IYGB - MATHEMATICAL METHODS 4 - PAPER C - QUESTION 1

USING THE TRANSFORMATION EQUATIONS GIVEN

$$\xi = x+y \quad \& \quad \eta = x-y$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial z}{\partial \xi}\times 1 + \frac{\partial z}{\partial \eta}\times 1 = \frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial z}{\partial \xi}\times 1 + \frac{\partial z}{\partial \eta}\times (-1) = \frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}$$

SUBSTITUTE INTO THE O.D.E

$$\Rightarrow \frac{\partial z}{\partial x} + \frac{\partial z}{\partial x} = 6(x+y)^2 z^2$$

$$\Rightarrow \left(\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}\right) - \left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}\right) = 6\xi^2 z^2$$

$$\Rightarrow 2\frac{\partial z}{\partial \eta} = 6\xi^2 z^2$$

$$\Rightarrow \frac{\partial z}{\partial \eta} = 3\xi^2 z^2$$

SOLVE BY SEPARATION OF VARIABLES & DIRECT INTEGRATION

$$\Rightarrow \frac{1}{z^2}\,\partial z = 3\xi^2\,\partial \eta$$

$$\Rightarrow -\frac{1}{z} = \xi^3 + f(\eta)$$

$$\Rightarrow -\dot{z} = \frac{1}{\xi^3 + f(\eta)}$$

$$\Rightarrow z = \frac{1}{-\xi^3 + f(\eta)}$$

$$\Rightarrow z(x,y) = \frac{1}{f(x-y) - (x+y)^3}$$

IYGB - MATHEMATICAL METHODS 4 - PAPER C - QUESTION 2

SOLVING THE WAVE EQUATION IN ONE DIMENSION

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{4}\frac{\partial^2 z}{\partial t^2} \quad \text{SUBJECT TO} \quad z(x,0) = e^{-x^2}, \; -\infty < x < \infty$$

$$\frac{\partial z}{\partial t}(x,0) = 0$$

ASSUMING D'ALEMBERT'S STANDARD SOLUTION FOR

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \Rightarrow z(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\zeta)\, d\zeta$$

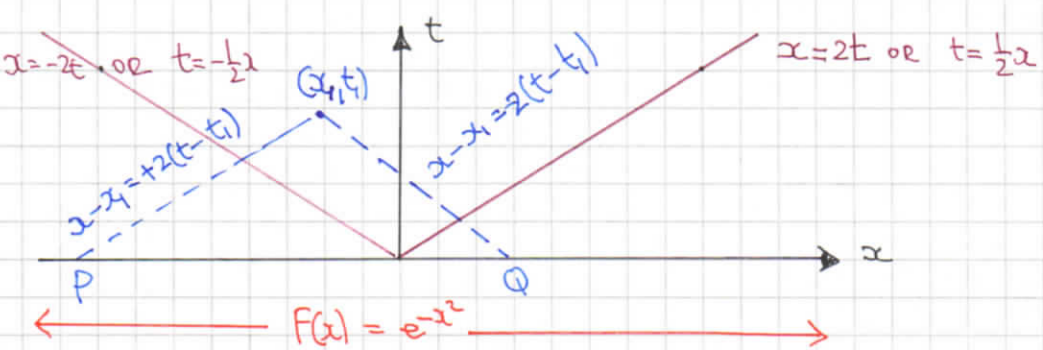
$$\text{WHERE} \quad F(x) = z(x,0)$$
$$G(x) = \frac{\partial z}{\partial t}(x,0)$$

HERE $c=2$, $F(x) = e^{-x^2}$ & $G(x) = 0$

$$z(x,t) = \frac{1}{2}e^{-(x-2t)^2} + \frac{1}{2}e^{-(x+2t)^2}$$

ALTERNATIVE BY CHARACTERISTICS ON THE x-t PLANE

Draw the characteristics, ie lines with gradient $\pm\frac{1}{c}$ (here $\pm\frac{1}{2}$) through any "keypoints" on the x axis — here there are no keypoints as $F(x) = e^{-x^2}$ for all x so we may draw them say through $x=0$



