=1$$

$$\Rightarrow \frac{1}{2u}\frac{\partial z}{\partial u}=1$$

$$\Rightarrow \frac{\partial z}{\partial u}=2u$$

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SOLVING BY DIRECT INTEGRATION

$$z(u,v) = u^2 + f(v)$$

$$z(x,y) = \left(\tfrac{1}{2}x + \tfrac{1}{2}y\right) + f\left(\sqrt{\tfrac{1}{2}x - \tfrac{1}{2}y}\right)$$

$$z(x,y) = \tfrac{1}{2}(x+y) + f\left(\sqrt{x-y}\right)$$

IYGB - MATHEMATICAL METHODS 4 - PAPER D - QUESTION 2

FOR THE STANDARD WAVE EQUATION

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \quad \text{SUBJECT TO} \quad z(x,0) = F(x) = \begin{cases} 1-|x| & |x|\leq 1 \\ 0 & |x|>1 \end{cases}$$

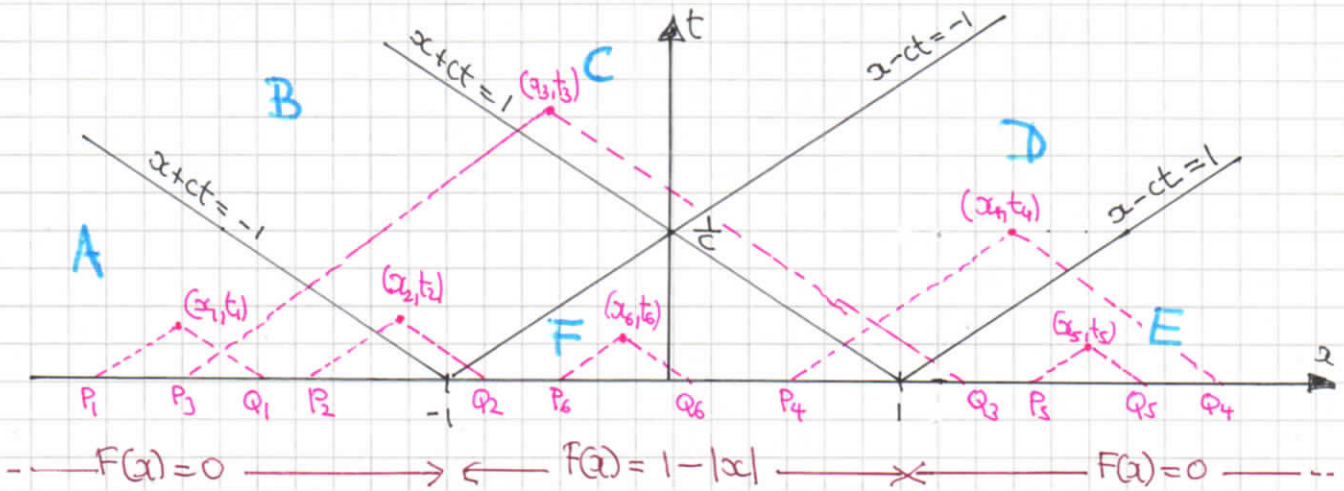
$$\frac{\partial z}{\partial t}(x,0) = G(x) = 0$$

D'ALEMBERT'S STANDARD SOLUTION IS

$$z(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$

ZERO HERE

DRAW THE CHARACTERISTICS, IE LINES WITH GRADIENT $\pm\frac{1}{c}$, THROUGH ANY CRITICAL POINTS, HERE AT $x=\pm 1$ WHERE $F(x)$ HAS DISCONTINUITIES



LOOKING AT THE x-t PLANE ABOVE THE SOLUTION IS $z(x,t) = \frac{1}{2}F(P) + \frac{1}{2}F(Q)$

- Regions A, C, E YIELD $z(x,t)=0$, AS $P_1, Q_1, P_3, Q_3, P_5, Q_5$ FALL INTO A REGION OF THE x AXIS WHERE $F(x)=0$
- Region B YIELDS $z(x_2,t_2) = \frac{1}{2}F(P_2) + \frac{1}{2}F(Q_2) = \frac{1}{2}F(x_2+ct_2) = \frac{1}{2}\left[1-|x_2+ct_2|\right]$

NOTE THAT THE EQUATION IS

$$\left.\begin{aligned} t-t_2 &= -\tfrac{1}{c}(x-x_2) \\ -ct+ct_2 &= x-x_2 \\ 0+ct_2 &= Q_2-x_2 \\ Q_2 &= x_2+ct_2 \end{aligned}\right\}$$

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- Region F yields $z(x,t) = \frac{1}{2}F(P_6) + \frac{1}{2}F(Q_6) = \frac{1}{2}F(x_6 - ct_6) + \frac{1}{2}F(x_6 + ct_6)$

EQUATIONS THROUGH (x_6, t_6) ARE

$t - t_6 = \frac{1}{c}(x - x_6)$	$t - t_6 = -\frac{1}{c}(x - x_6)$
$0 - t_6 = \frac{1}{c}(Q_6 - x_6)$	$0 - t_6 = -\frac{1}{c}(P_6 - x_6)$
$-ct_6 = Q_6 - x_6$	$ct_6 = P_6 - x_6$
$Q_6 = x_6 - ct_6$	$P_6 = x_6 + ct_6$

$$= \frac{1}{2}\left[1 - |x_6 - ct_6| + 1 - |x_6 + ct_6|\right]$$

$$= \frac{1}{2}\left[2 - |x_6 - ct_6| - |x_6 + ct_6|\right]$$

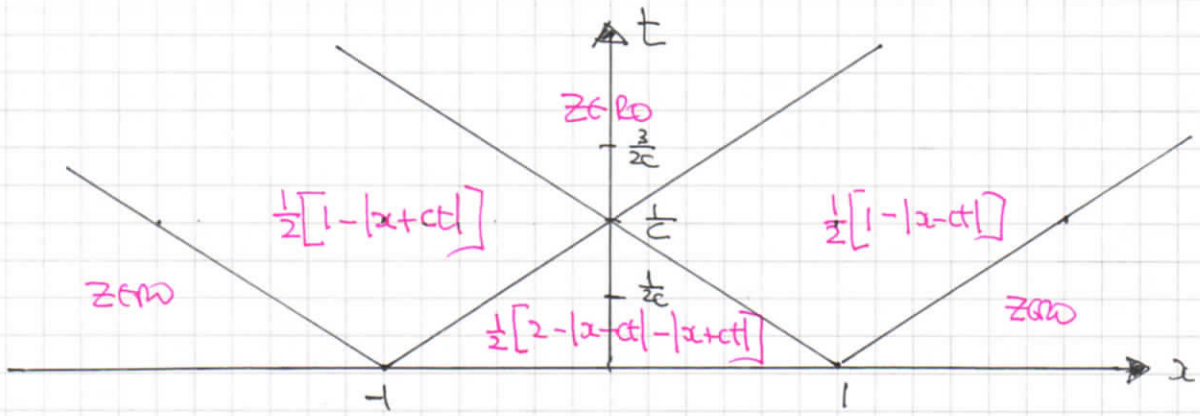
- Region D yields $z(x_4, t_4) = \frac{1}{2}F(P_4) + \frac{1}{2}F(Q_4) = \frac{1}{2}F(x_4 + ct_4)$

