IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 1

$$3\frac{\partial \psi}{\partial x} - 4\frac{\partial \psi}{\partial y} = x^2$$

USING THE TRANSFORMATIONS GIVEN

$$\xi = Ax + By \qquad AD - BC \neq 0$$
$$\eta = Cx + Dy$$

$$\frac{\partial \xi}{\partial x} = A \qquad \frac{\partial \xi}{\partial y} = B$$

$$\frac{\partial \eta}{\partial x} = C \qquad \frac{\partial \eta}{\partial y} = D$$

BY THE CHAIN RULE

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial x} = A\frac{\partial \psi}{\partial \xi} + C\frac{\partial \psi}{\partial \eta}$$

$$\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = B\frac{\partial \psi}{\partial \xi} + D\frac{\partial \psi}{\partial \eta}$$

SUBSTITUTE INTO THE P.D.E

$$\Rightarrow 3\left[A\frac{\partial \psi}{\partial \xi} + C\frac{\partial \psi}{\partial \eta}\right] - 4\left[B\frac{\partial \psi}{\partial \xi} + D\frac{\partial \psi}{\partial \eta}\right] = x^2$$

$$\Rightarrow (3A - 4B)\frac{\partial \psi}{\partial \xi} + (3C - 4D)\frac{\partial \psi}{\partial \eta} = x^2$$

"KNOCK OFF" $\frac{\partial \psi}{\partial \eta}$, FURTHER SIMPLIFYING THE P.D.E.

$$\left.\begin{array}{ll} A=1 & B=0 \\ C=4 & D=3 \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} \xi = x \\ \eta = 4x+3y \end{array}}$$

IYOB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 1

THE PDE NOW TRANSFORMS

$$\Rightarrow 3\frac{\partial \psi}{\partial \xi} = \xi^2$$

$$\Rightarrow \frac{\partial \psi}{\partial \xi} = \frac{1}{3}\xi^3$$

$$\Rightarrow \psi(\xi,\eta) = \frac{1}{9}\xi^3 + f(\eta)$$

REVERSING THE TRANSFORMATIONS

$$\Rightarrow \psi(x,y) = \frac{1}{9}x^3 + f(4x+3y)$$

IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 2

$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}+z^2=0\ ,\quad z=1\ \text{AT}\ xy=x+y$$

BY LAGRANGE'S METHOD

$$\Rightarrow \underset{\uparrow P(x,y,z)}{x^2}\frac{\partial z}{\partial x}+\underset{\uparrow Q(x,y,z)}{y^2}\frac{\partial z}{\partial y}=\underset{\uparrow R(x,y,z)}{-z^2}$$

$$\Rightarrow \frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{-z^2}$$

SOLVING TWO SIMPLE O.D.Es

$$\Rightarrow \frac{dx}{x^2}=\frac{dy}{y^2}$$

$$\Rightarrow -\frac{1}{x}=-\frac{1}{y}+C$$

$$\Rightarrow -\frac{1}{x}+\frac{1}{y}=C$$

$$\Rightarrow \frac{1}{x}-\frac{1}{y}=C$$

$$u(x,y,z)=C$$

$$\Rightarrow \frac{dx}{x^2}=-\frac{dz}{z^2}$$

$$\Rightarrow -\frac{1}{x}=\frac{1}{z}+k$$

$$\Rightarrow \frac{1}{z}+\frac{1}{x}=k$$

$$v(x,y,z)=k$$

SETTING UP A GENERAL SOLUTION

$$F(u,v)=0 \quad \text{or} \quad v=f(u) \quad \text{or} \quad u=g(v)$$

$$\frac{1}{z}+\frac{1}{x}=f\left(\frac{1}{x}-\frac{1}{y}\right)$$

$$\frac{1}{z}=-\frac{1}{x}+f\left(\frac{1}{x}-\frac{1}{y}\right)$$

IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 2

APPLY THE BOUNDARY CONDITION $Z=1$ AT $xy=x+y$

$$\Rightarrow xy = x+y$$
$$\Rightarrow 1 = \frac{x+y}{xy}$$
$$\Rightarrow 1 = \frac{x}{xy} + \frac{y}{xy}$$
$$\Rightarrow 1 = \frac{1}{y} + \frac{1}{x}$$
$$\Rightarrow -\frac{1}{y} = \frac{1}{x} - 1$$

$$\Rightarrow \frac{1}{z} = -\frac{1}{x} + f\left(\frac{1}{x} - \frac{1}{y}\right)$$
$$\Rightarrow \frac{1}{1} = -\frac{1}{x} + f\left(\frac{1}{x} + \frac{1}{x} - 1\right)$$
$$\Rightarrow 1 = -\frac{1}{x} + f\left(\frac{2}{x} - 1\right)$$

LET $u = \frac{2}{x} - 1$

$$u+1 = \frac{2}{x}$$
$$x = \frac{2}{u+1}$$
$$\frac{1}{x} = \frac{u+1}{2}$$
$$-\frac{1}{x} = -\frac{u+1}{2}$$

$$\Rightarrow 1 = -\frac{u+1}{2} + f(u)$$
$$\Rightarrow 1 + \frac{u+1}{2} = f(u)$$
$$\Rightarrow \frac{u+3}{2} = f(u)$$
$$\Rightarrow f(u) = \frac{1}{2}u + \frac{3}{2}$$
$$\Rightarrow f\left(\frac{1}{x} - \frac{1}{y}\right) = \frac{1}{2}\left(\frac{1}{x} - \frac{1}{y}\right) + \frac{3}{2}$$

FINALLY WE OBTAIN

$$\Rightarrow \frac{1}{z} = -\frac{1}{x} + \frac{1}{2}\left(\frac{1}{x} - \frac{1}{y}\right) + \frac{3}{2} = -\frac{1}{x} + \frac{1}{2x} - \frac{1}{2y} + \frac{3}{2}$$
$$\Rightarrow \frac{1}{z} = \frac{3}{2} - \frac{1}{2x} - \frac{1}{2y}$$
$$\Rightarrow \frac{1}{z} = \frac{3xy - y - x}{2xy}$$

$$\therefore z = \frac{2xy}{3xy - x - y}$$

IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 3

a) $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 z}{\partial t^2}$, SUBJECT TO THE INITIAL CONDITIONS $z(x,0) = F(x)$, $\dfrac{\partial z}{\partial t}(x,0) = G(x)$

STANDARD AUXILIARY EQUATION FOR A SECOND ORDER PDE

$$\lambda^2 = \frac{1}{c^2}$$
$$\lambda = \pm\frac{1}{c}$$

$\therefore$ GENERAL SOLUTION

$$z(x,t) = f\left(-\tfrac{1}{c}x + t\right) + g\left(\tfrac{1}{c}x + t\right)$$
$$z(x,t) = f(x-ct) + g(x+ct)$$