- consider an arbitrary point (x_1, t_1) anywhere on this plane
- draw lines with gradient $\pm\frac{1}{2}$ through that point
- determine the x intercepts, P & Q, by setting $t=0$

$$P(x_1 - 2t_1, 0) \text{ & } Q(x_1 + 2t_1)$$

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THUS WE NOW HAVE

$$z(x,t) = \frac{1}{2}F(P) + \frac{1}{2}F(Q)$$

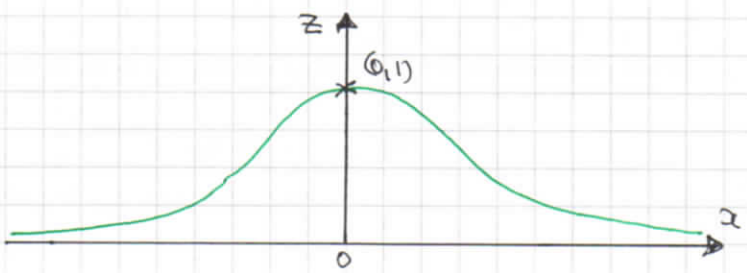
$$z(x_1,t_1) = \frac{1}{2}e^{-(x_1-2t_1)^2} + \frac{1}{2}e^{-(x_1+2t_1)^2}$$

AS THERE IS NOTHING SPECIAL ABOUT THE POINT WE MAY WRITE

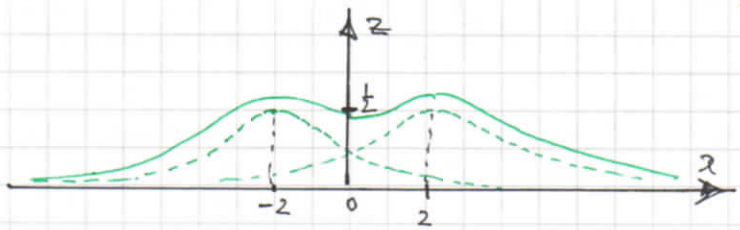
$$z(x,t) = \frac{1}{2}e^{-(x-2t)^2} + \frac{1}{2}e^{-(x+2t)^2}$$

b) DRAWING THE PROFILES FOR $t=0,1,2,3$

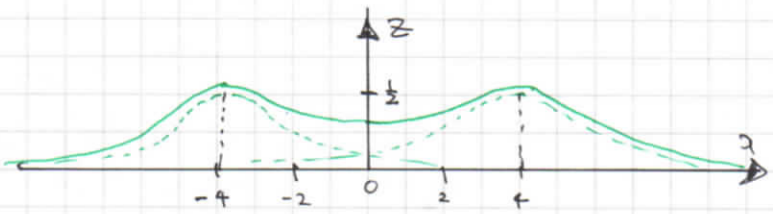
- $z(x,0) = e^{-x^2}$



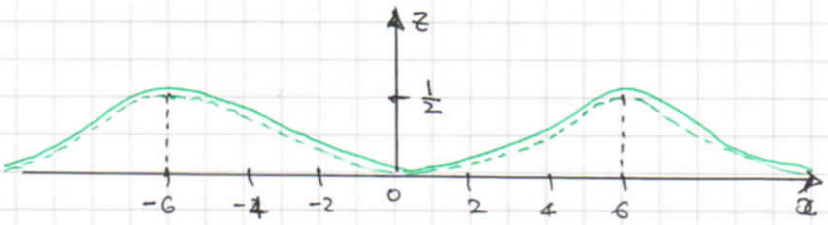
- $z(x,1) = \frac{1}{2}e^{-(x-2)^2} + \frac{1}{2}e^{-(x+2)^2}$



- $z(x,2) = \frac{1}{2}e^{-(x-4)^2} + \frac{1}{2}e^{-(x+4)^2}$



- $z(x,3) = \frac{1}{2}e^{-(x-6)^2} + \frac{1}{2}e^{-(x+6)^2}$

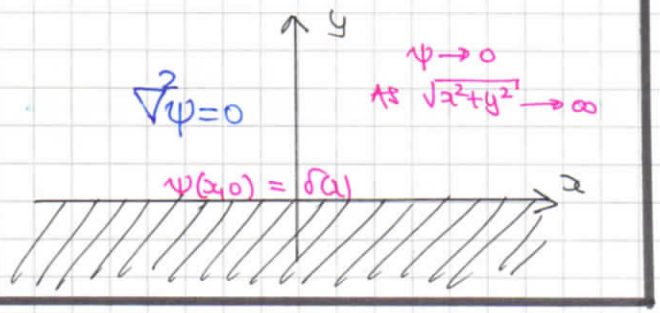


IYGB - MATHEMATICAL METHODS 4 - PAPER C - QUESTION 3

SOLVING LAPLACE'S EQUATION

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

- $\psi(x,0) = \delta(x)$
- $\psi(x,y) \to 0$ AS $\sqrt{x^2+y^2} \to \infty$



TAKING FOURIER TRANSFORM OF THE P.D.E IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 \psi}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 \psi}{\partial y^2}\right] = \mathcal{F}[0]$$