$$= \frac{1}{2}\left[1 - |x_4 + ct_4|\right]$$

EQUATION THROUGH (x_4, t_4)

$$t - t_4 = -\frac{1}{c}(x - x_4)$$
$$0 - t_4 = -\frac{1}{c}(Q_4 - x_4)$$
$$ct_4 = Q_4 - x_4$$
$$Q_4 = x_4 + ct_4$$

FINALLY DROPPING THE SUBSCRIPTS WE CAN ILLUSTRATE THE SOLUTION AT THE DIFFERENT REGIONS OF THE x-t PLANE



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b) IF $t=0$, $z(x,0)=\begin{cases} 1-|x| & |x|\leqslant 1 \\ 0 & |x|>1 \end{cases}$

IF $t=\frac{1}{2c}$, INTERSECTIONS WITH THE CHARACTERISTICS ARE AT $x=-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2}$

$$z\left(x,\tfrac{1}{2c}\right)=\begin{cases} \frac{1}{2}\left(1-\left|x+\frac{1}{2}\right|\right) & -\frac{3}{2}\leqslant x\leqslant -\frac{1}{2} \leftarrow \frac{1}{2}\left(1-\left(-x-\frac{1}{2}\right)\right)=\frac{1}{2}x+\frac{3}{4} \\ \frac{1}{2}\left[2-\left|x-\frac{1}{2}\right|-\left|x+\frac{1}{2}\right|\right] & -\frac{1}{2}\leqslant x\leqslant \frac{1}{2} \leftarrow \frac{1}{2}\left[2-\left(-x+\frac{1}{2}\right)-\left(x+\frac{1}{2}\right)\right]=\frac{1}{2} \\ \frac{1}{2}\left(1-\left|x-\frac{1}{2}\right|\right) & \frac{1}{2}\leqslant x\leqslant \frac{3}{2} \leftarrow \frac{1}{2}\left[1-\left(x-\frac{1}{2}\right)\right]=-\frac{1}{2}x+\frac{3}{4} \\ 0 & \text{OTHERWISE} \end{cases}$$

IF $t=\frac{1}{c}$, INTERSECTION WITH THE CHARACTERISTICS ARE AT $-2,0,2$

$$z\left(x,\tfrac{1}{c}\right)=\begin{cases} \frac{1}{2}\left[1-|x+1|\right] & -2\leqslant x\leqslant 0 \\ \frac{1}{2}\left[1-|x-1|\right] & 0\leqslant x\leqslant 2 \\ 0 & \text{OTHERWISE} \end{cases}$$

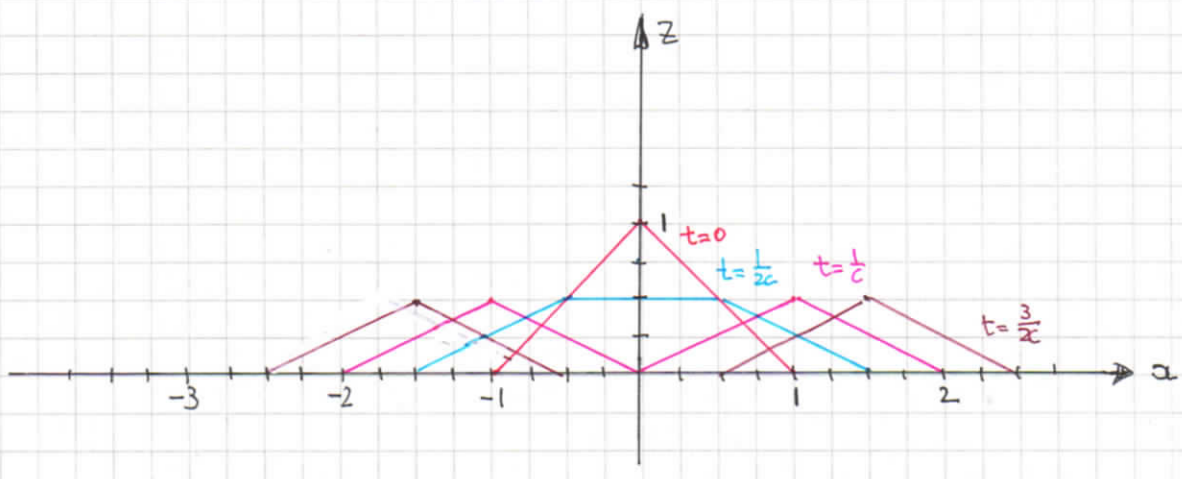
$\frac{1}{2}\left(1-(-x-1)\right)=\frac{1}{2}x+1$, $-2\leqslant x\leqslant -1$
$\frac{1}{2}\left(1-(x+1)\right)=-\frac{1}{2}x$ $-1\leqslant x\leqslant 0$
$\frac{1}{2}\left(1-(-x+1)\right)=\frac{1}{2}x$ $0\leqslant x\leqslant 1$
$\frac{1}{2}\left(1-(x-1)\right)=-\frac{1}{2}x+1$ $1\leqslant x\leqslant 2$

IF $t=\frac{3}{2c}$, INTERSECTIONS WITH THE CHARACTERISTICS ARE AT $-\frac{5}{2},-\frac{1}{2},\frac{1}{2},\frac{5}{2}$

$$z\left(x,\tfrac{3}{2c}\right)=\begin{cases} \frac{1}{2}\left[1-\left|x+\frac{3}{2}\right|\right] & -\frac{5}{2}\leqslant x\leqslant -\frac{1}{2} \\ \frac{1}{2}\left[1-\left|x-\frac{3}{2}\right|\right] & \frac{1}{2}\leqslant x\leqslant \frac{5}{2} \\ 0 & \text{OTHERWISE} \end{cases}$$