USING THE CONDITIONS

$$z(x,0) = F(x)$$
$$f(x) + g(x) = F(x)$$

↓ diff w.r.t x

$$\boxed{f'(x) + g'(x) = F'(x)}$$

$$\frac{\partial z}{\partial t}(x,t) = -cf'(x-ct) + cg'(x+ct)$$
$$\frac{\partial z}{\partial t}(x,0) = -cf'(x) + cg'(x)$$
$$G(x) = -cf'(x) + cg'(x)$$

$$\boxed{-f'(x) + g'(x) = \frac{1}{c}G(x)}$$

ADDING & SUBTRACTING THE ABOVE EQUATIONS YIELDS:

$$\left.\begin{aligned} 2g'(x) &= F'(x) + \tfrac{1}{c}G(x) \\ 2f'(x) &= F'(x) - \tfrac{1}{c}G(x) \end{aligned}\right\} \Rightarrow \begin{aligned} g'(x) &= \tfrac{1}{2}F'(x) + \tfrac{1}{2c}G(x) \\ f'(x) &= \tfrac{1}{2}F'(x) - \tfrac{1}{2c}G(x) \end{aligned}$$

INTEGRATE THESE EQUATIONS W.R.T x

$$g(x) = \frac{1}{2}F(x) + \frac{1}{2c}\int_0^x G(\xi)\,d\xi$$
$$f(x) = \frac{1}{2}F(x) - \frac{1}{2c}\int_0^x G(\xi)\,d\xi$$

$$\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$$

AS THE ABOVE EXPRESSIONS HOLD FOR ALL x, THEY WILL ALSO HOLD FOR $x \pm ct$

$$g(x+ct) = \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_0^{x+ct} G(\xi)\,d\xi$$
$$f(x-ct) = \frac{1}{2}F(x-ct) - \frac{1}{2c}\int_0^{x-ct} G(\xi)\,d\xi = \frac{1}{2}F(x-ct) + \frac{1}{2}\int_{x-ct}^{0} G(\xi)\,d\xi$$

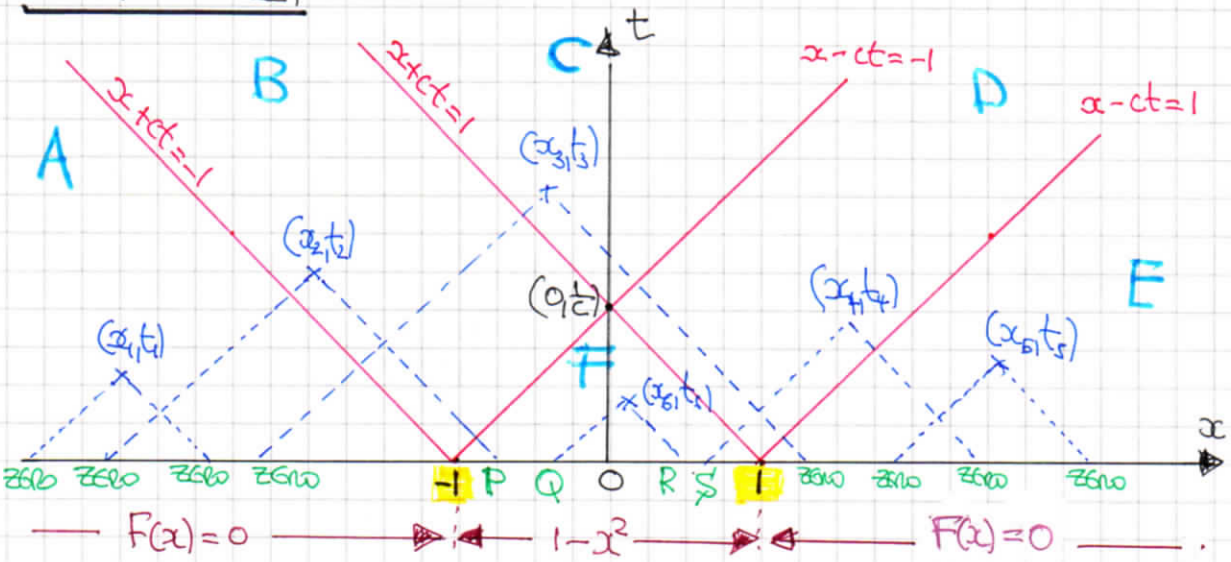
COMBINING WE HAVE

$$z(x,t) = f(x-ct) + g(x+ct) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$

IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 3

b) Now $G(x)=0$ & $F(x)=\begin{cases} 0 & |x|>1 \\ 1-x^2 & |x|\leqslant 0 \end{cases}$

DRAW THE CHARACTERISTICS IN AN x-t DIAGRAM, IE LINES WITH GRADIENT $\pm\frac{1}{c}$, PASSING THROUGH THE CRITICAL VALUES OF $F(x)$ IE AT $x=\pm 1$



IN REGIONS A, C, E, $z(x,t)=0$

IN REGION B

$t-t_2=-\frac{1}{c}(x-x_2)$

$t=0 \Rightarrow -ct_2=-x+x_2$

$\Rightarrow x=x_2+ct_2$ (P)

[THE OTHER x INTERCEPT YIELDS ZERO]

IN REGION D

$t-t_4=\frac{1}{c}(x-x_4)$

$t=0 \Rightarrow -ct_4=x-x_4$

$\Rightarrow x=x_4-ct_4$ (S)

[THE OTHER x INTERCEPT YIELDS ZERO]

SIMILARLY IN REGION F

$x=x_6+ct_6$ (R)

$x=x_6-ct_6$ (Q)

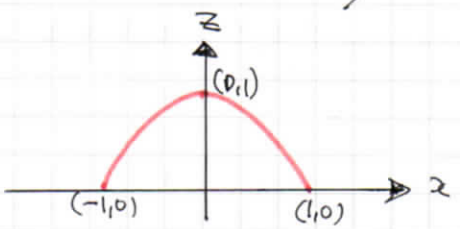
THERE IS NOTHING SPECIAL ABOUT THE POINTS PICKED, SO WE MAY "DROP" THE SUBSCRIPTS AND SUMMARIZE

$$z(x_i,t_i)=\frac{1}{2}\left[F(x_i+ct_i)+F(x_i-ct_i)\right]$$

1YGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 3

$$z(x,t)=\begin{cases}\frac{1}{2}\left[1-(x+ct)^2\right] \leftarrow \frac{1}{2}F(P) & \text{REGION B}\\ \frac{1}{2}\left[1-(x-ct)^2\right] \leftarrow \frac{1}{2}F(S) & \text{REGION D}\\ \frac{1}{2}\left[\left[1-(x+ct)^2\right]+\left[1-(x-ct)^2\right]\right] \leftarrow \frac{1}{2}F(Q)+\frac{1}{2}F(R) & \text{REGION F}\\ 0 \quad (\text{REGIONS } A, C, E)\end{cases}$$

c) $t=0 \quad z(x,0)=F(x)$



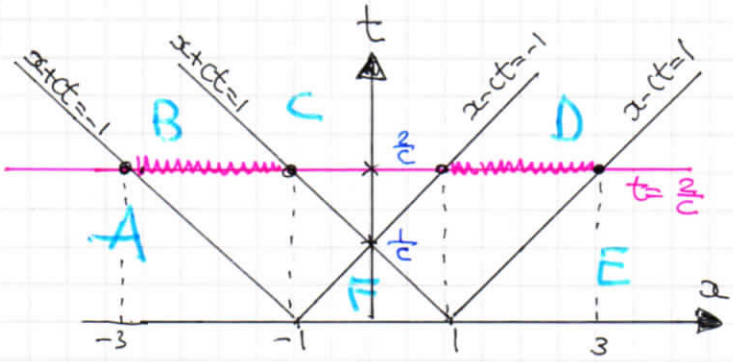
$t=\frac{2}{c}$ (FIND INTERSECTIONS)