$$\Rightarrow (ik)^2 \hat{\psi}(k,y) + \frac{\partial^2}{\partial y^2}[\hat{\psi}(k,y)] = 0$$

$$\Rightarrow \frac{\partial^2 \hat{\psi}}{\partial y^2} - k^2 \hat{\psi} = 0 \quad , \quad \hat{\psi} = \hat{\psi}(k,y)$$

SOLVING THE O.D.E, AS k IS A CONSTANT

$$\Rightarrow \hat{\psi}(k,y) = A(k)e^{-|k|y} + B(k)e^{|k|y}$$

AS $\psi(x,y)$ VANISHES AS "LARGE" DISTANCES, SO WOULD $\hat{\psi}(k,y)$, SO THIS IMPLIES THAT $B(k) = 0$

$$\Rightarrow \hat{\psi}(k,y) = A(k)e^{-|k|y}$$

NEXT WE TAKE THE FOURIER TRANSFORM OF THE CONDITION $\psi(x,0) = \delta(x)$

$$\psi(x,0) = \delta(x) \Rightarrow \hat{\psi}(k,0) = \mathcal{F}(\delta(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \times e^{-ikx0} = \frac{1}{\sqrt{2\pi}}$$

HENCE $\hat{\psi}(k,0) = \frac{1}{\sqrt{2\pi}}$

$$\frac{1}{\sqrt{2\pi}} = A(k)e^0$$

$$\boxed{A(k) = \frac{1}{\sqrt{2\pi}}}$$

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$$\Rightarrow \hat{\psi}(k,y) = \frac{1}{\sqrt{2\pi}} e^{-|k|y}$$

INVERTING THE TRANSFORM ABOVE

$$\Rightarrow \psi(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} e^{-|k|y} \right) e^{ikx} \, dk$$

$$\Rightarrow \psi(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|k|y} e^{ikx} \, dk$$

$$\Rightarrow \psi(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|k|y} (\cos kx + i \sin kx) \, dk$$

INTEGRATING THE EVEN PART TWICE FROM $k=0$ TO $k=\infty$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi} \int_{0}^{\infty} e^{-ky} \cos kx \, dk$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi} Re \left[\int_{0}^{\infty} e^{-ky} e^{ikx} \, dk \right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi} Re \left[\int_{0}^{\infty} e^{k(-y+ix)} \, dk \right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi} Re \left[\frac{1}{-y+ix} \left[e^{k(-y+ix)} \right]_0^{\infty} \right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi} Re \left[\frac{-y-ix}{y^2+x^2} \left[e^{-ky} (\cos kx + i \sin kx) \right]_0^{\infty} \right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi} Re \left[\frac{-y-ix}{y^2+x^2} [0 - 1] \right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi} Re \left[\frac{y+ix}{y^2+x^2} \right]$$

$$\Rightarrow \psi(x,y) = \frac{1}{\pi} \left(\frac{y}{x^2+y^2} \right)$$

IYGB - MATHEMATICAL METHODS 4 - PAPER D - QUESTION 4

SOLVE THE P.D.E BY "LAGRANGE'S METHOD"

$$e^x \frac{\partial z}{\partial x} + 1 \frac{\partial z}{\partial t} = 0$$

P Q R

$$\frac{dx}{P} = \frac{dt}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{e^x} = \frac{dt}{1} = \frac{dz}{0}$$