$\frac{1}{2}\left(1-(-x-\frac{3}{2})\right)=\frac{1}{2}x+\frac{5}{4}$, $-\frac{5}{2}\leqslant x\leqslant -\frac{3}{2}$
$\frac{1}{2}\left(1-(x+\frac{3}{2})\right)=-\frac{1}{2}x-\frac{1}{4}$, $-\frac{3}{2}\leqslant x\leqslant -\frac{1}{2}$
$\frac{1}{2}\left(1-(-x+\frac{3}{2})\right)=\frac{1}{2}x-\frac{1}{4}$, $\frac{1}{2}\leqslant x\leqslant \frac{3}{2}$
$\frac{1}{2}\left(1-(x-\frac{3}{2})\right)=-\frac{1}{2}x+\frac{5}{4}$, $\frac{3}{2}\leqslant x\leqslant \frac{5}{2}$

HENCE THE PROFILES CAN BE DRAWN



IYGB - MATHEMATICAL METHODS 4 - PAPER D - QUESTION 3

START BY TAKING THE LAPLACE TRANSFORM OF THE P.D.E, W.R.T t

$$\Rightarrow \frac{\partial u}{\partial t} + y\frac{\partial u}{\partial y} = y$$

$$\Rightarrow \mathcal{L}\left[\frac{\partial u}{\partial t}\right] + \mathcal{L}\left[y\frac{\partial u}{\partial y}\right] = \mathcal{L}[y]$$

$$\Rightarrow \left[s\bar{u}(s,y) - u(0,y)\right] + y\frac{\partial}{\partial y}\left(\bar{u}(s,y)\right) = y\,\mathcal{L}[1]$$

$$\Rightarrow s\bar{u} - (1+y^2) + y\frac{\partial \bar{u}}{\partial y} = \frac{y}{s}$$

$$\Rightarrow y\frac{\partial \bar{u}}{\partial y} + s\bar{u} = 1 + y^2 + \frac{y}{s}$$

$$\Rightarrow \frac{\partial \bar{u}}{\partial y} + \frac{s}{y}\bar{u} = \frac{1}{y} + y + \frac{1}{sy}$$

TREAT THE ABOVE AS AN O.D.E FOR $\bar{u} = f(y)$, AS s IS A CONSTANT, AND LOOK FOR AN INTEGRATING FACTOR

$$e^{\int \frac{s}{y}\,dy} = e^{s\ln y} = e^{\ln y^s} = y^s$$

THUS WE NOW HAVE

$$\Rightarrow \frac{\partial}{\partial y}\left[\bar{u}y^s\right] = y^s\left(\frac{1}{y} + y + \frac{1}{sy}\right)$$

$$\Rightarrow \frac{\partial}{\partial y}\left[\bar{u}y^s\right] = y^{s-1} + y^{s+1} + \frac{y^s}{s}$$

$$\Rightarrow \bar{u}y^s = \int y^{s-1} + y^{s+1} + \frac{y^s}{s}\,dy$$

$$\Rightarrow \bar{u}y^s = \frac{1}{s}y^s + \frac{1}{s+2}y^{s+2} + \frac{1}{s(s+1)}y^{s+1} + A(s)$$

$$\Rightarrow \boxed{\bar{u}(s,y) = \frac{1}{s} + \frac{y^2}{s+2} + \frac{y}{s(s+1)} + A(s)y^{-s}}$$

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NEXT WE APPLY THE BOUNDARY CONDITION $u(t,0)=1$

$$\Rightarrow u(t,0)=1$$

$$\Rightarrow \bar{u}(s,0)=\frac{1}{s}$$

$$\Rightarrow \frac{1}{s}=\frac{1}{s}+\frac{1}{s+2}\times 0^2+\frac{1}{s(1+s)}\times 0+A(s)\times 0^{-s}$$

$$\times \frac{1}{0^s}\to\infty$$

$$\therefore A(s)=0$$

$$\therefore \boxed{\bar{u}(s,y)=\frac{1}{s}+\frac{1}{s+2}y^2+\frac{1}{s(s+1)}y}$$

INVERTING BY PARTIAL FRACTIONS & INSPECTION

$$\bar{u}(s,y)=\frac{1}{s}+\frac{1}{s+2}y^2+\left(\frac{1}{s}-\frac{1}{s+1}\right)y$$

$$\bar{u}(s,y)=\frac{1}{s}(1+y)+\frac{1}{s+2}y^2-\frac{1}{s+1}y$$

$$u(t,y)=1+y+e^{-2t}y^2-e^{-t}y$$

// AS REQUIRED

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REWRITE THE P.D.E & SOLVE BY "LAGRANGE'S METHOD"

$$\underset{\uparrow P}{xz}\frac{\partial z}{\partial x} + \underset{\uparrow Q}{yz}\frac{\partial z}{\partial y} = \underset{\uparrow R}{-xy}$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \underset{①}{\frac{dx}{xz}} = \underset{②}{\frac{dy}{yz}} = \underset{③}{\frac{dz}{-xy}}$$

SOLVING ① = ②

$$\Rightarrow \frac{dx}{xz} = \frac{dy}{yz}$$
$$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$$
$$\Rightarrow \ln x = \ln y + \ln A$$
$$\Rightarrow \ln x = \ln(Ay)$$
$$\Rightarrow x = Ay$$
$$\Rightarrow \frac{y}{x} = C_1$$

SOLVING ① = ③

$$\Rightarrow \frac{dx}{xz} = \frac{dz}{-xy}$$
$$\Rightarrow \frac{dx}{z} = -\frac{dz}{y}$$
$$\Rightarrow y\,dx = -z\,dz$$
$$\Rightarrow C_1 x\,dx = -z\,dz$$
$$\Rightarrow \frac{1}{2}C_1x^2 + C_2 = -\frac{1}{2}z^2$$
$$\Rightarrow \frac{1}{2}\left(\frac{y}{x}\right)x^2 + C_2 = -\frac{1}{2}z^2$$
$$\Rightarrow \frac{1}{2}xy + C_2 = -\frac{1}{2}z^2$$
$$\Rightarrow -xy + C_2 = z^2$$
$$\Rightarrow z^2 + xy = C_2$$