$x+c\left(\frac{2}{c}\right)=-1 \Rightarrow x=-3$

$x+c\left(\frac{2}{c}\right)=1 \Rightarrow x=-1$

$x-c\left(\frac{2}{c}\right)=-1 \Rightarrow x=1$

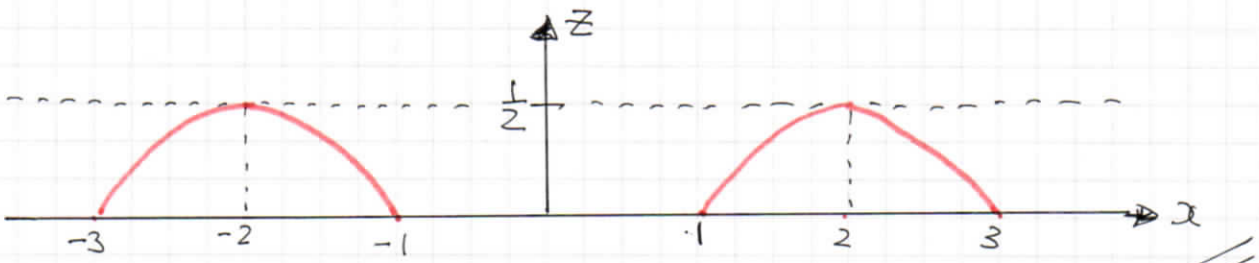
$x-c\left(\frac{2}{c}\right)=1 \Rightarrow x=3$



USING PART (b) WITH $t=\frac{2}{c}$ FOR REGIONS B & D

$$z\left(x,\tfrac{2}{c}\right)=\tfrac{1}{2}\left[1-\left(x+c\times\tfrac{2}{c}\right)^2\right]=\tfrac{1}{2}\left[1-(x+2)^2\right]$$

$$z\left(x,\tfrac{2}{c}\right)=\tfrac{1}{2}\left[1-\left(x-c\times\tfrac{2}{c}\right)^2\right]=\tfrac{1}{2}\left[1-(x-2)^2\right]$$



IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 4

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \qquad \text{SUBJECT TO} \qquad y\geqslant 0$$

$$\phi(x,y)\to 0, \text{ as } \sqrt{x^2+y^2}\to\infty$$

$$\phi(x,0)=\begin{cases}\frac{1}{2} & |x|<1\\ 0 & |x|>1\end{cases}$$

START BY TAKING THE FOURIER TRANSFORM OF THE P.D.E. IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2\phi}{\partial x^2}\right]+\mathcal{F}\left[\frac{\partial^2\phi}{\partial y^2}\right]=\mathcal{F}[0]$$

$$\Rightarrow (ik)^2\hat{\phi}(k,y)+\frac{\partial^2}{\partial y^2}\hat{\phi}(k,y)=0$$

$$\Rightarrow \frac{\partial^2\hat{\phi}}{\partial y^2}-k^2\hat{\phi}=0$$

$$\Rightarrow \hat{\phi}(k,y)=A(k)e^{-|k|y}+B(k)e^{|k|y}$$

As $\sqrt{x^2+y^2}\to\infty$, $\phi(x,y)\to 0$ $\Rightarrow$ As $\sqrt{k^2+y^2}\to\infty$, $\hat{\phi}(k,y)\to 0$

$$\Rightarrow B(k)=0$$

$$\Rightarrow \hat{\phi}(k,y)=A(k)e^{-|k|y}$$

NEXT WE TAKING THE FOURIER TRANSFORM OF $\phi(x,0)=g(x)$

$$\Rightarrow \hat{\phi}(k,0)=\hat{g}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x)e^{-ikx}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\frac{1}{2}e^{-ikx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\frac{1}{2}\cos kx-\frac{1}{2}i\sin kx\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{0}^{1}\cos kx\,dx=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{k}\sin kx\right]_0^1$$

$$=\frac{1}{\sqrt{2\pi}}\frac{\sin k}{k}$$

$$\Rightarrow \hat{\phi}(k,0)=A(k)=\frac{1}{\sqrt{2\pi}}\frac{\sin k}{k}$$

1Y6B - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 4

INVERTING $\hat{\phi}(k,y)$ DIRECTLY FROM THE DEFINITION

$$\Rightarrow \hat{\phi}(k,y) = \frac{1}{\sqrt{2\pi}} \frac{\sin k}{k} e^{-|k|y}$$

$$\Rightarrow \phi(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\underbrace{\frac{1}{\sqrt{2\pi}} \frac{\sin k}{k} e^{-|k|y}}_{\text{EVEN}} \right] e^{ikx} \, dk$$

ONLY EVEN PART SURVIVES ×2

$$\Rightarrow \phi(x,y) = \frac{1}{2\pi} \int_{0}^{\infty} 2 \times \left(\frac{\sin k}{k} e^{-ky} \right) \cos kx \, dk$$

$$\Rightarrow \phi(x,y) = \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{k} e^{-ky} \sin k \cos kx \, dk$$

// ✱ REQUIRED

FINALLY FIND $\phi(\pm 1, 0)$

$$\Rightarrow \phi(\pm 1,0) = \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{k} \times 1 \times \sin k \times \underbrace{\cos(\pm k)}_{\text{(EVEN)}} \, dk$$

$$\Rightarrow \phi(\pm 1,0) = \frac{1}{2\pi} \int_{0}^{\infty} \frac{2 \sin k \cos k}{k} \, dk$$

$$\Rightarrow \phi(\pm 1,0) = \frac{1}{2\pi} \int_{0}^{\infty} \frac{\sin 2k}{k} \, dk$$

PROCEED BY SUBSTITUTION

$$\Rightarrow \phi(\pm 1,0) = \frac{1}{2\pi} \int_{0}^{\infty} \frac{\sin t}{\frac{1}{2}t} \frac{1}{2} dt$$

$$\Rightarrow \phi(\pm 1,0) = \frac{1}{2\pi} \int_{0}^{\infty} \frac{\sin t}{t} \, dt$$

$$\Rightarrow \phi(\pm 1,0) = \frac{1}{2\pi} \times \frac{\pi}{2}$$

$$\Rightarrow \underline{\phi(\pm 1,0) = \frac{1}{4}}$$

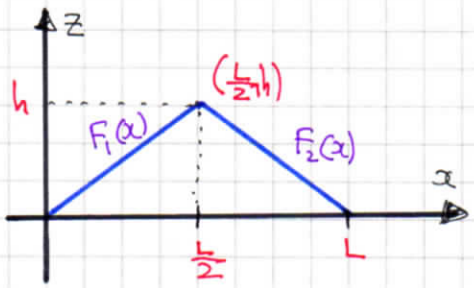
$t = 2k$
$k = \frac{1}{2}t$
$dk = \frac{1}{2} dt$
LIMITS UNCHANGED

IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTIONS

START WITH THE INITIAL CONFIGURATION OF THE STRING (DIAGRAM)

$$z(x,0) = F(x) = \begin{cases} \frac{2h}{L}x & 0 \leqslant x \leqslant \frac{L}{2} \\ -\frac{2h}{L}x + 2h & \frac{L}{2} < x \leqslant L \end{cases}$$

$$z(x,0) = F(x) = \begin{cases} F_1(x) & 0 \leqslant x \leqslant \frac{L}{2} \\ F_2(x) & \frac{L}{2} < x \leqslant L \end{cases}$$



$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$ SUBJECT TO

$$z(0,t) = 0 \ , \ \forall t \geqslant 0 \quad (1)$$

$$z(L,t) = 0 \ , \ \forall t \geqslant 0 \quad (2)$$

$$z(x,0) = f(x), \ 0 \leqslant x \leqslant L \quad (3)$$

$$\frac{\partial z}{\partial t}(x,0) = 0, \ 0 \leqslant x \leqslant L \quad (4)$$

(FROM REST)

ASSUME A SOLUTION IN VARIABLE SEPARATE FORM

$$\Rightarrow z(x,t) = X(x)T(t)$$

$$\Rightarrow \frac{\partial^2 z}{\partial t^2} = X(x)T''(t)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = X''(x)T(t)$$

SUBSTITUTE INTO THE P.D.E

$$\Rightarrow X''(x)T(t) = \frac{1}{c^2}X(x)T''(t)$$

$$\Rightarrow \frac{X''(x)T(t)}{X(x)T(t)} = \frac{1}{c^2}\frac{X(x)T''(t)}{X(x)T(t)}$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}$$

AS THE L.H.S IS A FUNCTION OF x ONLY AND THE R.H.S IS A FUNCTION OF t ONLY, BOTH SIDES MUST BE AT MOST A CONSTANT, SAY λ

1YGB – MATHEMATICAL METHODS 4 – PAPER B – QUESTION 5

$$\Rightarrow \frac{X''(x)}{X(x)} = \lambda \qquad \text{AND} \qquad \Rightarrow \frac{T''(t)}{T(t)} = \lambda c^2$$

$$\Rightarrow X''(x) = \lambda X(x) \qquad \Rightarrow T''(t) = c^2 \lambda T(t)$$

AS THE DISPLACEMENT z IS ZERO AT THE ENDPOINTS, IE AT $x=0$ AND AT $x=L$, WE NEED PERIODIC SOLUTIONS IN x – THIS IS ONLY ACHIEVABLE IF $\lambda < 0$

IF $\lambda > 0$ $\quad X(x) = \alpha e^{\sqrt{\lambda} x} + \beta e^{-\sqrt{\lambda} x}$

IF $\lambda = 0$ $\quad X(x) = \alpha x + \beta$

IF $\lambda < 0$ $\quad X(x) = \alpha \cos\sqrt{\lambda} x + \beta \sin\sqrt{\lambda} x$

LET $\lambda < 0$, SAY $\lambda = -p^2$

$$\Rightarrow X''(x) = -p^2 X(x) \qquad \text{AND} \qquad \Rightarrow T''(t) = -p^2 c^2 T(t)$$

$$\Rightarrow X(x) = A\cos px + B\sin px \qquad \Rightarrow T(t) = D\cos pct + E\sin pct$$

$$z(x,t) = X(x)T(t) = [A\cos px + B\sin px][D\cos pct + E\sin pct]$$

APPLY CONDITION ①, $z(0,t) = 0$

$$A(D\cos pct + E\sin pct) = 0 \;, \forall t \quad \Rightarrow \quad A = 0$$

ABSORBING B INTO D & E, WE OBTAIN

$$z(x,t) = [D\cos pct + E\sin pct]\sin px$$

APPLY CONDITION ②, $z(L,t) = 0$

$$\Rightarrow 0 = [D\cos pct + E\sin pct]\sin pL = 0, \forall t$$

$$\Rightarrow pL = n\pi \qquad n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow p = \frac{n\pi}{L}$$

IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTIONS

$$\Rightarrow Z_n(x,t) = \left[D_n \cos\frac{n\pi ct}{L} + E_n \sin\frac{n\pi ct}{L}\right]\sin\frac{n\pi x}{L},\ n=0,1,2,3,\ldots$$

IGNORING $n=0$, AS IT YIELDS $Z=0$

$$\Rightarrow Z(x,t) = \sum_{n=1}^{\infty}\left[\sin\frac{n\pi x}{L}\left[D_n\cos\frac{n\pi ct}{L} + E_n\sin\frac{n\pi ct}{L}\right]\right]$$

NEXT DIFFERENTIATE WITH RESPECT TO t

$$\Rightarrow \frac{\partial z}{\partial t} = \sum_{n=1}^{\infty}\left[\sin\frac{n\pi x}{L}\left[-\frac{n\pi c}{L}D_n\sin\frac{n\pi ct}{L} + \frac{n\pi c}{L}E_n\cos\frac{n\pi ct}{L}\right]\right]$$

APPLY CONDITION (4), $\frac{\partial z}{\partial t}(x,0)=0$

$$\Rightarrow 0 = \sum_{n=1}^{\infty}\left[\sin\frac{n\pi x}{L}\times E_n\frac{n\pi c}{L}\right],\ \forall x$$

$$\Rightarrow E_n = 0$$

$$\Rightarrow Z(x,t) = \sum_{n=1}^{\infty}\left[D_n\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}\right]$$

FINALLY APPLY CONDITION (3), $Z(x,0)=F(x)$, $0\le x\le L$

$$\Rightarrow \sum_{n=1}^{\infty}\left(D_n\sin\frac{n\pi x}{L}\right) = F(x)$$

WHICH IS A FOURIER EXPANSION IN SINES ONLY, FOR $0\le x\le L$

$$\Rightarrow D_n = \frac{1}{L/2}\int_0^L F(x)\sin\frac{n\pi x}{L}\,dx$$

$$\Rightarrow D_n = \frac{2}{L}\int_0^{L/2}\frac{2h}{L}x\sin\frac{n\pi x}{L}\,dx + \frac{2}{L}\int_{L/2}^{L}\left(2h-\frac{2h}{L}x\right)\sin\frac{n\pi x}{L}\,dx$$

IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 5

$$\Rightarrow D_n = \frac{4h}{L^2}\int_0^{\frac{L}{2}} x\sin\frac{n\pi x}{L}\,dx + \frac{4h}{L}\int_{\frac{L}{2}}^{L}\sin\frac{n\pi x}{L}\,dx - \frac{4h}{L^2}\int_{\frac{L}{2}}^{L} x\sin\frac{n\pi x}{L}\,dx$$

$$\Rightarrow D_n = \frac{4h}{L}\int_{\frac{L}{2}}^{L}\sin\frac{n\pi x}{L}\,dx + \frac{4h}{L^2}\left[\int_0^{\frac{L}{2}} x\sin\frac{n\pi x}{L}\,dx - \int_{\frac{L}{2}}^{L} x\sin\frac{n\pi x}{L}\,dx\right]$$