① ② ③

EQUATING ① & ② YIELDS

$$\Rightarrow \frac{dx}{e^x} = \frac{dt}{1}$$

$$\Rightarrow \int e^{-x}\,dx = \int 1\,dt$$

$$\Rightarrow -e^{-x} = t + C_1$$

$$\Rightarrow t + e^{-x} = C$$

$$\therefore u(x,t) = t + e^{-x}$$

EQUATION ③ INVOLVES DIVISION BY ZERO

$$\Rightarrow dz = 0$$ (OTHERWISE THE RATIO IS MEANINGLESS)

$$\Rightarrow z = C_2$$

$$\therefore v(z) = z$$

SINCE WE HAVE A GENERAL SOLUTION $F(u,v) = 0$

$$\therefore v = f(u) \quad \text{or} \quad u = g(v)$$

$$\therefore z = f(t + e^{-x})$$

IYGB - MATHEMATICAL METHODS 4 - PAPER C - QUESTION 4

APPLY THE INITIAL CONDITION, $t=0 \quad z=\tanh x$

$$\Rightarrow \tanh x = f(e^{-x})$$

LET $w = e^{-x}$

$$\frac{1}{w} = e^{x}$$

$$x = \ln \frac{1}{w}$$

$$x = -\ln w$$

$$\Rightarrow \tanh(-\ln w) = f(w)$$

$$\Rightarrow f(w) = \tanh(-\ln w)$$

$$\Rightarrow f(w) = -\tanh(\ln w)$$

$$\Rightarrow f(t+e^{-x}) = -\tanh\left[\ln(t+e^{-x})\right]$$

FINALLY WE OBTAIN

$$z = -\tanh\left[\ln\left(t+e^{-x}\right)\right]$$

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SOLVING THE STANDARD WAVE EQUATION SUBJECT TO THESE CONDITIONS

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

①	$z(0,t) = 0$	$\forall t \geqslant 0$
②	$z(L,t) = 0$	$\forall t \geqslant 0$
③	$z(x,0) = f(x)$	$\forall x : 0 \leqslant x \leqslant L$
④	$\frac{\partial z}{\partial t}(x,0) = G(x)$	$\forall x : 0 \leqslant x \leqslant L$

ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM

$$\left.\begin{aligned} z(x,t) &= X(x)T(t) \\ \frac{\partial^2 z}{\partial x^2} &= X''(x)T(t) \\ \frac{\partial^2 z}{\partial t^2} &= X(x)T''(t) \end{aligned}\right\}$$ SUBSTITUTE INTO THE P.D.E

$$\Rightarrow X''(x)T(t) = \frac{1}{c^2}X(x)T''(t)$$

$$\Rightarrow \frac{X''(x)T(t)}{X(x)T(t)} = \frac{1}{c^2}\frac{X(x)T''(t)}{X(x)T(t)}$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}$$

AS THE L.H.S IS A FUNCTION OF x ONLY AND THE R.H.S IS A FUNCTION OF t ONLY, BOTH SIDES MUST BE EQUAL TO AT MOST A CONSTANT, SAY λ

IF $\lambda > 0$, SAY $\lambda = p^2$

- $\frac{X''(x)}{X(x)} = p^2$

  $X''(x) = p^2 X(x)$

  $X(x) = A\cosh px + B\sinh px$ (or exponentials)

- $\frac{1}{c^2}\frac{T''(t)}{T(t)} = p^2$

  $T''(t) = p^2c^2T(t)$

  $T(t) = D\cosh(pct) + E\sinh(pct)$ (or exponentials)

$$\therefore z(x,t) = (A\cosh px + B\sinh px)(D\cosh(pct) + E\sinh(pct))$$

IYGB - MATHEMATICAL METHODS 4 - PAPER C - QUESTION 5

IF $\lambda=0$

- $\frac{X''(x)}{X(x)}=0$

$X''(x)=0$

$X(x)=Ax+B$

- $\frac{1}{c^2}\frac{T''(t)}{T(t)}=0$

$T''(t)=0$

$T(t)=Dt+E$

$\therefore z(x,t)=(Ax+B)(Dt+E)$

IF $\lambda<0$, SAY $\lambda=-p^2$

- $\frac{X''(x)}{X(x)}=-p^2$

$X''(x)=-p^2X(x)$

$X(x)=A\cos px+B\sin px$

- $\frac{1}{c^2}\frac{T''(t)}{T(t)}=-p^2$

$T''(t)=-c^2p^2T(t)$

$T(t)=D\cos cpt+E\sin cpt$

$\therefore z(x,t)=(A\cos px+B\sin px)(D\cos cpt+E\sin cpt)$

BECAUSE OF THE BOUNDARY CONDITIONS, $z(0,t)=z(4,t)$, WE REQUIRE A SOLUTION WHICH GIVES THE SAME VALUE OF z FOR TWO DIFFERENT VALUES OF x — CONSEQUENTLY WE REQUIRE A CIRCULAR SOLUTION, OR A CONSTANT SOLUTION WHICH OF COURSE IS ALSO INCLUDED IN THE TRIGONOMETRIC PART