THE GENERAL SOLUTION IS GIVEN BY

$$F(u,v) = 0 \quad \text{WHERE} \quad u(x,y,z) = \frac{y}{x}$$
$$v(x,y,z) = z^2 + xy$$

$$\therefore u = f(v) \quad \text{OR} \quad v = g(u)$$

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$$\Rightarrow z^2+xy = f\left(\frac{y}{x}\right)$$

$$\Rightarrow z^2 = f\left(\frac{y}{x}\right) - xy$$

APPLY BOUNDARY CONDITIONS NEXT, $xy=1$ AT $z=x$ $\forall x$,

$$\Rightarrow z^2 = f\left(\frac{y}{x}\right) - xy$$
$$\Rightarrow x^2 = f\left(\frac{1/x}{x}\right) - 1$$
$$\Rightarrow x^2+1 = f\left(\frac{1}{x^2}\right)$$

LET $u=\frac{1}{x^2}$

$x^2=\frac{1}{u}$

$$\Rightarrow \frac{1}{u}+1 = f(u)$$
$$\Rightarrow f(u) = \frac{1}{u}+1$$
$$\Rightarrow f\left(\frac{y}{x}\right) = \frac{1}{y/x}+1$$
$$\Rightarrow f\left(\frac{y}{x}\right) = \frac{x}{y}+1$$

HENCE WE NOW HAVE

$$z^2 = \frac{x}{y}+1-xy$$

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ASSUME A SOLUTION IN UNPLIABLE SEPARABLE FORM, DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E.

$$z(x,t) = X(x)T(t) \implies \frac{\partial^2 z}{\partial x^2} = X''(x)T(t)$$

$$\frac{\partial z}{\partial t} = X(x)T'(t)$$

$$\implies \frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t}$$

$$\implies X''(x)T(t) = X(x)T'(t)$$

$$\implies \frac{X''(x)T(t)}{X(x)T(t)} = \frac{X(x)T'(t)}{X(x)T(t)}$$

$$\implies \frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}$$

AS THE L.H.S IS A FUNCTION OF x ONLY AND THE R.H.S IS A FUNCTION OF t ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ

IF $\lambda = 0$

- $\frac{X''(x)}{X(x)} = 0$

$$X''(x) = 0$$

$$X(x) = Ax + B$$

- $\frac{T'(t)}{T(t)} = 0$

$$T'(t) = C$$

$$\therefore z(x,t) = C(Ax+B) = Px + Q$$

I.E. STEADY STATE WHICH IS NOT APPROPRIATE IN THIS PROBLEM

IF $\lambda > 0$, $\lambda = p^2$

- $\frac{X''(x)}{X(x)} = p^2$

$$X''(x) = p^2 X(x)$$

$$X(x) = A\cosh px + B\sinh px$$

(OR EXPONENTIALS)

- $\frac{T'(t)}{T(t)} = p^2$

$$T'(t) = p^2 T(t)$$

$$T(t) = Ce^{p^2 t}$$

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$$\therefore z(x,t) = Ce^{p^2t}(A\cosh px + B\sinh px)$$

$$\underline{\underline{z(x,t) = e^{p^2t}(P\cosh px + Q\sinh px)}}$$

WHICH IS ALSO INAPPROPRIATE SINCE IT IS UNBOUNDED AS $t \to \infty$

IF $\lambda < 0$, $\lambda = -p^2$

- $\dfrac{X''(x)}{X(x)} = -p^2$

  $X''(x) = -p^2X(x)$

  $X(x) = A\cos px + B\sin px$

- $\dfrac{T'(t)}{T(t)} = -p^2$

  $T'(t) = -p^2T(t)$

  $T(t) = Ce^{-p^2t}$

$$\therefore z(x,t) = Ce^{-p^2t}(A\cos px + B\sin px)$$

$$\underline{\underline{z(x,t) = e^{-p^2t}(P\cos px + Q\sin px)}} \quad \text{WHICH IS OK}$$

APPLY BOUNDARY CONDITION, $z(0,t) = 0$

$$\Rightarrow 0 = e^{-p^2t} \times P$$

$$\Rightarrow P = 0$$

$$\therefore \underline{\underline{z(x,t) = Qe^{-p^2t}\sin px}}$$

APPLY BOUNDARY CONDITION, $z(2,t) = 0$

$$\Rightarrow 0 = Qe^{-p^2t}\sin 2p$$

$$\Rightarrow \sin 2p = 0 \qquad (Q \neq 0, \text{ OTHERWISE SOLUTION IS TRIVIAL})$$

$$\Rightarrow 2p = n\pi \qquad n = 0,1,2,3,\ldots$$

$$\Rightarrow p = \frac{n\pi}{2}$$

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$$\therefore Z_n(x,t) = Q_n e^{-\frac{n^2\pi^2 t}{4}} \sin\left(\frac{n\pi x}{2}\right)$$

$$Z(x,t) = \sum_{n=1}^{\infty}\left[Q_n e^{-\frac{n^2\pi^2 t}{4}} \sin\left(\frac{n\pi x}{2}\right)\right]$$

NOTE $n=0$, YIELDS ZERO, SO OMITTED

APPLY INITIAL CONDITION $Z(x,0) = 20$

$$\Rightarrow 20 = \sum_{n=1}^{\infty}\left[Q_n \times 1 \times \sin\left(\frac{n\pi x}{2}\right)\right]$$

I.E A FOURIER SERIES IN x IN $(0,2)$, PERIOD 2

$$\Rightarrow Q_n = \frac{1}{1}\int_0^2 20\sin\left(\frac{n\pi x}{2}\right)\,dx$$

$$\Rightarrow Q_n = \left[-\frac{20\times 2}{n\pi}\cos\frac{n\pi x}{2}\right]_0^2$$

$$\Rightarrow Q_n = \left[\frac{40}{n\pi}\cos\frac{n\pi x}{2}\right]_2^0$$

$$\Rightarrow Q_n = \frac{40}{n\pi} - \frac{40}{n\pi}\cos(n\pi)$$

$$\Rightarrow Q_n = \frac{40}{n\pi} - \frac{40}{n\pi}(-1)^n$$

$$\Rightarrow Q_n = \begin{cases} 0 & \text{IF } n \text{ IS EVEN} \\ \frac{80}{n\pi} & \text{IF } n \text{ IS ODD} \end{cases}$$

LET $n = 2m-1$, $m = 1,2,3,4,5,\ldots$

$$Z(x,t) = \sum_{m=1}^{\infty}\left[\frac{80}{(2m-1)\pi} e^{-\frac{(2m-1)^2\pi^2 t}{4}} \sin\left(\frac{(2m-1)\pi x}{2}\right)\right]$$