BY PART & IGNORING LIMITS

$$\int x\sin\frac{n\pi x}{L}\,dx = -\frac{L}{n\pi}x\cos\frac{n\pi x}{L} + \frac{L}{n\pi}\int\cos\frac{n\pi x}{L}\,dx$$

	x	1
	$-\frac{L}{n\pi}\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$

$$\int x\sin\frac{n\pi x}{L}\,dx = -\frac{L}{n\pi}x\cos\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}$$

$$\Rightarrow D_n = \frac{4h}{n\pi}\left[-\cos\frac{n\pi x}{L}\right]_{\frac{L}{2}}^{L}$$

$$+\frac{4h}{L^2}\left[\left[-\frac{L}{n\pi}x\cos\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right]_0^{\frac{L}{2}} - \left[-\frac{L}{n\pi}x\cos\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right]_{\frac{L}{2}}^{L}\right]$$

$$\Rightarrow D_n = \frac{4h}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

$$+\frac{4h}{L^2}\left[-\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2} - \left[-\frac{L^2}{n\pi}\cos n\pi + 0 - \left(-\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]\right]$$

$\sin n\pi = 0$

$$\Rightarrow D_n = \frac{4h}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

$$+\frac{4h}{L^2}\left[-\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\cos n\pi - \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 5

$$\Rightarrow D_n = \frac{4h}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right] + \frac{4h}{L^2}\left[\frac{2L^2}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{L^2}{n\pi}\cos\frac{n\pi}{2} + \frac{L^3}{n^2\pi^2}\cos n\pi\right]$$

$$\Rightarrow D_n = \cancel{\frac{4h}{n\pi}\cos\frac{n\pi}{2}} - \cancel{\frac{4h}{n\pi}\cos n\pi} + \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2} - \cancel{\frac{4h}{n\pi}\cos\frac{n\pi}{2}} + \cancel{\frac{4h}{n^2\pi^2}\cos n\pi}$$

$$\Rightarrow D_n = \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}$$

- IF n IS EVEN $\quad D_n = 0$
- IF n IS $1, 5, 9, 13, \ldots \quad D_n = \frac{8h}{n^2\pi^2}$
- IF n IS $3, 7, 11, 15, \ldots \quad D_n = -\frac{8h}{n^2\pi^2}$

$$\therefore D_{2m-1} = \frac{8h(-1)^{m+1}}{(2m-1)^2\pi^2}$$

FINALLY WE HAVE A SOLUTION

$$z(x,t) = \sum_{m=1}^{\infty}\left[\frac{8h(-1)^{m+1}}{\pi^2(2m-1)^2}\sin\left[\frac{(2m-1)\pi x}{L}\right]\cos\left[\frac{(2m-1)\pi ct}{L}\right]\right]$$

$$z(x,t) = \frac{8h}{\pi^2}\sum_{m=1}^{\infty}\left[\frac{(-1)^{m+1}}{(2m-1)^2}\sin\left[\frac{(2m-1)\pi x}{L}\right]\cos\left[\frac{(2m-1)\pi ct}{L}\right]\right]$$

IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 6

LOOK FOR A SOLUTION IN VARIABLE SEPARABLE FORM

$$\theta(x,y) = X(x)Y(y)$$

$$\frac{\partial^2\theta}{\partial x^2}(x,y) = X''(x)Y(y) \quad \& \quad \frac{\partial^2\theta}{\partial y^2} = X(x)Y''(y)$$

SUBSTITUTE INTO THE P.D.E AND REARRANGE

$$\Rightarrow \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

$$\Rightarrow X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\Rightarrow X''(x)Y(y) = -X(x)Y''(y) = 0$$

$$\Rightarrow \frac{X''(x)Y(y)}{X(x)Y(y)} = -\frac{X(x)Y''(y)}{X(x)Y(y)}$$

$$\Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$$

AS THE L.H.S IS A FUNCTION OF x ONLY & THE RHS IS A FUNCTION OF y ONLY, THEN BOTH SIDES ARE AT MOST A CONSTANT, CALL IT λ, WHICH CAN BE POSITIVE, ZERO OR NEGATIVE — CONSIDER SOLUTIONS FOR ALL CASES

IF $\lambda = 0$

- $X''(x) = 0$

  $X(x) = Ax + B$

- $Y''(y) = 0$

  $Y(y) = Cy + D$

$$\boxed{\theta(x,y) = (Ax+B)(Cy+D)}$$

IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 6

IF $\lambda > 0$, SAY $\lambda = p^2$

- $\dfrac{X''(x)}{X(x)} = p^2$

  $X''(x) = p^2 X(x)$

  $X(x) = A\cosh px + B\sinh px$

  (OR EXPONENTIALS)

- $\dfrac{Y''(y)}{Y(y)} = -p^2$

  $Y''(y) = -p^2 Y(y)$

  $Y(y) = C\cos py + D\sin py$

$$\theta(x,y) = (A\cosh px + B\sinh px)(C\cos py + D\sin py)$$

IF $\lambda < 0$, SAY $\lambda = -p^2$

- $\dfrac{X''(x)}{X(x)} = -p^2$

  $X''(x) = -p^2 X(x)$

  $X(x) = A\cos px + B\sin px$

- $\dfrac{Y''(y)}{Y(y)} = +p^2$

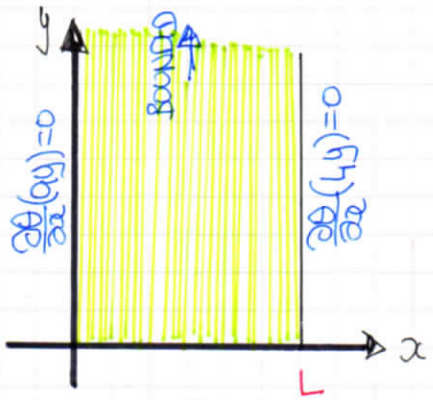
  $Y''(y) = p^2 Y(y)$

  $Y(y) = C\cosh py + D\sinh py$

  (OR EXPONENTIALS)

$$\theta(x,y) = (A\cos px + B\sin px)(C\cosh py + D\sinh py)$$

THE SOLUTION NEEDED IS DICTATED BY THE CONDITIONS OF THIS PROBLEM



$\theta(x,0) = f(x) = \dfrac{x(L-x)}{L^2}$

① $\dfrac{\partial\theta}{\partial x}(0,y) = 0$, $y \geqslant 0$

② $\dfrac{\partial\theta}{\partial x}(L,y) = 0$, $y \geqslant 0$

③ $\theta(x,0) = f(x) = \dfrac{x(L-x)}{L^2}$, $0 \leqslant x \leqslant L$

④ $\theta(x,y)$ IS BOUNDED AS $y \to \infty$

IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 6

FROM THE BOUNDARY CONDITIONS ① & ② WE REQUIRE A CIRCULAR SOLUTION IN x, AS WE REQUIRE $\frac{\partial \theta}{\partial x}=0$ AT TWO DIFFERENT VALUES OF x.