$\therefore \boxed{z(x,t)=[A\cos px+B\sin px][D\cos pct+E\sin pct]}$

APPLY CONDITION ①, $z(0,t)=0$

$\Rightarrow 0=(A+0)(D\cos pct+E\sin pct)$

$\Rightarrow A=0$ ($D=E=0$ IS TRIVIAL, AS IT GIVES $z(x,t)=0$)

$\Rightarrow z(x,t)=(B\sin px)(D\cos pct+E\sin pct)$

$\therefore \boxed{z(x,t)=\sin px[P\cos pct+E\sin pct]}$

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APPLY CONDITION ②, $Z(L,t)=0$

$$\Rightarrow 0 = \sin pL\left[P\cos pct + Q\sin pct\right]$$

$$\Rightarrow pL = n\pi \qquad \left[P=Q=0 \text{ IS TRIVIAL AS IT GIVES } Z(x,t)=0\right]$$

$$\Rightarrow p = \frac{n\pi}{L} \qquad n=0,1,2,3,\ldots$$

$$\therefore Z_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[P\cos\left(\frac{n\pi ct}{L}\right) + Q\sin\left(\frac{n\pi ct}{L}\right)\right] \qquad n=1,2,3,4,\ldots$$

$n\neq 0$ AS IT IS TRIVIAL

$$\therefore \boxed{Z(x,t) = \sum_{n=1}^{\infty}\left[\sin\left(\frac{n\pi x}{L}\right)\left[P_n\cos\left(\frac{n\pi ct}{L}\right) + Q_n\sin\left(\frac{n\pi ct}{L}\right)\right]\right]}$$

DIFFERENTIATE W.R.T t IN ORDER TO APPLY ④, $\frac{\partial Z}{\partial t}(x,0) = G(x)$

$$\frac{\partial Z}{\partial t}(x,t) = \sum_{n=1}^{\infty}\left[\sin\left(\frac{n\pi x}{L}\right)\left[-\frac{n\pi c}{L}P_n\sin\left(\frac{n\pi ct}{L}\right) + \frac{n\pi c}{L}Q_n\cos\left(\frac{n\pi ct}{L}\right)\right]\right]$$

$$\frac{\partial Z}{\partial t}(x,0) = G(x) = \sum_{n=1}^{\infty}\left[\left[\sin\frac{n\pi x}{L}\right]\left[\frac{n\pi c}{L}Q_n\right]\right]$$

$$G(x) = \sum_{n=1}^{\infty}\left[\frac{n\pi c Q_n}{L}\sin\frac{n\pi x}{L}\right]$$

APPLY ③, $Z(x,0) = F(x)$

$$F(x) = \sum_{n=1}^{\infty}\left[\sin\left(\frac{n\pi x}{L}\right)\right]\left[P_n\right]$$

$$F(x) = \sum_{n=1}^{\infty}\left[P_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

THESE ARE FOURIER SINE EXPRESSIONS IN x, OVER THE RANGE 0 TO L, I.E HALF PERIOD $\frac{L}{2}$

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THUS WE HAVE FINALLY

$$P_n = \frac{1}{L/2}\int_0^L F(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$P_n = \frac{2}{L}\int_0^L F(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

AND SIMILARLY

$$\frac{n\pi c Q_n}{L} = \frac{1}{L/2}\int_0^L G(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

↓ RELABELLING THE CONSTANT

$$Q_n = \frac{2}{L}\int_0^L G(x)\sin\frac{n\pi x}{L}\,dx$$

$$\therefore \quad z(x,t) = \sum_{n=1}^{\infty}\left[\left[\sin\left(\frac{n\pi x}{L}\right)\right]\left[P_n\cos\left(\frac{n\pi ct}{L}\right) + Q_n\sin\left(\frac{n\pi ct}{L}\right)\right]\right]$$

where $P_n = \frac{2}{L}\int_0^L F(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$Q_n = \frac{2}{L}\int G(x)\sin\left(\frac{n\pi x}{L}\right)dx$

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ASSUME A SOLUTION IN VARIABLE FORM, DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E.