NOTE THAT AS $t\to+\infty$, $Z(x,t)\to 0$ SO WE DO NOT NEED AN "AUXILIARY BUILT UP FUNCTION"

IYGB - MATHEMATICAL METHODS 4 - PAPER D - QUESTION 6

ASSUME A SOLUTION IN VARIABLE SEPARATE FORM

$$u(x,t) = X(x)T(t) \Rightarrow \frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$$

$$\frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$$

SUBSTITUTE INTO THE P.D.E.

$$\Rightarrow X''(x)T(t) = \frac{1}{4}X(x)T''(t)$$

$$\Rightarrow \frac{X''(x)T(t)}{X(x)T(t)} = \frac{1}{4}\frac{X(x)T''(t)}{X(x)T(t)}$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{T''(t)}{4T(t)}$$

AS THE L.H.S. IS A FUNCTION OF x ONLY AND THE RHS IS FUNCTION OF t ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ

IF $\lambda > 0$, $\lambda = p^2$

$$\Rightarrow \frac{X''(x)}{X(x)} = p^2$$

$$\Rightarrow X''(x) = p^2 X(x)$$

$$\Rightarrow X(x) = A\cosh px + B\sinh px$$

(OR EXPONENTIALS)

$$\Rightarrow \frac{T''(t)}{4T(t)} = p^2$$

$$\Rightarrow T''(t) = 4p^2 T(t)$$

$$\Rightarrow T(t) = D\cosh 2pt + E\sinh 2pt$$

(OR EXPONENTIALS)

IF $\lambda = 0$

$$\Rightarrow \frac{X''(x)}{X(x)} = 0$$

$$\Rightarrow X''(x) = 0$$

$$\Rightarrow X(x) = Ax + B$$

$$\Rightarrow \frac{T''(t)}{4T(t)} = 0$$

$$\Rightarrow T''(t) = 0$$

$$\Rightarrow T(t) = At + B$$

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IF $\lambda < 0$, $\lambda = -p^2$

$$\Rightarrow \frac{X''(x)}{X(x)} = -p^2 \qquad \Rightarrow \frac{T''(t)}{4T(t)} = -p^2$$

$$\Rightarrow X''(x) = -p^2 X(x) \qquad \Rightarrow T''(t) = -4p^2 T(t)$$

$$\Rightarrow X(x) = A\cos px + B\sin px \qquad \Rightarrow T(t) = D\cos 2pt + E\sin 2pt$$

AS WE REQUIRE A SOLUTION WHICH GIVES THE SAME VALUE OF $u(x,t)$ FOR THE TWO DISTINCT VALUES OF x AT THE ENDPOINTS ($x = \pm 1$) WE CAN ONLY PICK THE "TRIGONOMETRIC SOLUTION" & DISCARD THE OTHER TWO

$$\therefore \boxed{u(x,t) = [A\cos px + B\sin px][D\cos 2pt + E\sin 2pt]}$$

APPLY CONDITION $u(x,0) = 0$ (UNDISTURBED INITIALLY)

$$\Rightarrow 0 = [A\cos px + B\sin px] \times D$$

$$\Rightarrow D = 0 \quad \text{(otherwise trivial solution } A = B = 0\text{)}$$

ABSORBING "E" INTO "A" AND "B"

$$\therefore \boxed{u(x,t) = [A\cos px + B\sin px]\sin 2pt}$$

DIFFERENTIATE W.R.T t AND APPLY $\frac{\partial u}{\partial t}(x,0) = 1 - x^2$ (INITIAL TRANSVERSE VELOCITY)

$$\Rightarrow \frac{\partial u}{\partial t}(x,t) = 2p[A\cos px + B\sin px]\cos 2pt$$

$$\Rightarrow 1 - x^2 = 2p[A\cos px + B\sin px] \times 1$$

$$\Rightarrow B = 0 \quad \text{(AS THE LHS. IS AN EVEN FUNCTION IN } x\text{)}$$

$$\therefore \boxed{u(x,t) = A\cos px \sin 2pt}$$

IYGB - MATHEMATICAL METHODS 4 - PAPER D - QUESTION 6

APPLY BOUNDARY CONDITIONS $u(-1,t) = u(1,+) = 0$

$$\left.\begin{aligned} 0 &= A\cos(-p)\sin 2pt \\ 0 &= A\cos p\sin 2pt \end{aligned}\right\} \Rightarrow A\cos p\sin 2pt = 0 \quad \forall t \geqslant 0$$

$$\Rightarrow \cos p = 0 \quad (A \neq 0)$$

$$\Rightarrow p = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$$

$$\Rightarrow p = \frac{2n-1}{2}\pi \qquad n = 1, 2, 3, \cdots$$

$$\Rightarrow u_n(x,t) = A_n \cos\left[\frac{(2n-1)\pi x}{2}\right]\sin\left[(2n-1)\pi t\right]$$

$$\therefore \quad \boxed{u(x,t) = \sum_{n=1}^{n} A_n \cos\left[\frac{(2n-1)\pi x}{2}\right]\sin\left[(2n-1)\pi t\right]}$$

ONE MORE CONSTANT TO EVALUATE, SO REAPPLY $\frac{\partial u}{\partial t}(x,0) = 1 - x^2$, $-1 \leq x \leq 1$

$$\Rightarrow \frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty}\left[\left[A_n(2n-1)\pi\right]\cos\left[\frac{(2n-1)\pi x}{2}\right]\cos\left[(2n-1)\pi t\right]\right]$$

$$\Rightarrow 1 - x^2 = \sum_{n=1}^{\infty}\left[\left[A_n(2n-1)\pi\right]\cos\left[\frac{(2n-1)\pi x}{2}\right]\right]$$

THIS IS A FOURIER SERIES IN x, IN $-1 \leq x \leq 1$

$$A_n(2n-1)\pi = \frac{1}{1}\int_{-1}^{1}(1-x^2)\cos\left[\frac{(2n-1)\pi x}{2}\right]dx$$

$$A_n(2n-1)\pi = 2\int_{0}^{1}(1-x^2)\cos\left[\frac{(2n-1)\pi x}{2}\right]dx$$

$$A_n(2n-1)\pi = \int_{0}^{2} 2\cos\left[\frac{(2n-1)\pi x}{2}\right]dx - \int_{0}^{2} 2x^2\cos\left[\frac{(2n-1)\pi x}{2}\right]dx$$