HENCE THE ONLY PLAUSIBLE SOLUTION IS THAT PRODUCED BY $\lambda=-p^2$ (OR THE CONSTANT TERM PRODUCED BY $\lambda=0$)

$$\theta(x,y)=(A\cos px+B\sin px)(Ce^{py}+De^{-py})+E$$

USED EXPONENTIALS INSTEAD OF HYPERBOLICS

↑ ALREADY INCLUDED

BY ④ $C=0$, SINCE e^{py} IS UNBOUNDED AS $y\to\infty$ $(p>0)$

$$\theta(x,y)=De^{-py}(A\cos px+B\sin px)$$

$$\theta(x,y)=e^{-py}(A\cos px+B\sin px)$$

$$\frac{\partial \theta}{\partial x}(x,y)=e^{-py}(-Ap\sin px+Bp\cos px)$$

BY ① $\frac{\partial \theta}{\partial x}(0,y)=0$ $\Rightarrow$ $0=e^{-py}Bp$

$\Rightarrow$ $B=0$ $(p\neq 0)$

$\Rightarrow$ $\theta(x,y)=Ae^{-py}\cos px$

$\Rightarrow$ $\frac{\partial \theta}{\partial x}(x,y)=-Ape^{-py}\sin px$

IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 6

USING CONDITION ②, $\frac{\partial \Theta}{\partial x}(L,y)=0$ $\Rightarrow 0=-Ape^{-py}\sin pL$

$\Rightarrow pL=n\pi,\ n\in\mathbb{Z}$

$\Rightarrow p=\frac{n\pi}{L}$

$$\Theta(x,y)=\sum_{n=0}^{\infty}\left[A_n e^{\frac{-n\pi y}{L}}\cos\frac{n\pi x}{L}\right]$$

$$\boxed{\Theta(x,y)=A_0+\sum_{n=1}^{\infty}\left[A_n e^{\frac{-n\pi y}{L}}\cos\frac{n\pi x}{L}\right]}$$

USING CONDITION ③

$$\Rightarrow \frac{x(L-x)}{L^2}=A_0+\sum_{n=1}^{\infty}\left[A_n\cos\frac{n\pi x}{L}\right]$$

$$\Rightarrow \frac{1}{L}x-\frac{x^2}{L^2}=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{L}\right]$$

BUILDING EVEN EXTENSION - HALF PERIOD IS L

- $a_0=\frac{1}{L}\int_{-L}^{L}f(x)\,dx=\ldots\text{EVEN}\ldots\frac{2}{L}\int_0^L\frac{1}{L}x-\frac{x^2}{L^2}\,dx$

$=\frac{2}{L}\left[\frac{1}{2L}x^2-\frac{x^3}{3L^2}\right]_0^L=\frac{2}{L}\left[\frac{1}{2}L-\frac{1}{3}L\right]=\frac{1}{3}$

- $a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}\,dx=\ldots\text{EVEN}\ldots$

$=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx=\frac{2}{L}\int_0^L\left(\frac{1}{L}x-\frac{x^2}{L^2}\right)\cos\frac{n\pi x}{L}\,dx$

$=\frac{2}{L^3}\int_0^L(Lx-x^2)\cos\frac{n\pi x}{L}\,dx=\ldots\text{BY PARTS}$

IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 6

$Lx-x^2$	$L-2x$
$\frac{L}{n\pi}\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$

$$\cdots=\frac{2}{L^3}\left\{\left[(Lx-x^2)\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right]_0^L-\frac{L}{n\pi}\int_0^L(L-2x)\sin\frac{n\pi x}{L}\,dx\right\}$$

$$=-\frac{2}{n\pi L^2}\int_0^L(L-2x)\sin\frac{n\pi x}{L}\,dx$$

BY PARTS AGAIN

$L-2x$	-2
$-\frac{L}{n\pi}\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$

$$\cdots=-\frac{2}{n\pi L^2}\left\{\left[\frac{L}{n\pi}(2x-L)\cos\frac{n\pi x}{L}\right]_0^L+\frac{2L}{n\pi}\int_0^L\cos\frac{n\pi x}{L}\,dx\right\}$$

$$=\left[\frac{2}{n^2\pi^2L}(L-2x)\cos\frac{n\pi x}{L}\right]_0^L-\frac{4}{n^2\pi^2L}\times\frac{L}{n\pi}\left[\sin\frac{n\pi x}{L}\right]_0^L$$

$$=\frac{2}{n^2\pi^2L}\left[-L\cos n\pi-L\right]=-\frac{2}{n^2\pi^2}\left[1+\cos n\pi\right]$$

$$=-\frac{2}{n^2\pi^2}\left[1+(-1)^n\right]=\begin{cases}-\frac{4}{n^2\pi^2} & \text{IF } n \text{ IS EVEN}\\ 0 & \text{IF } n \text{ IS ODD}\end{cases}$$

AND LETTING $n=2m$

$$a_m=-\frac{4}{(2m)^2\pi^2}=-\frac{1}{m^2\pi^2}$$

$$\Rightarrow\theta(x,y)=\frac{1}{3}+\sum_{m=1}^{\infty}\left[\frac{-1}{m^2\pi^2}e^{-\frac{2m\pi y}{L}}\cos\left(\frac{2m\pi x}{L}\right)\right]$$

$$\Rightarrow\boxed{\theta(x,y)=\frac{1}{6}-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}e^{-\frac{2n\pi y}{L}}\cos\left(\frac{2n\pi x}{L}\right)\right]}$$

IYGB – MATHEMATICAL METHODS 4 – PAPER B – QUESTION 6

FINALLY ALONG THE LINE $x=\frac{1}{2}L$

$$\theta\left(\tfrac{1}{2}L,y\right)=\frac{1}{6}-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}e^{\frac{-2n\pi y}{L}}\cos\left(\frac{2n\pi\left(\frac{1}{2}L\right)}{L}\right)\right]$$

$$\theta\left(\tfrac{1}{2}L,y\right)=\frac{1}{6}-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}e^{\frac{-2n\pi y}{L}}\cos\left(n\pi\right)\right]$$

$$\theta\left(\tfrac{1}{2}L,y\right)=\frac{1}{6}-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}e^{\frac{-2n\pi y}{L}}\right]$$

AS REQUIRED

IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 7

a)

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\sigma}\frac{\partial T}{\partial t} \quad \text{SUBJECT TO} \quad \begin{aligned} T(x,0) &= 0 \\ T(0,t) &= 1 \\ T(x,t) &\to 0, \text{ AS } x \to \infty \end{aligned}$$

REARRANGE THE O.D.E AND TAKE LAPLACE TRANSFORMS IN t

$$\Rightarrow \sigma\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

$$\Rightarrow \mathcal{L}\left[\sigma\frac{\partial^2 T}{\partial x^2}\right] = \mathcal{L}\left[\frac{\partial T}{\partial t}\right]$$

$$\Rightarrow \sigma\frac{\partial^2 \bar{T}}{\partial x^2} = s\bar{T} - \cancel{T(x,0)}$$

$$\Rightarrow \frac{\partial^2 \bar{T}}{\partial x^2} = \frac{s}{\sigma}\bar{T}$$

THIS IS A SECOND ORDER O.D.E WITH EXPONENTIAL SOLUTIONS

$$\Rightarrow \boxed{\bar{T}(x,s) = A(s)e^{-\sqrt{\frac{s}{\sigma}}x} + B(s)e^{\sqrt{\frac{s}{\sigma}}x}}$$