$$u(x,y) = X(x)Y(y) \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = X''(x)Y(y)$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = X(x)Y''(y)$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Rightarrow \quad X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\Rightarrow \quad \frac{X''(x)Y(y)}{X(x)Y(y)} + \frac{X(x)Y''(y)}{X(x)Y(y)} = \frac{0}{X(x)Y(y)}$$

$$\Rightarrow \quad \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

$$\Rightarrow \quad \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$$

AS THE L.H.S OF THE ABOVE EQUATION IS A FUNCTION OF x ONLY & THE R.H.S. IS A FUNCTION OF y ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ – LOOKING AT THE BOUNDARY CONDITIONS

(1) $u(0,y)=0$

(2) $u(2,y)=0$

(3) $u(x,0)=0$

(4) $u(x,1) = \begin{cases} x & 0 \leqslant x \leqslant 1 \\ 2-x & 1 \leqslant x \leqslant 2 \end{cases}$

LOOKING AT (1) & (2), WE OBSERVE THAT WE NEED A PERIODIC SOLUTION IN x (AS $u(0,y) = u(2,y)$), SO THE CONSTANT λ MUST BE NEGATIVE, $\lambda = -p^2$

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$$\Rightarrow \frac{X''(x)}{X(x)} = -p^2$$

$$\Rightarrow X''(x) = -p^2 X(x)$$

$$\Rightarrow X(x) = A\cos px + B\sin px$$

$$\Rightarrow -\frac{Y''(y)}{Y(y)} = -p^2$$

$$\Rightarrow Y''(y) = p^2 Y(y)$$

$$\Rightarrow Y(y) = C\cosh py + D\sinh py$$

(OR EXPONENTIALS)

THUS THE GENERAL SOLUTION IS

$$u(x,y) = X(x)Y(y) = [A\cos px + B\sin px][C\cosh py + D\sinh py]$$

- APPLY (1), $u(0,y) = 0$

$$\Rightarrow 0 = A[C\cosh py + D\sinh py), \quad 0 \leq y \leq 1$$

$$\Rightarrow A = 0$$

ABSORBING A INTO C AND D, YIELDS

$$u(x,y) = (\sin px)(C\cosh py + D\sinh py)$$

- APPLY (3), $u(x,0) = 0$

$$\Rightarrow 0 = (\sin px)C$$

$$\Rightarrow C = 0$$

$$\Rightarrow u(x,y) = D\sin px \sinh py$$

- APPLY (2), $u(2,y) = 0$

$$\Rightarrow 0 = D\sin 2p \sinh py$$

$$\Rightarrow D \neq 0$$, OTHERWISE WE HAVE A TRIVIAL SOLUTION

$$\Rightarrow \sin 2p = 0$$

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$$\Rightarrow 2p = n\pi \qquad n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow p = \frac{n\pi}{2} \qquad n = 0, 1, 2, 3, \ldots$$

$$\therefore \quad \boxed{u_n(x,y) = D_n \sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi y}{2}\right)}$$

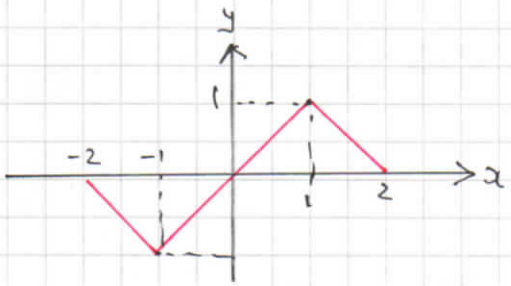
$$\boxed{u(x,y) = \sum_{n=1}^{\infty}\left[D_n \sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi y}{2}\right)\right]}$$

- FINALLY APPLY (4) $u(x,1) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \end{cases}$

$$\Rightarrow u(x,1) = \sum_{n=1}^{\infty}\left[D_n \sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi}{2}\right)\right]$$

$$\Rightarrow u(x,1) = \sum_{n=1}^{\infty}\left[\left[D_n \sinh\frac{n\pi}{2}\right]\sin\left(\frac{n\pi x}{2}\right)\right]$$

THIS IS A FOURIER SERIES IN x — BUILT AN ODD EXTENSION SO WE GET AN EVEN INTEGRAND WITH PERIOD 4



$$\Rightarrow D_n \sinh\frac{n\pi}{2} = \frac{1}{L}\int_{-L}^{L} \overset{\text{ODD}}{f(x)} \overset{\text{ODD}}{\sin\frac{n\pi x}{L}}\, dx$$

$$\Rightarrow D_n \sinh\frac{n\pi}{2} = \frac{1}{2}\int_{-2}^{2} u(x,1)\sin\frac{n\pi x}{2}\, dx$$

$$\Rightarrow D_n \sinh\frac{n\pi}{2} = \int_{0}^{2} u(x,1)\sin\frac{n\pi x}{2}\, dx$$

$$\Rightarrow D_n \sinh\frac{n\pi}{2} = \int_{0}^{1} x\sin\frac{n\pi x}{2}\, dx + \int_{1}^{2}(2-x)\sin\frac{n\pi x}{2}\, dx$$