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CARRYING OUT THE INTEGRATIONS

- $$\int_0^1 2\cos\left[\frac{(2n-1)\pi x}{2}\right]dx = \left[2\times\frac{2}{(2n-1)\pi}\times\sin\left[\frac{(2n-1)\pi x}{2}\right]\right]_0^1$$

$$= \frac{4}{\pi(2n-1)}\sin\left[\frac{(2n-1)\pi}{2}\right] = \frac{4}{\pi(2n-1)}(-1)^{n+1}$$

- $$\int_0^1 -2x^2\cos\left[\frac{(2n-1)\pi x}{2}\right]dx = \ldots$$ INTEGRATION BY PARTS

$-2x^2$	$-4x$
$\frac{2}{(2n-1)\pi}\sin\left[\frac{(2n-1)\pi x}{2}\right]$	$\cos\left[\frac{(2n-1)\pi x}{2}\right]$

$$= \left[\frac{-4x^2}{(2n-1)\pi}\sin\left[\frac{(2n-1)\pi x}{2}\right]\right]_0^1 + \frac{8}{(2n-1)\pi}\int_0^1 x\sin\left[\frac{(2n-1)\pi x}{2}\right]dx$$

$$= \frac{-4}{(2n-1)\pi}\sin\left[\frac{(2n-1)\pi}{2}\right] - 0 + \frac{8}{(2n-1)\pi}\int_0^1 x\sin\left[\frac{(2n-1)\pi x}{2}\right]dx$$

$$= \frac{-4(-1)^{n+1}}{(2n-1)\pi} + \frac{8}{(2n-1)\pi}\int_0^1 x\sin\left[\frac{(2n-1)\pi x}{2}\right]dx$$

BY PARTS AGAIN

x	1
$-\frac{2}{(2n-1)\pi}\cos\left[\frac{(2n-1)\pi x}{2}\right]$	$\sin\left[\frac{(2n-1)\pi x}{2}\right]$

$$= \frac{-4(-1)^{n+1}}{(2n-1)\pi} + \frac{8}{(2n-1)\pi}\left\{\left[-\frac{2x}{(2n-1)\pi}\cos\left[\frac{(2n-1)\pi x}{2}\right]\right]_0^1 + \frac{2}{(2n-1)\pi}\int_0^1\cos\left[\frac{(2n-1)\pi x}{2}\right]\right\}$$

$$= \frac{-4(-1)^{n+1}}{(2n-1)\pi} + \frac{16}{(2n-1)^2\pi^2}\int_0^1\cos\left[\frac{(2n-1)\pi x}{2}\right]dx$$

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$$= \frac{-4(-1)^{n+1}}{(2n-1)\pi} + \frac{16}{(2n-1)^2\pi^2} \times \frac{2}{(2n-1)\pi} \left[\sin\left[\frac{(2n-1)\pi x}{2}\right] \right]_0^1$$

$$= \frac{-4(-1)^{n+1}}{(2n-1)\pi} + \frac{32}{(2n-1)^3\pi^3} \times (-1)^{n+1}$$

COLLECTING THE INTEGRATION RESULTS

$$A_n(2n-1)\pi = \frac{4}{\pi(2n-1)}(-1)^{n+1} - \frac{4(-1)^{n+1}}{(2n-1)\pi} + \frac{32(-1)^{n+1}}{(2n-1)^3\pi^3}$$

$$A_n = \frac{32(-1)^{n+1}}{(2n-1)^4\pi^4}$$

HENCE THE SOLUTION IS GIVEN BY

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{32(-1)^{n+1}}{(2n-1)^4\pi^4} \cos\left[\frac{(2n-1)\pi x}{2}\right] \sin\left[(2n-1)\pi t\right] \right]$$

& THE FREQUENCIES OF NORMAL MODES OF VIBRATION ARE

$$f_n = \frac{w_n}{2\pi} \leftarrow \text{coefficient of } t$$

$$f_n = \frac{(2n-1)\pi}{2\pi}$$

$$f_n = n - \frac{1}{2} \qquad n = 1, 2, 3, 4, \ldots$$

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ASSUME A SOLUTION IN VARIABLE SEPARATE FORM - DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E

$$\Phi(r,\theta) = R(r)\,\theta(\theta) \implies \frac{\partial \Phi}{\partial r} = R'(r)\,\theta(\theta)$$
$$\implies \frac{\partial^2 \Phi}{\partial r^2} = R''(r)\,\theta(\theta)$$
$$\implies \frac{\partial^2 \Phi}{\partial \theta^2} = R(r)\,\theta''(\theta)$$

$$\implies \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0$$

$$\implies R''(r)\theta(\theta) + \frac{1}{r}R'(r)\theta(\theta) + \frac{1}{r^2}R(r)\theta''(\theta) = 0$$

$$\implies r^2 R''(r)\theta(\theta) + r R'(r)\theta(\theta) + R(r)\theta''(\theta) = 0$$

$$\implies \frac{r^2 R''(r)\cancel{\theta(\theta)}}{R(r)\cancel{\theta(\theta)}} + \frac{r R'(r)\cancel{\theta(\theta)}}{R(r)\cancel{\theta(\theta)}} + \frac{\cancel{R(r)}\theta''(\theta)}{\cancel{R(r)}\theta(\theta)} = \frac{0}{R(r)\theta(\theta)}$$

$$\implies \frac{r^2 R''(r)}{R(r)} + \frac{r R'(r)}{R(r)} = -\frac{\theta''(\theta)}{\theta(\theta)}$$

AS THE L.H.S IS A FUNCTION OF r ONLY AND THE R.H.S IS A FUNCTION OF θ ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ

LOOKING AT THE R.H.S OF THE ABOVE EXPRESSION, WE REQUIRE TRIGONOMETRIC SOLUTIONS IN θ [DUE TO THE POLAR SYSTEM REPEATING EVERY 2π]

GIVEN FURTHER THAT THERE IS ALREADY A MINUS SIGN IN THE R.H.S, WE DEDUCE $\lambda > 0$, SAY p^2

$$\implies -\frac{\theta''(\theta)}{\theta(\theta)} = p^2$$

$$\implies \theta''(\theta) = -p^2\theta(\theta)$$

$$\implies \boxed{\theta(\theta) = A\cos(p\theta) + B\sin(p\theta)}$$

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EXAMINING FURTHER THE "θ SOLUTION" WITH SAY SINES (OR COSINES)

$$\sin\theta = \sin(\theta+2\pi) \Rightarrow \sin(p\theta) = \sin\left[p(\theta+2\pi)\right]$$
$$= \sin(p\theta + 2p\pi)$$
$$\Rightarrow p = n = \text{INTEGER} = 0,1,2,3,4,\ldots$$