- APPLYING $T(x,t) \to 0$ AS $x \to \infty$

$$\Rightarrow \bar{T}(x,s) \to 0 \text{ AS } x \to \infty$$

$$\Rightarrow B(s) = 0 \text{ AS THE EXPONENTIAL WILL BE UNBOUNDED}$$

$$\Rightarrow \boxed{\bar{T}(x,s) = A(s)e^{-\sqrt{\frac{s}{\sigma}}x}}$$

TRANSFORMING THE FINAL CONDITION

$$\Rightarrow T(0,t) = 1$$

$$\Rightarrow \bar{T}(0,s) = \frac{1}{s}$$

USE IT THE ABOVE SOLUTION YIELDS $A(s) = \frac{1}{s}$

$$\Rightarrow \boxed{\bar{T}(x,s) = \frac{1}{s}e^{-\sqrt{\frac{s}{\sigma}}x}}$$

// AS REQUIRED

1YGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 7

a) PROCEED BY INVERTING VIA CONTOUR INTEGRATION THE TEMPERATURE GRADIENT, AS SUGGESTED

[THIS IS BECAUSE $\bar{T}(x,s)$ HAS A NON INTEGRABLE SINGULARITY AT $s=0$, WHICH WILL BECOME A PROBLEM EVENTUALLY IF WE WENT AHEAD TRYING TO INVERT $\bar{T}(x,s)$]

$$\frac{\partial \bar{T}}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{s}e^{-\sqrt{\frac{s}{\alpha}}x}\right] = -\frac{1}{\sqrt{s\alpha}}e^{-\sqrt{\frac{s}{\alpha}}x}$$

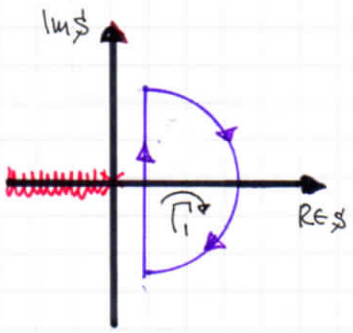
WE SHALL INVERT $\frac{\partial \bar{T}}{\partial x}$ TO GET THE TEMPERATURE GRADIENT $\frac{\partial T}{\partial x}$, NOTING THAT THE TRANSFORM IS RESPECT TO t

USING THE BROMWICH FORMULA

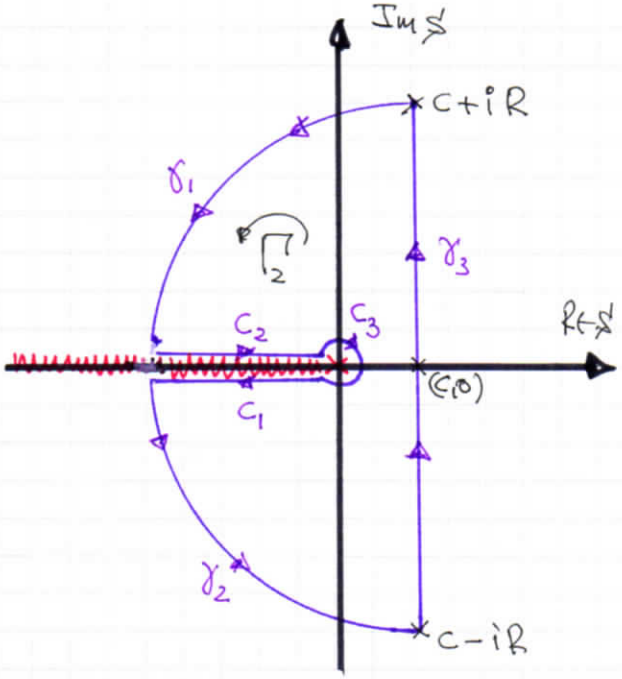
$$\Rightarrow \frac{\partial T}{\partial x} = \frac{1}{2\pi i}\int_\gamma \left[-\frac{1}{\sqrt{s\alpha}}e^{-\sqrt{\frac{s}{\alpha}}x}\right]e^{st}\,ds = \frac{1}{2\pi i\sqrt{\alpha}}\int_\gamma \frac{e^{-\sqrt{\frac{s}{\alpha}}x+st}}{\sqrt{s}}\,ds$$

THE INTEGRAND CONTAINS A NON INTEGER POWER IN s, SO WE HAVE A BRANCH POINT AT $s=0$, AND HENCE A BRANCH CUT

IF $t<0$



IF $t>0$



INVERTING FOR $t<0$ YIELDS

$$\frac{\partial T}{\partial x} = 0$$

$$\Rightarrow T(x,t) = \text{CONSTANT} = k = T_0$$

WE WILL GET THIS T_0 FROM THE OTHER SECTION

NOTE THAT BY CAUCHY THEOREM AS THERE ARE NO POLES, $\int_{\Gamma_1} = 0$ AND ARC DOES NOT CONTRIBUTE AS $R\to\infty$. THUS THE STRAIGHT LINE YIELDS ZERO

IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 7

INVERTING FOR $t>0$ & IGNORING $\frac{1}{2\pi i\sqrt{\sigma}}$ AT THE FRONT

- NO POLES INSIDE Γ_2, SO BY CAUCHY'S THEOREM THE INTEGRAL IS ZERO
- IT CAN BE SHOWN THAT THE CONTRIBUTION FROM γ_1 & γ_2 VANISHES AS $R\to\infty$ (STANDARD RESULT ON THIS TYPE OF CONTOUR SIMILAR TO JORDAN'S LEMMA)

PARAMETERIZE C_3 $\quad s=\varepsilon e^{i\theta}$, $\pi\leq\theta<-\pi$

$ds=\varepsilon i e^{i\theta}d\theta$

$$\left|\int_{C_3}\frac{e^{st}\,e^{-\frac{x}{\sqrt{\sigma}}\sqrt{s}}}{\sqrt{s}}\,ds\right|\leq\int_{\pi}^{-\pi}\left|\frac{e^{\varepsilon te^{i\theta}}\,e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}e^{i\frac{\theta}{2}}}}{\varepsilon^{\frac{1}{2}}e^{i\frac{\theta}{2}}}\left(\varepsilon i e^{i\theta}d\theta\right)\right|$$

$$=\int_{\pi}^{-\pi}\left|\frac{e^{\varepsilon t\cos\theta}\times e^{i\varepsilon t\sin\theta}\times e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}\cos\frac{\theta}{2}}\times e^{-i\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}\sin\frac{\theta}{2}}\times\varepsilon i e^{i\theta}}{\varepsilon^{\frac{1}{2}}e^{i\frac{\theta}{2}}}\right|d\theta$$

$$=\int_{\pi}^{-\pi}\frac{\left|e^{\varepsilon t\cos\theta}\right|\left|e^{i\varepsilon t\sin\theta}\right|\left|e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}\cos\frac{\theta}{2}}\right|\left|e^{-i\frac{x}{\sqrt{\sigma}}\sin\frac{\theta}{2}}\right|\left|\varepsilon i e^{i\theta}\right|}{\left|\varepsilon^{\frac{1}{2}}e^{i\frac{\theta}{2}}\right|}d\theta$$

$$=\int_{\pi}^{-\pi}\frac{\left|e^{\varepsilon t\cos\theta}\right|\times 1\times\left|e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}\cos\frac{\theta}{2}}\right|\times 1\times\varepsilon}{\varepsilon^{\frac{1}{2}}}\,d\theta$$

$$\leq\int_{\pi}^{-\pi}\frac{e^{\varepsilon t}\times e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}}\times\varepsilon}{\varepsilon^{\frac{1}{2}}}\,d\theta$$

$$=e^{\varepsilon t}\times e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}}\times\varepsilon^{\frac{1}{2}}\int_{\pi}^{-\pi}1\,d\theta$$

$$=e^{\varepsilon t}\times e^{-\frac{x}{\sqrt{\sigma}}\varepsilon^{\frac{1}{2}}}\times\varepsilon^{\frac{1}{2}}\times(-2\pi)$$

$$=O\left(\varepsilon^{\frac{1}{2}}\right)\to 0\quad\text{AS}\quad\varepsilon\to 0$$