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INTEGRATE EACH SEPARATELY (BY PARTS)

x	1
$-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$	$\sin\left(\frac{n\pi x}{2}\right)$

$2-x$	-1
$-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$	$\sin\left(\frac{n\pi x}{2}\right)$

$$\Rightarrow D_n \sinh\left(\frac{n\pi}{2}\right) = \left[-\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_0^1 + \frac{2}{n\pi}\int_0^1 \cos\left(\frac{n\pi x}{2}\right)dx$$
$$+\left[\frac{2(x-2)}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_1^2 - \frac{2}{n\pi}\int_1^2 \cos\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow D_n \sinh\left(\frac{n\pi}{2}\right) = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi}\int_0^1 \cos\left(\frac{n\pi x}{2}\right)dx$$
$$+\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\int_1^2 \cos\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow D_n \sinh\left(\frac{n\pi}{2}\right) = \frac{4}{n^2\pi^2}\left[\sinh\left(\frac{n\pi x}{2}\right)\right]_0^1 - \frac{4}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{2}\right)\right]_1^2$$

$$\Rightarrow D_n \sinh\left(\frac{n\pi}{2}\right) = \frac{4}{n^2\pi^2}\sinh\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2}\left[-\sin\left(\frac{n\pi}{2}\right)\right]$$

$$\Rightarrow D_n \sinh\left(\frac{n\pi}{2}\right) = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\Rightarrow D_n = \frac{8\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2\sinh\left(\frac{n\pi}{2}\right)}$$

FINALLY WE HAVE A GENERAL SOLUTION

$$\therefore \quad \boxed{u(x,y) = \sum_{n=1}^{\infty}\left[\frac{8}{n^2\pi^2}\frac{\sin\left(\frac{n\pi}{2}\right)}{\sinh\left(\frac{n\pi}{2}\right)}\sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi y}{2}\right)\right]}$$

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ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM - DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E.

- $\Phi(r,\theta) = R(r)\Theta(\theta) \Rightarrow \dfrac{\partial \Phi}{\partial r} = R'(r)\Theta(\theta)$

$$\Rightarrow \frac{\partial^2 \phi}{\partial r^2} = R''(r)\Theta(\theta)$$

$$\Rightarrow \frac{\partial^2 \Phi}{\partial \theta^2} = R(r)\Theta''(\theta)$$

- $\dfrac{\partial^2 \phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2} = 0$

$$\Rightarrow R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

$$\Rightarrow \frac{R''(r)\Theta(\theta)}{R(r)\Theta(\theta)} + \frac{1}{r}\frac{R'(r)\Theta(\theta)}{R(r)\Theta(\theta)} + \frac{1}{r^2}\frac{R(r)\Theta''(\theta)}{R(r)\Theta(\theta)} = 0$$

$$\Rightarrow \frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$

$$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$

$$\Rightarrow \boxed{r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)}}$$

AS THE L.H.S. OF THE ABOVE EXPRESSION IS A FUNCTION OF r ONLY AND THE R.H.S IS A FUNCTION OF θ ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ

LOOKING AT THE R.H.S OF THE ABOVE EXPRESSION, WE REQUIRE CIRCULAR SOLUTIONS IN θ, DUE TO THE POLAR SYSTEM REPEATING EVERY 2π - GIVEN THAT THERE IS A MINUS IN THE R.H.S, WE DEDUCE THAT $\lambda > 0$, SAY p^2

$$\Rightarrow -\frac{\Theta''(\theta)}{\Theta(\theta)} = p^2$$

$$\Rightarrow \Theta''(\theta) = -p^2\Theta(\theta)$$

$$\Rightarrow \boxed{\Theta(\theta) = A\cos(p\theta) + B\sin(p\theta)}$$

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LOOKING AT THE "θ-SOLUTION" WITH SAY SINES (OR COSINES)

$$\sin\theta = \sin(\theta+2\pi) \Rightarrow \sin(p\theta) = \sin(p(\theta+2\pi))$$
$$= \sin(p\theta + 2p\pi)$$
$$\Rightarrow p = n = \text{INTEGER} = 0,1,2,3,4,\ldots$$