NOTES

- $n=0$ IS O.K AS IT PRODUCES A CONSTANT SOLUTION
- $n<0$ NEED NOT TO BE CONSIDERED, AS AT THIS STAGE WILL BE ABSORBED INTO THE CONSTANTS, BUT THEY WILL ACTUALLY APPEAR AT THE END

$$\therefore \boxed{\Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)}$$

RETURNING TO THE L.H.S (FUNCTION OF r ONLY) WITH $\lambda = n = \text{INTEGER}$

$$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = n^2, \quad n = 0,1,2,3,4,\ldots$$

- IF $n=0$

$$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = 0$$
$$\Rightarrow rR''(r) + R'(r) = 0$$
$$\Rightarrow rR''(r) = -R'(r)$$
$$\Rightarrow \frac{R''(r)}{R'(r)} = -\frac{1}{r}$$

INTEGRATE W.R.T r

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$$\Rightarrow \ln|R'(r)| = -\ln r + \ln C$$

$$\Rightarrow \ln|R'(r)| = \ln\left|\frac{C}{r}\right|$$

$$\Rightarrow R'(r) = \frac{C}{r}$$

$$\Rightarrow R(r) = C\ln r + D$$

IF $n = 1,2,3,4,\ldots$

$$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = n^2$$

$$\Rightarrow r^2R''(r) + rR'(r) - n^2R(r) = 0$$

THIS IS A CAUCHY EULER O.D.E.

LET $R(r) = r^\lambda$

$R'(r) = \lambda r^{\lambda-1}$

$R''(r) = \lambda(\lambda-1)r^{\lambda-2}$

SUBSTITUTE INTO THE O.D.E.

$$\lambda(\lambda-1)r^\lambda + \lambda r^\lambda - n^2r^\lambda = 0$$

$$\lambda^2 - \lambda + \lambda - n^2 = 0$$

$$\lambda^2 = n^2$$

$$\lambda = \pm n$$

$$R_n(r) = \alpha_n r^n + \beta_n r^{-n}, \quad n = 1,2,3,4,\ldots$$

COMBINING RESULTS

$n = 0$ $\quad \Theta(\theta) = B$

$R(r) = C\ln r + D$

$n = 1,2,3,4,\ldots$ $\quad \Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta$

$R_n(r) = \alpha_n r^n + \beta_n r^{-n}$

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HENCE THE GENERAL SOLUTION IS

$$\Phi(r,\theta)=\left[C\ln r+D\right]+\sum_{n=1}^{\infty}\left[\left(A_n\cos n\theta+B_n\sin n\theta\right)\left(\alpha_n r^n+\beta_n r^{-n}\right)\right]$$

ABSORBING AND RELABELING CONSTANTS WE OBTAIN

$$\Phi(r,\theta)=A+B\ln r+\sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta+D_n r^{-n}\cos n\theta+E_n r^n\sin n\theta+F_n r^{-n}\sin n\theta\right]$$

b) NOW WE HAVE THREE FUNCTIONS IN THE 3 DIFFERENT REGIONS OF r, AND CONDITIONS TO BE APPLIED

$$\Phi_1(r,\theta)=A+B\ln r+\sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta+D_n r^{-n}\cos n\theta+E_n r^n\sin n\theta+F_n r^{-n}\sin n\theta\right]$$

$$\Phi_2(r,\theta)=G+H\ln r+\sum_{n=1}^{\infty}\left[J_n r^n\cos n\theta+K_n r^{-n}\cos n\theta+L_n r^n\sin n\theta+M_n r^{-n}\sin n\theta\right]$$

$$\Phi_3(r,\theta)=N+P\ln r+\sum_{n=1}^{\infty}\left[Q_n r^n\cos n\theta+R_n r^{-n}\cos n\theta+T_n r^n\sin n\theta+W_n r^{-n}\sin n\theta\right]$$

APPLY CONDITION AS $r\to\infty$ $\quad\Phi_1(r,\theta)\longrightarrow r\cos\theta$

HENCE $\quad A=B=0$

$E_n=0$

$C_1=1$, $C_n=0$ $\quad n\geqslant 2$

D_n, F_n ARE UNDETERMINED

$$\Phi_1(r,\theta)=r\cos\theta+\sum_{n=1}^{\infty}\left[D_n r^{-n}\cos n\theta+F_n r^{-n}\sin n\theta\right]$$

DIFFERENTIATE W.R.T r AND APPLY $\dfrac{\partial\Phi_1}{\partial r}=2\cos\theta$ AT $r=2$

$$\frac{\partial\Phi_1}{\partial r}(r,\theta)=\cos\theta-\sum_{n=1}^{\infty}\left[nD_n r^{-n-1}\cos n\theta+nF_n r^{-n-1}\sin n\theta\right]$$

$$2\cos\theta=\cos\theta-\sum_{n=1}^{\infty}\left[nD_n 2^{-n-1}\cos n\theta+nF_n 2^{-n-1}\sin n\theta\right]$$

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HENCE $F_n = 0$

$D_n = 0 \quad n = 2,3,4,5,\ldots$

BUT IF $n=1$

$$2\cos\theta = \cos\theta - 1 \times D_1 \times 2\cos\theta$$
$$2\cos\theta = \cos\theta - \tfrac{1}{4}D_1\cos\theta$$
$$2 = 1 - \tfrac{1}{4}D_1$$
$$\tfrac{1}{4}D_1 = -1$$
$$D_1 = -4$$

$$\therefore \Phi_1(r,\theta) = r\cos\theta - \frac{4}{r}\cos\theta$$

APPLY CONDITION $\Phi_2(r,\theta) = \cos\theta$ AT $r=1$

$$\Rightarrow \cos\theta = G + H\ln 1 + \sum_{n=1}^{\infty}\left[J_n\cos n\theta + K_n\cos n\theta + L_n\sin n\theta + M_n\sin n\theta\right]$$
$$\Rightarrow \cos\theta = G + \sum_{n=1}^{\infty}\left[(J_n+K_n)\cos n\theta + (L_n+M_n)\sin n\theta\right]$$