$|e^{i\alpha}|=1$ IF $\alpha\in\mathbb{R}$

iVGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 7

THUS THE REQUIRED ANSWER (CONTRIBUTION OF γ_3) MUST SATISFY

$$\Rightarrow \int_{\gamma_3} + \int_{C_2} + \int_{C_1} = 0 \qquad (\text{AS } \gamma_1, \gamma_2, C_3 \text{ VANISH})$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{1}{2\pi i\sqrt{\sigma}} \int_{\gamma_3} \frac{e^{s't} e^{-\frac{x}{\sqrt{\sigma}}\sqrt{s'}}}{\sqrt{s'}}\, ds' = +\frac{1}{2\pi i\sqrt{\sigma}} \int_{C_1+C_2} \frac{e^{s't} e^{-\frac{x}{\sqrt{\sigma}}\sqrt{s'}}}{\sqrt{s'}}\, ds'$$

PARAMETRIZE EACH SECTION (C_1 & C_2)

● C_2: $s' = ue^{i\pi}$

$ds' = e^{i\pi}\, du$

$ds' = -du$

u FROM $+\infty$ TO 0

● C_1: $s' = ue^{-i\pi}$

$ds' = e^{-i\pi}\, du$

$ds' = -du$

u FROM 0 TO $+\infty$

$$\Rightarrow \frac{\partial T}{\partial x} = \frac{1}{2\pi i\sqrt{\sigma}} \left[\int_{\infty}^{0} \frac{e^{ute^{i\pi}} e^{-\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}e^{i\frac{\pi}{2}}}}{u^{\frac{1}{2}}e^{i\frac{\pi}{2}}} (-du) + \int_{0}^{\infty} \frac{e^{ute^{-i\pi}} e^{-\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}e^{-i\frac{\pi}{2}}}}{u^{\frac{1}{2}}e^{-i\frac{\pi}{2}}} (-du) \right]$$

$$\Rightarrow \frac{\partial T}{\partial x} = \frac{1}{2\pi i\sqrt{\sigma}} \left[\int_{0}^{\infty} \frac{e^{-ut} e^{-i\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}}}{iu^{\frac{1}{2}}} du - \int_{0}^{\infty} \frac{e^{-ut} e^{i\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}}}{-iu^{\frac{1}{2}}} du \right]$$

NOTE THAT $e^{\pm i\pi} = -1$, $e^{i\frac{\pi}{2}} = i$, $e^{-i\frac{\pi}{2}} = -i$

$$\Rightarrow \frac{\partial T}{\partial x} = \frac{1}{2\pi i\sqrt{\sigma}} \int_{0}^{\infty} \frac{e^{-ut}}{iu^{\frac{1}{2}}} \left(e^{i\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}} + e^{-i\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}} \right) du$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{1}{2\pi\sqrt{\sigma}} \int_{0}^{\infty} \frac{e^{-ut}}{u^{\frac{1}{2}}} \times 2\cos\left(\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}\right) du$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{1}{\pi\sqrt{\sigma}} \int_{0}^{\infty} \frac{e^{-ut}}{u^{\frac{1}{2}}} \cos\left(\frac{x}{\sqrt{\sigma}}u^{\frac{1}{2}}\right) du$$

IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 7

PROCEED BY A SUBSTITUTION

$u^{\frac{1}{2}} = \sqrt{u} = v$

$u = v^2$

$du = 2v\,dv$

LIMITS UNCHANGED

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{1}{\pi\sqrt{\sigma}} \int_0^{\infty} \frac{e^{-v^2 t}}{v} \cos\left(\frac{xv}{\sqrt{\sigma}}\right) (2v\,dv)$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{2}{\pi\sqrt{\sigma}} \int_0^{\infty} e^{-tv^2} \cos\left(\frac{xv}{\sqrt{\sigma}}\right) dv$$

USING THE RESULT GIVEN

$$\int_0^{\infty} e^{-ax^2} \cos kx\,dx = \sqrt{\frac{\pi}{4a}}\, \text{Exp}\left(-\frac{k^2}{4a}\right)$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{2}{\pi\sigma} \times \sqrt{\frac{\pi}{4t}}\, \text{EXP}\left(-\frac{x^2}{4\sigma t}\right)$$

$$\Rightarrow \frac{\partial T}{\partial x} = -\frac{1}{\sqrt{\pi\sigma t}}\, e^{-\frac{x^2}{4\sigma t}}$$

INTEGRATE WITH RESPECT TO x, NOTING THAT $T(0,t) = 1$

$$\Rightarrow T = C - \frac{1}{\sqrt{\pi\sigma t}} \int_0^{x} e^{-\frac{q^2}{4\sigma t}}\, dq$$

when $x=0$, $T=1$

$\therefore C = 1$

$$\Rightarrow T = 1 - \frac{1}{\sqrt{\pi\sigma t}} \int_0^{x} e^{-\frac{q^2}{4\sigma t}}\, dq$$

USING A SUBSTITUTION TO PRODUCE THE ERROR FUNCTION

$$\left.\begin{aligned} \xi^2 &= \frac{q^2}{4\sigma t} \\ \xi &= \frac{q}{2\sqrt{\sigma t}} \end{aligned}\right\} \Rightarrow d\xi = \frac{1}{2\sqrt{\sigma t}}\, dq$$

IYGB - MATHEMATICAL METHODS 4 - PAPER B - QUESTION 7

$$\Rightarrow d\phi = 2\sqrt{\sigma t}\, d\xi$$

AND THE LIMITS

WHEN $\phi = 0 \longmapsto \xi = 0$

WHEN $\phi = x \longmapsto \xi = \frac{x}{2\sqrt{\sigma t}}$

SO WE FINALLY OBTAIN

$$\Rightarrow T = 1 - \frac{1}{\sqrt{\sigma\pi t}} \int_0^{\frac{x}{\sqrt{4\sigma t}}} e^{-\xi^2} \left(2\sqrt{\sigma t}\, d\xi\right)$$

$$\Rightarrow T = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4\sigma t}}} e^{-\xi^2} d\xi$$

$$\Rightarrow T = 1 - \operatorname{erf}\left(\frac{x}{\sqrt{4\sigma t}}\right)$$ // ∎ REQUIRED

$$\left[T = \operatorname{erf}_c\left(\frac{x}{\sqrt{4\sigma t}}\right) \right]$$