NOTES

$n=0$ IS O.K. AS IT PRODUCES A CONSTANT SOLUTION

$n<0$ NEED NOT BE CONSIDERED SEPARATELY AS THEY WILL BE ABSORBED IN THE CONSTANTS AT THIS STAGE, BUT THEY WILL ACTUALLY APPEAR AGAIN AT THE END

$$\therefore \boxed{\Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta) \qquad n=0,1,2,3,4,\ldots}$$

NEXT WE RETURN TO THE L.H.S (FUNCTION of r ONLY), WITH n INTEGER

$$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = n^2, \quad n=0,1,2,3,4\ldots$$

IF $n=0$

$$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = 0$$
$$\Rightarrow rR''(r) + R'(r) = 0$$
$$\Rightarrow rR''(r) = -R'(r)$$
$$\Rightarrow \frac{R''(r)}{R'(r)} = -\frac{1}{r}$$

INTEGRATE W.R.T. r

$$\Rightarrow \ln|R'(r)| = -\ln r + \ln C$$
$$\Rightarrow \ln|R'(r)| = \ln\left|\frac{C}{r}\right|$$
$$\Rightarrow R'(r) = \frac{C}{r}$$

INTEGRATE W.R.T. r

$$\Rightarrow \underline{\underline{R(r) = C\ln r + D}}$$

IF $n=1,2,3,4\ldots$

$$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = n^2$$
$$\Rightarrow r^2R''(r) + rR'(r) - n^2R(r)$$

THIS IS A CAUCHY-EULER O.D.E.

LET $R(r) = r^k$, $R'(r) = kr^{k-1}$, $R''(r) = k(k-1)r^{k-2}$ } SUBSTITUTE THE O.D.E.

$$\Rightarrow k(k-1)r^k + kr^k - n^2r^k = 0$$
$$\Rightarrow k^2 - k + k - n^2 = 0$$
$$\Rightarrow k^2 = n^2$$
$$\Rightarrow k = \pm n$$

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$$\Rightarrow R_n(r) = \alpha_n r^n + \beta_n r^{-n}$$

COLLECTING ALL THE RESULTS

IF $n=0$

$$\left.\begin{aligned}\Theta_0(\theta) &= A_0 \\ R_0(r) &= C\ln r + D\end{aligned}\right\} \text{ ABSORBING CONSTANTS}$$

$$\boxed{\Phi_0(r,\theta) = \Theta_0(\theta)R_0(r) = C\ln r + D}$$

IF $n=1,2,3,4,\ldots$

$$\Theta_n(\theta) = A_n\cos n\theta + B_n \sin n\theta$$
$$R_n(r) = \alpha_n r^n + \beta_n r^{-n}$$

FINALLY WE HAVE A GENERAL SOLUTION

$$\Phi(r,\theta) = \left[C\ln r + D\right] + \sum_{n=1}^{\infty}\left[\left[A_n\cos n\theta + B_n\sin n\theta\right]\left[\alpha_n r^n + \beta_n r^{-n}\right]\right]$$

· ABSORBING & RELABELLING THE CONSTANTS

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + D_n r^{-n}\cos n\theta + E_n r^n \sin n\theta + F_n r^{-n}\sin n\theta\right]}$$

NEXT WE APPLY CONDITIONS, NOTING THAT $\Phi(r,\theta)$ MUST BE FINITE IN ALL PARTS OF THE DISC AND IN PARTICULAR AT $r=0$

$$\therefore B=0 \;,\; D_n=0 \;,\; F_n=0$$

$$\boxed{\Phi(r,\theta) = A + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + E_n r^n \sin n\theta\right]}$$

ALSO $\Phi(1,\theta) = \sin 2\theta$

$$\Rightarrow \sin 2\theta = A + \sum_{n=1}^{\infty}\left[C_n\cos n\theta + E_n\sin n\theta\right]$$

$\therefore A=0\ ,\ C_4=0,\quad E_2=1\ ,\quad E_4=0\quad n\neq 2$

$$\Rightarrow \Phi(r,\theta)= r^2\sin 2\theta$$
$$= r^2(2\sin\theta\cos\theta)$$
$$= 2(r\sin\theta)(r\cos\theta)$$
$$= 2xy$$