HENCE $G = 0$

H IS UNDETERMINED

$J_1 + K_1 = 1$

$J_n + K_n = 0 \quad n = 2,3,4,5,\ldots$

$L_n + M_n = 0 \quad n = 1,2,3,4,\ldots$

$$\Phi_2(r,\theta) = H\ln r + \sum_{n=1}^{\infty}\left[J_n r^n\cos n\theta + K_n r^{-n}\cos n\theta + L_n r^n\sin n\theta + M_n r^{-n}\sin n\theta\right]$$

DIFFERENTIATE Φ_2 W.R.T r AND APPLY $\frac{\partial\Phi_2}{\partial r} = 2\cos\theta$ AT $r=2$

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$$\Rightarrow \frac{\partial \phi_2}{\partial r} = \frac{H}{r} + \sum_{n=1}^{\infty}\left[nJ_n r^{n-1}\cos n\theta - nk_n r^{-n-1}\cos n\theta + nL_n r^{n-1}\sin n\theta - nM_n r^{-n-1}\sin n\theta \right]$$

$$\Rightarrow 2\cos\theta = \frac{H}{2} + \sum_{n=1}^{\infty}\left[nJ_n 2^{n-1}\cos n\theta - \frac{nk_n}{2^{n+1}}\cos n\theta + nL_n 2^{n-1}\sin n\theta - \frac{nM_n}{2^{n+1}}\sin n\theta \right]$$

HENCE $H = 0$

IF $n=1$

$$2\cos\theta = J_1\cos\theta - \frac{k_1}{4}\cos\theta$$

$$2 = J_1 - \frac{1}{4}k_1$$

IF $n = 2,3,4,5,\ldots$

$$0 = nJ_n 2^{n-1}\cos n\theta - \frac{nk_n}{2^{n+1}}\cos n\theta$$

$$0 = J_n 2^{n-1} - \frac{k_n}{2^{n+1}}$$

IF $n = 1,2,3,4,\ldots$

$$0 = nL_n 2^{n-1}\sin n\theta - \frac{nM_n}{2^{n+1}}\sin n\theta$$

$$0 = L_n 2^{n-1} - \frac{M_n}{2^{n+1}}$$

FROM EARLIER $J_1 + k_1 = 1$

$$\left.\begin{aligned} J_1 + k_1 &= 1 \\ J_1 - \tfrac{1}{4}k_1 &= 2 \end{aligned}\right\} \Rightarrow \tfrac{5}{4}k_1 = -1$$

$$\Rightarrow k_1 = -\frac{4}{5} \quad \& \quad J_1 = \frac{9}{5}$$

ALSO FROM EARLIER $J_n + k_n = 0$, $n = 2,3,4,5,\ldots$

$$\left.\begin{aligned} J_n + k_n &= 0 \\ J_n 2^{n-1} - \frac{k_n}{2^{n+1}} &= 0 \end{aligned}\right\} \Rightarrow J_n = -k_n$$

$$\Rightarrow -k_n 2^{n-1} - \frac{k_n}{2^{n+1}} = 0$$

$$\Rightarrow -k_n\left(2^{n-1} - \frac{1}{2^{n+1}}\right) = 0$$

$$\therefore k_n = 0 \quad \& \quad J_n = 0 \qquad n = 2,3,4,5,\ldots$$

AND IN ANALOGY $L_n = M_n = 0$, $n = 1,2,3,4,\ldots$

$$\Rightarrow \Phi_2(r,\theta) = \frac{9}{5}r\cos\theta - \frac{4}{5r}\cos\theta$$

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$$\Rightarrow \Phi_2(r,\theta) = \frac{1}{5}\left(qr - \frac{4}{r}\right)\cos\theta$$

Now $\Phi_3(r,\theta) = \cos\theta$ at $r=1$

$$\Rightarrow \cos\theta = N + P\ln r + \sum_{n=1}^{\infty}\left[Q_n\cos n\theta + R_n\cos n\theta + T_n\sin n\theta + W_n\sin n\theta\right]$$

$$\Rightarrow \cos\theta = N = \sum_{n=1}^{\infty}\left[(Q_n+R_n)\cos n\theta + (T_n+W_n)\sin n\theta\right]$$

Hence $N=0$

P is undetermined

$T_n + W_n = 0$, $n=1,2,3,4,5,\ldots$

$Q_1 + R_1 = 1$

$Q_n + R_n = 0$, $n=2,3,4,5,\ldots$

Differentiate Φ_3 w.r.t r and apply $r\frac{\partial \Phi_3}{\partial r} \to 1$ as $r \to 0$

$$\Phi_3(r,\theta) = P\ln r + \sum_{n=1}^{\infty}\left[Q_n r^n\cos n\theta + R_n r^{-n}\cos n\theta + T_n r^n\sin n\theta + W_n r^{-n}\sin n\theta\right]$$

$$\frac{\partial \Phi_3}{\partial r}(r,\theta) = \frac{P}{r} + \sum_{n=1}^{\infty}\left[nQ_n r^{n-1}\cos n\theta - nR_n r^{-n-1}\cos n\theta + nT_n r^{n-1}\sin n\theta - nW_n r^{-n-1}\sin n\theta\right]$$

$$r\frac{\partial \Phi_3}{\partial r}(r,\theta) = P + \sum_{n=1}^{\infty}\left[nQ_n r^n\cos n\theta - nR_n r^{-n}\cos n\theta + nT_n r^n\sin n\theta - nW_n r^{-n}\sin n\theta\right]$$

Now as $r\to 0$ $\quad r\frac{\partial \Phi_3}{\partial r} \to 1$

$P=1$ $\quad R_n = W_n = 0$, $n = 1,2,3,4,\ldots$

But from earlier

$T_n + W_n = 0$ & now $W_n = 0$ $\Rightarrow$ $T_n = 0$ $(n=1,2,3,4\ldots)$

$Q_n + R_n = 0$ & now $R_n = 0$ $\Rightarrow$ $Q_n = 0$ $(n=2,3,4,5,\ldots)$

$Q_1 + R_1 = 1$ & now $R_1 = 0$ $\Rightarrow$ $Q_1 = 1$

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$$\therefore \Phi_3(r,\theta) = \ln r + r\cos\theta$$

COLLECTING ALL 3 RESULTS WE HAVE

$$\Phi(r,\theta) = \begin{cases} \left(r - \frac{4}{r}\right)\cos\theta & r > 2 \\ \frac{1}{5}\left(9r - \frac{4}{r}\right)\cos\theta & 1 < r < 2 \\ \ln r + r\cos\theta & 0 < r < 1 \end{cases}$$